Bogoliubov and Mitropolsky Theory in the Realm of Ultracold Quantum Gases

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Abstract

The present paper addresses the topical issue of collective modesin ultracold quantum gases. At first, we discuss the effects of the modulation interaction for a positive s-wave scattering length in order to study the resonance curve which has been observed in the experiment**.** Furthermore, we use a variational Gaussian ansatz for a spherical-symmetric trap and obtain the corresponding equation of motion for the condensate width in order to revisit the problem of the modulation interaction and prominent nonlinear effects, including resonance in the collective mode frequency.

Keywords: Bose-Einstein condensation, Gross-Piteavskii equation, and nonlinear effects,

نظرية بوكيليوبف و متروبوسكي لغازات الكم شديدة البرودة

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الخلاصة

يتناول البحث دراسة الترددات المتهيجة لمكثف بوز-اينشتاين الذي يعد من المواضيع المهمة لغازات الكم شديدة البرودة.أذ يتم دراسة الترددات المتهيجة من خلال تاثير تضمين (تعديل)لطول موجة الاستطارة الموجب sالذي من خلاله يمكن دراسة منحني الرنين الذي شوهد في التجربة. علاوة على ذلك نظرية كاوس للمتغيرات لفرق جهد اسطواني استخدمت متناظر أذ من خلال هذه الطريقة وجدت معادلات الحركة لعرض المكثف التي من خلالها يمكن اعادة النظر لمشكلة التضمين او تغيير في الشكل الهندسي لفرق جهد المكثف و دراسة الظواهر غير الخطية التي من ضمنها الرنين في الترددات المتهيجة.

الكلمات المفتاحية: مكثف بوز-اينشتاين, معادلة كروس-بيتافيسكي, الظواهر غير الخطية.

I. Introduction

Einstein predicted in 1925 that a gas of massive bosonic particles will undergo a phasetransition into a Bose-Einstein condensate (BEC) if it is cooled below a critical temperature**[1]**. It is now possible to create such a BEC in laboratory by combining laser cooling andevaporative cooling **[2, 3]**. The first cooling technique relies on trapping the atoms due to aZeeman shift **[2]** in a magneto-optical trap and cooling them to about10 μK. Afterwards,they are bombarded by photons of counter-propagating laser beams in all three spatialdirections. The second cooling technique is performed by removing the high-energy tail ofthe thermal distribution from the trap, thus lowering the temperature below1 μK**[3]**. Theabove mentioned cooling techniques have paved the way for numerous experimental andtheoretical works to study and understand ultracold quantum gases which can be regardedas a new state of matter. Many experiments focused on investigating collective excitations ofharmonically trapped BECs as they can be measured very accurately and, therefore, allow forextracting the respective system parameters **[4]**. They are described by the time-dependentGross-Piteavskii equation for a macroscopic wave function of a BEC at zero temperature **[5]**.Either the Gross-Piteavskii equation is solved numerically **[6]** or it is solved variationally by assuming a Gaussian ansatz for the wave function **[5]**. In a recent experiment led by V. S. Bagnato and R. G. Hulet a quadrupole mode of a ^{7}Li condensate was excited by modulating **[7]** the scattering length through a broad Feschbach resonance **[8]**. The experiment was done in the following way: A BEC of ^{7}Li was produced by Zeeman decelerating the atoms and confining them in optical trap. Coaxial to the trap laser there were coils which allow to manipulate the s-wave scattering length. In the first step the scattering length had to be large in order to accelerate the thermalisation of the atoms and enhancing the evaporation process. After obtaining a condensate of about 10^5atoms the magnetic field was lowered slowly to the desired value. The modulation of the scattering length was now realized by adding a small AC component to the bias field of the trap.

