Affine Fuzzy Set

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Abstract. In this paper, we introduce a new concept that is the affine fuzzy set and we give some properties of it .

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1. Introduction

There are many kind of fuzzy sets in a fuzzy vector space . Katsaras in [1] was defined a convex fuzzy set, a balanced fuzzy set and absorbing fuzzy set. In this paper we defined and introduced a new kind of fuzzy sets that is affine fuzzy sets by dependence on a concept of affine set which introduced in [2] . Some properties of affine fuzzy sets was prove it in this paper.

2. Preliminaries

Let X be a non-empty set. A fuzzy set in X is the element of the set I^X of all functions from X into the unit interval I =[0,1]. If $C_\alpha : X \to I$ is a function defined by $C_\alpha(x) = \alpha$ for all $x \in X$, $\alpha \in I$, then C_α is called a constant fuzzy set. Let X be a vector space over a field F , where F is the space of either the real or the complex numbers. If A_1, A_2, …, A_n are fuzzy sets in X , then the sum $A_1 + A_2 + \cdots + A_n$ (see [1]) is the fuzzy set A in X defined by

$A(x) = \sup\limits_{x_1+x_2+\cdots+x_n=x} \min\{A_1(x_1), A_2(x_2),..., A_n(x_n)\}$. Also, if A is a fuzzy set in X and

$\alpha \in F$, then αA is a fuzzy defined by $\alpha A(x) = \begin{cases} A(x/\alpha) & if\ \alpha \neq 0\ for\ x \in X \\ 0 & if\ \alpha = 0, x \neq 0 \\ \sup\limits_{y\in X} A(y) & if\ \alpha = 0, x = 0 \end{cases}$.

Let X, Y be non-empty sets and $f: X \to Y$ be a function. If A_1 and A_2 are fuzzy sets in X, then $f(A_1) \subset f(A_2)$.

Let A be a subset of a vector space X on a field F. Then A is an affine set in X (see [2]) if $\lambda A + (1-\lambda)A \subset A$ for all $\lambda \in F$. The smallest affine set of X which contains a subset A of X is called the affine spanned of A (in simple $aff(A)$) and

$$aff(A) = \{\sum_{k=1}^{n} \lambda_k x_k : \lambda_k \in F, x_k \in A, \sum_{k=1}^{n} \lambda_k = 1\}.$$

3. Main Results

Theorem 3.1. [2]

Let A, B be fuzzy sets in a vector space X over F, and let $\lambda, \alpha \in F$. Then,

(1) $\alpha(\lambda A) = \lambda(\alpha A) = (\alpha\lambda)A$;

(2) If $A \subseteq B$, then $\lambda A \subseteq \lambda B$.

(3) $\alpha(A+B) = \alpha A + \alpha B$

Definition 3.2.

Let X be a vector space over F. A fuzzy set A in X is called an affine fuzzy set if $\lambda A + (1-\lambda)A \subset A$ for each $\lambda \in F$.

Theorem 3.3.

Let A be a fuzzy set in a vector space X over F and $\lambda \in F$, then the following statement are equivalent.

(1) A is an affine fuzzy set.

(2) for all x, y in X, we have $A(\lambda x + (1-\lambda)y) \geq \min\{A(x), A(y)\}$.

Proof :

$(1) \Rightarrow (2)$

Suppose that A is an affine fuzzy set, by (Definition 3.2) we have $\lambda A + (1-\lambda)A \subset A$ for each $\lambda \in F$. Now, for each $x, y \in X$

$$A(\lambda x + (1-\lambda)y) \geq (\lambda A + (1-\lambda)A)(\lambda x + (1-\lambda)y)$$

$$= \sup_{\lambda x + (1-\lambda)y = x_1 + y_1} \min\{(\lambda A)(x_1), ((1-\lambda)A)(y_1)\}$$

$$\geq \min\{(\lambda A)(\lambda x),((1-\lambda)A)((1-\lambda)y)\} \geq \min\{A(x),A(y)\}.$$

$(2) \Rightarrow (1)$

Let $x \in X$, $(\lambda A+(1-\lambda)A)(x) = \sup\limits_{x_1+x_2=x} \min\{(\lambda A)(x_1),((1-\lambda)A)(x_2)\}$

(a) If $\lambda \neq 0$ and $1-\lambda \neq 0$, then

$$(\lambda A)(x_1) = A(\frac{1}{\lambda}x_1) \text{ and } ((1-\lambda)A)(x_2) = A(\frac{1}{(1-\lambda)}x_2)$$

Thus, $(\lambda A+(1-\lambda)A)(x) = \sup\limits_{x_1+x_2=x} \min\{A(\frac{1}{\lambda}x_1), A(\frac{1}{(1-\lambda)}x_2)\}$

But, $\min\{A(\frac{1}{\lambda}x_1), A(\frac{1}{(1-\lambda)}x_2)\} \leq A(\lambda(\frac{1}{\lambda}x_1)+(1-\lambda)(\frac{1}{(1-\lambda)}x_2)) = A(x_1+x_2) = A(x)$.