II. Method

Many experiments focused on investigating collective excitations of harmonically trapped BECs as they can be measured very accurately and, therefore, allow for extracting the respective system parameters **[5].** They are described by the time-dependent Gross-Piteavskii equation for a macroscopic wave function of a BEC at zero temperature **[6]**:

$$i \frac{\partial}{\partial t}\psi(\boldsymbol{r},t) = \left\{ \quad \frac{^2}{2M}\nabla^2 + V(\boldsymbol{r})\right\}\psi(\boldsymbol{r},t)$$

$$+g(t)N|\psi(\boldsymbol{r},t)|^2\psi(\boldsymbol{r},t) \ldots\ldots (1)$$

Where $\psi(\boldsymbol{r},t)$ denotes a condensate wave function normalized to unity, and N is the total number of atoms in the condensate. On the right-hand side of the above equation we have a kinetic energy term, an external spherical-symmetric harmonic trap potential $V(\boldsymbol{r}) = \frac{1}{2}M\omega_\rho^2 r^2$, with the interaction strength of two-body interaction$g(t)$ which is proportional to the s-wave scattering length $a(t)$, and is given by$g(t) = \frac{4\pi \quad ^2 a(t)}{M}$, where$M$denotes the mass of the correspondingatomic species.Either the Gross-Pitaevskii (GP) equation is solved numerically **[7]** or it is solved variationally by assuming a Gaussian ansatz for the wave function **[9, 10]**. In the latter case the equation of motion for the condensate width in spherical-symmetric harmonic trap$w(t)$ read in dimensionless form as follow

$$w(t) + w(t) \quad \frac{1}{w^3(t)} \quad \frac{P(t)}{w^4(t)} = 0 \ldots\ldots (2)$$

Here $P(t) = \sqrt{\pi/2}\,\frac{Na(t)}{l}$ denotes the dimensionless interaction strength with the particle number N, $a(t)$ is the time dependent s-wave scattering length, and the oscillator length$l = \sqrt{\ /M\omega_p}$. Aquadrupole mode of a [7]Li condensate was excited by modulating**[1]**the scattering length through a broad Feshbach resonance **[8]** according to

$$P(t) = P_0 + P_1 \sin \Omega t \dots \dots (3).$$

This finding is surprising as one would expect, according to the general theory of parametric resonance, an inverted peak structure of the experimental result of Ref. **[7]**: the peak at the twice quadrupole mode frequency should be larger than the peak at the quadrupole mode frequency as well as particularly if one would excite one mode the other modes will be excited eventually. Therefore, we develop an analytical method to study and describe the dynamics of the system which will be discussed in the next section.

III. Bogoliubov and Mitropolsky Method in the Theory of Nonlinear Oscillation

To this end, we assume that the reason for this typical result is due to the anisotropy of theharmonic trap. In order to check this working hypothesis we have examine a parametricmodel system with one degree of freedom for an isotropic spherical trapusing Bogoliubov and Mitropolskytheory, where the equationof motion for the condensate $w(t)$ width reads **[11]**

$$w(t) + w(t) \quad \frac{1}{w^3(t)} \quad \frac{P_0}{w^4(t)} \quad \frac{P_1}{w^4(t)} \sin \Omega t = 0, \dots \dots (4)$$

With $P_0$is the dimensionless of two-body interaction, P_1 is driving amplitude, and Ω is driving frequency. The ansatz to solve Eq. (4) is

$$w(t) = w_0 + \delta w_1(t), \dots \dots (5)$$

where$w_1(t)$ is a deviation from the equilibrium position w_0, which is determined by

$$w_0^5 = w_0 \quad P_0, \dots \dots (6)$$

yielding the equation of motion

$$w_1(t) + \omega_0^2 w_1(t) = \varepsilon f(w_1, \Omega t), \dots \dots (7)$$

where

$$\omega_0 = \sqrt{\frac{w_0^5 + 3\, w_0 \quad 4P_0}{w_0^5}}, \dots \dots (8)$$

The right-hand side of Eq. (7) is given by

$$\varepsilon f(w_1, \Omega t) = \frac{1}{\left(w_0 + w_1(t)\right)^3} + \frac{P_0}{\left(w_0 + w_1(t)\right)^4} + \frac{P_1 \sin \Omega\, t}{\left(w_0 + w_1(t)\right)^4} + \frac{3}{w_0^4} w_1(t) \quad \frac{1}{w_0^3} w_1(t) \quad \frac{P_0}{w_0^4} w_1(t) \frac{4\, P_0}{w_0^5} w_1(t) \dots \dots (9)$$

The solution of Eq. (7) is given as a perturbation in the smallness parameter ε:

$$w_1(t) = a \cos(\Omega\, t + \theta) + \varepsilon u^1(\Omega\, t + \theta) + \varepsilon^2 u^2(\Omega\, t + \theta) + \quad \dots (10)$$

Furthermore, a and θ are functions of time and can be determined perturbatively following a systematic procedure developed in Ref. **[12]**. Here a and θ are defined as solutions of differential equation in the form

$$\frac{da}{dt} = \varepsilon A_1(a, \theta) + \varepsilon^2 A_2(a, \theta) \dots, \dots \dots (11)$$

$$\frac{d\theta}{dt} = \varepsilon B_1(a, \theta) + \varepsilon^2 B_2(a, \theta) \dots, \dots \dots (12)$$

where the functions on the right-hand side are periodic functions in the angular variable θ. Up to first order in ε we have

$$A_1(a, \theta) = \frac{q}{4\pi p \nu} \sum_{\sigma} e^{iq\sigma\theta} \times \int_0^{2\pi} \int_0^{2\pi} d\psi d\theta F(a, \theta, \psi) e^{-iq\sigma\theta} (\psi \quad \theta) \sin \psi, \dots \dots (13)$$

$$B_1(a, \theta) = \frac{\Delta q}{p\nu} \quad \frac{q}{4\pi p\nu} \times$$

$$\sum_{\sigma} e^{iq\sigma\theta}\int_0^{2\pi}\int_0^{2\pi} d\psi d\theta F(a,\theta,\psi)e^{-iq\sigma\theta}(\psi \quad \theta)\cos\psi \ldots\ldots(14)$$

Here p and q are integers, $\Delta = \omega \quad \frac{p}{q}\nu$, and the function $F(a,\theta,\psi)$ is defined by

$$F(a,\theta,\psi) = f(a,\theta,\psi) + g(a,\theta,\psi)\sin\theta, \ldots\ldots(15)$$

With the functions

$$f(a,\theta,\psi) = \frac{1}{(w_0 + a\cos\psi)^3} + \frac{P_0}{(w_0 + \mathrm{a}\cos\psi)^4} + \frac{3a\cos\psi}{w_0^4} + \frac{4\,P_0 a\cos\psi}{w_0^5} \quad \frac{P_0}{w_0^4} \quad \frac{1}{w_0^3} \ldots\ldots(16)$$

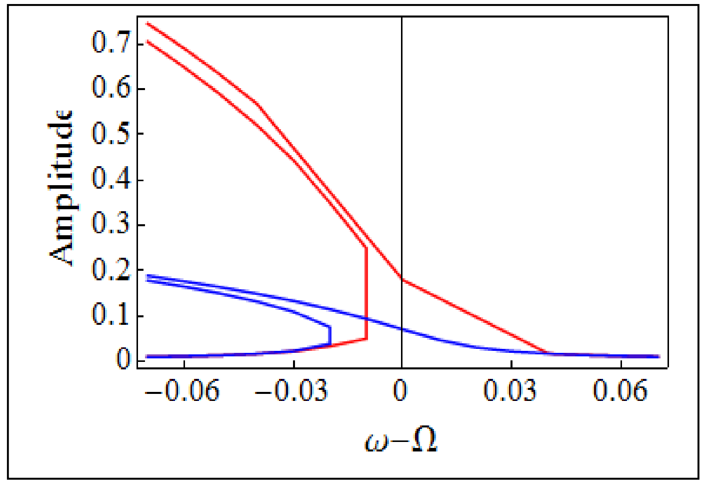
$$g(a,\theta,\psi) = \frac{P_1}{(w_0 + \mathrm{a}\cos\psi)^4} \ldots\ldots(17)$$