(b) If $\lambda \neq 0$ and $1-\lambda = 0$, then $(\lambda A)(x_1) = A(x_1)$ and

$$((1-\lambda)A)(x_2) = \begin{cases} 0 & ,x_2 \neq 0 \\ \sup\limits_{z\in X} A(z) & ,x_2 = 0 \end{cases}$$

(i) If $x_2 \neq 0$, then $((1-\lambda)A)(x_2) = 0$ and

$(\lambda A+(1-\lambda)A)(x) = \sup\limits_{x_1+x_2=x} \min\{A(x_1),0\} = 0$

Since $((1-\lambda)A)(x_2) \geq A(x_2)$, then $A(x_2) = 0$ and we get $\min\{A(x_1),A(x_2)\} = 0$.

(ii) If $x_2 = 0$, then $((1-\lambda)A)(x_2) = \sup\limits_{z\in X} A(z)$ and

$(\lambda A+(1-\lambda)A)(x) = \sup\limits_{x_1=x} \min\{\lambda A(x_1), \sup\limits_{z\in X} A(z)\} \leq \sup\limits_{x_1=x} A(x_1) = A(x)$.

(c) If $\lambda = 0$ and $1-\lambda \neq 0$, by the same way in (b).

(d) if $\lambda = 0$ and $1-\lambda = 0$, its impossible.

Theorem 3.4.

Let X be a vector space over a field F and $x_\circ \in X$, then

(1) the constant fuzzy set C_α , $\alpha \in I$ is an affine fuzzy set.

(2) the characteristic function of $\{x_\circ\}$ (in symbol $\chi_{\{x_\circ\}}$) is an affine fuzzy set.

Proof :

(1) clear.

(2) Let $x, y \in X$, $\lambda \in F$ and suppose that $\chi_{\{x_\circ\}}(\lambda x+(1-\lambda)y) < \min\{\chi_{\{x_\circ\}}(x), \chi_{\{x_\circ\}}(y)\}$

Then $\chi_{\{x_\circ\}}(\lambda x+(1-\lambda)y)=0$ and $\min\{\chi_{\{x_\circ\}}(x),\chi_{\{x_\circ\}}(y)\}=1$. Thus, $\lambda x+(1-\lambda)y\neq x_\circ$ and $x=y=x_\circ$ implies that $x_\circ\neq x_\circ$ which is impossible. Hence, $\chi_{\{x_\circ\}}(\lambda x+(1-\lambda)y)\geq \min\{\chi_{\{x_\circ\}}(x),\chi_{\{x_\circ\}}(y)\}$. Using (Theorem 3.3), we get the result.

Theorem 3.5.

Let A be a fuzzy set in a vector space X over F. Then, A is an affine fuzzy set in X if and only if $A_{[\alpha]}=\{x\in X : A(x)\geq\alpha\}$ is an affine set in X for all $0\leq\alpha\leq 1$.

Proof :

Suppose that A is an affine fuzzy set in X. Let $x,y\in A_{[\alpha]}$, then $A(x)\geq\alpha$ and $A(y)\geq\alpha$, thus, $\min\{A(x),A(y)\}\geq\alpha$. Since A is an affine fuzzy set in X, then $A(\lambda x+(1-\lambda)y)\geq\min\{A(x),A(y)\}\geq\alpha$, for all $\lambda\in F$. Thus, $\lambda x+(1-\lambda)y\in A_{[\alpha]}$. By other word $A_{[\alpha]}$ is an affine set in X.

The converse :

Suppose that $A_{[\alpha]}$ is an affine set in X for all $\alpha\in[0,1]$ and let $x,y\in X$, $\lambda\in F$. Let $\alpha=\min\{A(x),A(y)\}$, then $\alpha\in[0,1]$. Now, $A(x)\geq\min\{A(x),A(y)\}=\alpha$ and $A(y)\geq\min\{A(x),A(y)\}=\alpha$

That is mean $x,y\in A_{[\alpha]}$. But $A_{[\alpha]}$ is an affine set , then $\lambda x+(1-\lambda)y\in A_{[\alpha]}$. Thus, $A(\lambda x+(1-\lambda)y)\geq\alpha=\min\{A(x),A(y)\}$. From (Theorem 3.3.), A is an affine fuzzy set in X.