In the case of the main resonance, when we have $p = q = 1$, the functions $A_1(a,\theta)$ and $B_1(a,\theta)$ turn out to be determined by

$$A_1(a,\theta) = \quad \frac{P_1\cos\theta}{2\nu w_0}\frac{a}{\left(1 \quad \frac{a^2}{w_0^2}\right)^{\frac{5}{2}}} \ldots\ldots(18)$$

$$B_1(a,\theta) = \Delta \quad \frac{3a}{2\nu w_0^4\left(1 \quad \frac{a^2}{w_0^2}\right)^{\frac{5}{2}}} + \frac{P_1\sin\theta(1+4a^2)}{2\nu w_0^4 a\left(1 \quad \frac{a^2}{w_0^2}\right)^{\frac{7}{2}}} + \frac{P_0(4+a^2)}{2\nu w_0^5\left(1 \quad \frac{a^2}{w_0^2}\right)^{\frac{7}{2}}} \quad \frac{3}{2\nu w_0^4} \quad \frac{2P_0}{\nu w_0^5} \ldots\ldots(19)$$

For stationary state we have to demand $\frac{d\theta}{dt} = \frac{da}{dt} = 0$. From Eq. (18) we determine $\theta = \pm\frac{\pi}{2}$. Inserting this into Eq. (19) yields the following two branches for the resonance curves:



$$\Delta \quad \frac{3}{2\nu w_0^4\left(1 \quad \frac{a^2}{w_0^2}\right)^{\frac{5}{2}}} + \frac{P_1(1+4a^2)}{2\nu w_0^4 a\left(1 \quad \frac{a^2}{w_0^2}\right)^{\frac{7}{2}}} + \frac{P_0(4+a^2)}{2\nu w_0^5\left(1 \quad \frac{a^2}{w_0^2}\right)^{\frac{7}{2}}} \quad \frac{3}{2\nu w_0^4} \quad \frac{2P_0}{\nu w_0^5} = 0 \ \ldots\ldots(20)$$

Figure 1: Comparison of our results Eq. (20) for $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ (blue curves) with the finding of Ref. **[11]** **(**red curves).

Note that for small driving amplitude a the resonance curve Eq. (20) reproduces the result in Ref. **[12]** up to the first order:

$$3Ba^3 \pm 3Ea^2 \pm 4C + 8\Omega\Delta a = 0 \ldots\ldots(21)$$

Here the respective abbreviations read

$$B = \frac{10}{w_0^6} + \frac{20\,P_0}{w_0^7}, \qquad E = \frac{10\,P_1}{w_0^6},$$

$$C = \frac{P_1}{w_0^4}, \qquad \Delta = \omega_0 \quad \Omega \ldots\ldots(22)$$

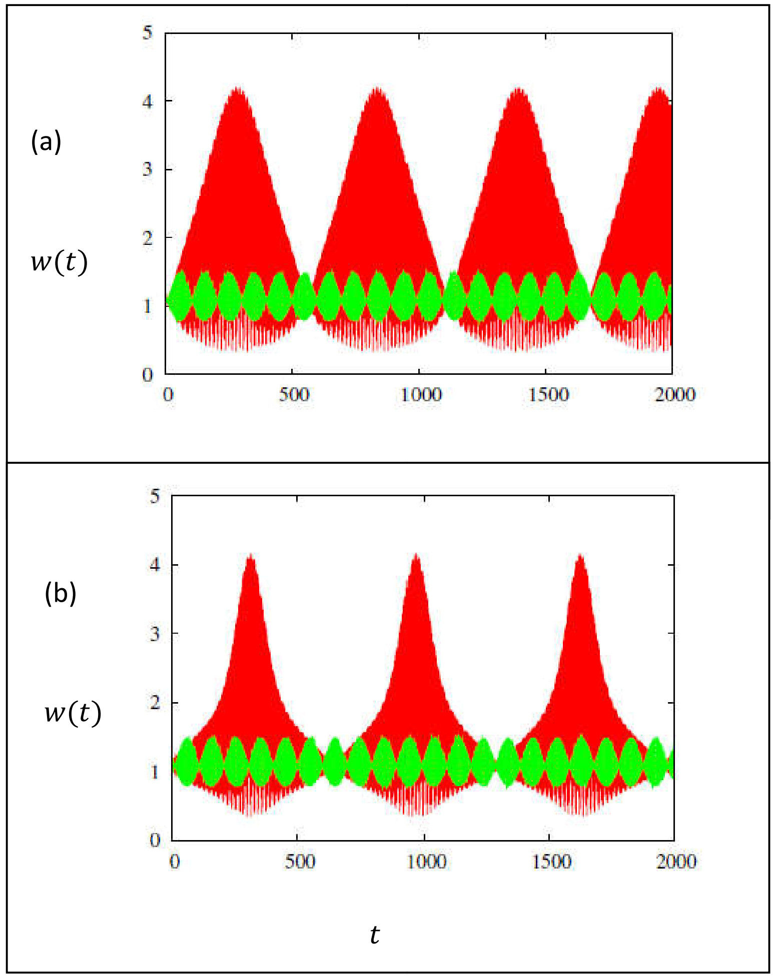
Obviously, Fig. 1 does not yet explain the experimental resonances curve of Ref. **[7]**. The reason is that the method of Ref. **[12]** takes in Eq. (10) only an oscillation with frequency Ω into account. Therefore, we have to extend this procedure in view of the real time dependence of Ref. **[7]**by allowing, in addition, an oscillation with frequencyω_0.

IV. Numerical Simulations

Numerical solution for the time interval$(0, T)$ is then analyzed using the discrete Fouriertransform. We take numerical values obtained with the time step t and preform the discrete transformation from time to frequency domain. Maximal ωaccessible this way is given by$\omega_{max}\omega_{max} = \pi/t$. The resolution of the obtained spectrum is determined by $\omega = 2\pi/T$.

We start by analyzing complete Fourier spectrafor $\Omega = 2.08$usingT $= 8000$and $t = 0.1$,yielding $\omega = 0.0008$. In further calculations, we set $P_0 = 1$and $P_1 = 0.1$. Time-independent solution of the previousequation is given by$w_0 = 1.222098$ .From the linear response analysis, we determine the breathing modefrequency $\omega_0 = 2.11220$. As initial conditions for solving the differential equation (4) we use$w_0(0) = 0$and $w_0(0) = 0$. Examples ofthe time-dependent solution are shown in Figure 1. Clearly, large amplitude oscillations are present for$\Omega = 2.08$and $\Omega = 4.08$, anexample close to a resonance.

Figure 3 shows the amplitude ratios for the driving $\frac{w_{\omega_0}}{w_0}$ (green curve) and breathing $\frac{w_\Omega}{w_0}$ (red curve) modes obtained numerically by solving Eq. (4). Its solution possessesprecisely two modes,$\omega = \omega_0$ and$\omega = \Omega$.We denote these modes as basic modes. Main resonance is located at$\omega_0 = \Omega$.



To this end, Fig. 4 shows positions of all the prominent peaks in the vicinity of ω_0 for different driving frequency Ω values . These peaks appear to be equidistant and by fitting linear function which is obtained from numerical data.

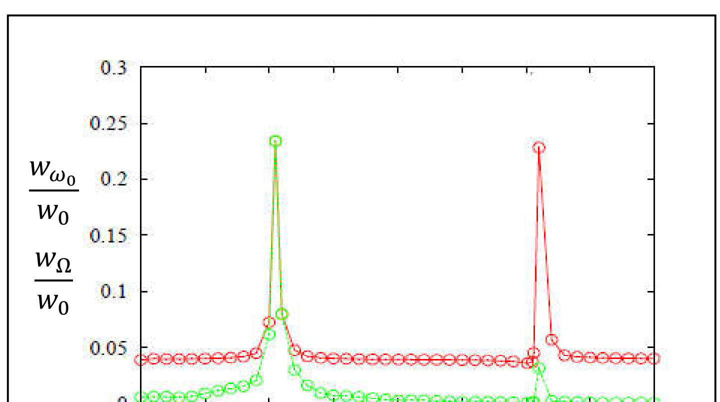


Figure 3:Amplitude ratios for the driving$\frac{w_{\omega_0}}{w_0}$ (green curve) and breathing $\frac{w_\Omega}{w_0}$ (red curve) modes, obtained using the low-resolution Fourier transformation,$T = 200$, $\Delta t = 0.1$, and $\Delta\omega = 0.3$.