Theorem 3.6.

Let A and B be affine fuzzy sets in a vector space X over F, and let $\alpha\in F$. Then, αA, $A\cap B$ and $A+B$ are affine fuzzy sets in X.

Proof :

(1) To prove αA is an affine fuzzy set in X. From (Theorem 3.1.(1),(3)) , for $\lambda\in F$

$\lambda(\alpha A)+(1-\lambda)(\alpha A)=\alpha(\lambda A)+\alpha((1-\lambda)A)=\alpha(\lambda A+(1-\lambda)A)\subset\alpha A$.

(2) To prove $A\cap B$ is an affine fuzzy set in X. Let $\lambda\in F$ and Suppose that for all $x,y\in X$, let

$g_1 : X\to X,\ g_1(x)=\lambda x$;

$g_2 : X \to X$, $g_2(x) = (1-\lambda)x$;

$g_3 : X \to X$, $g_3(x) = x$;

are functions. Since A and B are affine fuzzy sets in X, then $\lambda A + (1-\lambda)A \subset A$ and $\lambda B + (1-\lambda)B \subset B$. In other words $g_1(A) + g_2(A) \subset g_3(A)$ and $g_1(B) + g_2(B) \subset g_3(B)$. Now, $g_1(A \cap B) \subset g_1(A)$ and $g_2(A \cap B) \subset g_2(A)$. Likewise $g_1(A \cap B) \subset g_1(B)$ and $g_2(A \cap B) \subset g_2(B)$. Thus,

$g_1(A \cap B) + g_2(A \cap B) \subset g_1(A) + g_2(A) \subset g_3(A)$ and

$g_1(A \cap B) + g_2(A \cap B) \subset g_1(B) + g_2(B) \subset g_3(B)$. Subsequently

$\lambda(A \cap B) + (1-\lambda)(A \cap B) = g_1(A \cap B) + g_2(A \cap B) \subset g_3(A) \cap g_3(B) = A \cap B$.

<u>(3) To prove</u> $A + B$ is an affine fuzzy set in X. By using the same style in (2). For $\lambda \in F$

$$\begin{aligned}\lambda(A+B) + (1-\lambda)(A+B) &= \lambda A + (1-\lambda)A + \lambda B + (1-\lambda)B \\ &= g_1(A) + g_2(A) + g_1(B) + g_2(B) \\ &\subset g_3(A) + g_3(B) = A + B.\end{aligned}$$

Definition 3.7.

Let A be a fuzzy set in a vector space X over F, then the smallest affine fuzzy set which contains A is called the affine spanned of A (in symbol $Span(A)$).

Theorem 3.8.

Let A be a fuzzy set of a vector space X over F. Then the affine spanned of A ($Span(A)$) is the set $B = \cup\{\ \mu_j = \sum_{i=1}^{n} \lambda_{ij} A : \lambda_{ij} \in F, \sum_{i=1}^{n} \lambda_{ij} = 1\}$.

Proof :

It is clear that $A \subset B \subset Span(A)$. We will finish the proof by showing that B is an affine fuzzy set. Let now $\lambda \in F$, $\mu_{k_1}, \mu_{k_2} \subset B$, then there is $\lambda_{ik_1}, \lambda_{jk_2} \in F$, $\begin{matrix} i = 1,\dots,n \\ j = 1,\dots,m \end{matrix}$ such that $\mu_{k_1} = \sum_{i=1}^{n} \lambda_{ik_1} A$, $\mu_{k_2} = \sum_{j=1}^{m} \lambda_{jk_2} A$ and $\sum_{i=1}^{n} \lambda_{ik_1} = 1$, $\sum_{j=1}^{m} \lambda_{jk_2} = 1$.

Now,

$$\lambda\mu_{k_1} + (1-\lambda)\mu_{k_2} = \lambda\left(\sum_{i=1}^{n} \lambda_{ik_1} A\right) + (1-\lambda)\left(\sum_{j=1}^{m} \lambda_{jk_2} A\right)$$

$$= \lambda\lambda_{1k_1}A + \cdots + \lambda\lambda_{nk_1}A + (1-\lambda)\lambda_{1k_2}A + \cdots + (1-\lambda)\lambda_{mk_2}A \subset B$$

Moreover, $\sum_{i=1}^{n}\lambda\lambda_{ik_1} + \sum_{j=1}^{m}(1-\lambda)\lambda_{jk_2} = 1$. It follows that $\lambda B + (1-\lambda)B \subset B$, which complete the proof.

References

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