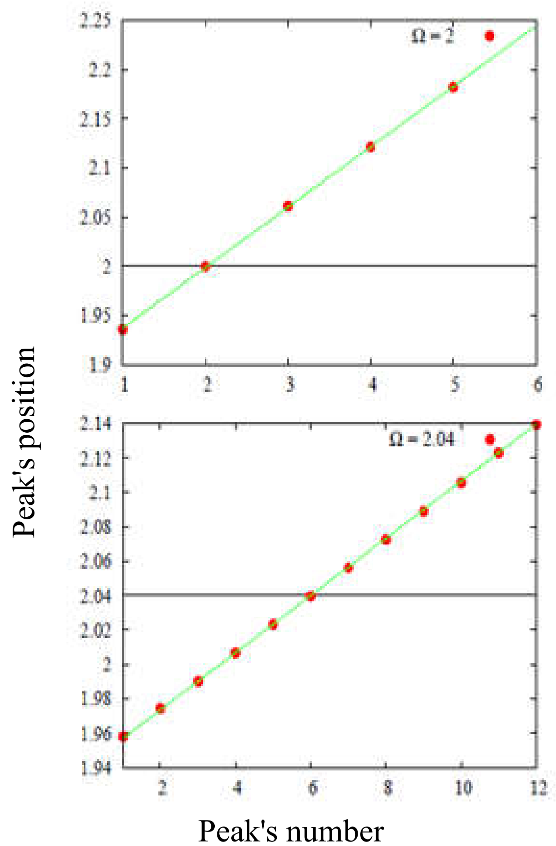


Figure 4:Positions of prominent peaks in the Fourier spectrum of the condensate width in the vicinity of$\omega_0$fordifferent values of Ωclose toω_0:,$T = 8000$, $\Delta t = 0.1$.

V. Conclusions

We have studied in detail Bogoliubov and Mitropolskytheory in the radially symmetric BEC assuming that time dependence of the condensate width Eq. (4) is captured by an ordinary differential equation.We have studied the dynamics and collective excitations of a BEC for modulating the interaction strength at zero temperature. All results are obtained using Mathematica and C^{++} programs.

References

1. A. Einstein, Sitzungsber. Preuss. Akad. Wiss.,**18** (1925).
2. M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E.A. Cornell, Science **269**, 198 (1995).
3. K. B. Davis, M. O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M.Kurn, and W. Ketterle, Phys.Rev. Lett. **75**, 3969 (1995).
4. D. M. Stamper-Kurn, H. J. Miesner, S. Inouye, M. R. Anderws, and W. Ketterle,Phys. Rev.Lett. **88**, 500 (1998).
5. E. P. Gross, NuovoCimento**20**, 454 (1961), L. P. Pitaevskii, Sov. Phys. JETP **13**, 451 (1961).
6. L. You, W. Hoston, and M. Lewenstein, Phys. Rev. A **55**, R1581 (1997).
7. S. E. Pollack, D. Dries, and R. G. Hulet, K. M. F. Magalhaes, E. A. L. Henn, E. R. F.Romas, M. A. Caracanhas, and V. S. Bagnato, Phys. Rev. A 81, 053627 (2010).
8. S. E. Pollack, D. Dries, M. Junker, Y. P. Chen, T. A. Corcovilos, and R. G. Hulet,Phys. Rev. Lett.**102**, 090402 (2009).
9. V. M. Perez, H. Michinel, J. I. Cirac, M. Lewenstein, P. Zoller, Phys. Rev. A **56**, 1424 (1997).
10. V. M. Perez, H. Michinel, J. I. Cirac, M. Lewenstein, P. Zoller, Phys. Rev. **77**, 5320 (1996).
11. F. K. Abdulleav, R. M. Galimzyanov, M. Brtka, and R. A.

Kraenkel, J. Phys. B 37, **3535** (2004).

12. N. N. Bogoliubov, Y. A. Mitropolsky, *Asympotitic Methods in the Theory of Nonlinear Oscillation* (Hindustan P. Corp. Delhi-6, 1961).