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RESEARCH ARTICLE

SOME RESULTS OF RIESZ REPRESENTATION FOR FUZZY NORMED SPACES

***Noori F. Al-Mayahi, Hind W. Twair**
Department of Mathematics College of Computer Science and Mathematics
University of Al-Qadisiya, Diwaniya, Iraq

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******Corresponding Author***

Noori F. Al-Mayahi

Abstract

The main aim of this paper is to consider the fuzzy norm , define the fuzzy normed spaces , and prove some theorems in these spaces and study some basic results on finite dimensional fuzzy normed spaces.

INTRODUCTION

The notion of fuzzy norm on a linear space was introduced by Katsaras [7] in 1984. Later on many other Mathematicians like Felbin [5] in 1992 ,Cheng and Mordeson [4] in 1994 , Bag and Samanta [2] in 2003 etc , have given different definitions of fuzzy normed spaces . In this paper we have been able to establish some important results involving compactness of finite dimensional fuzzy normed linear spaces including Riesz Lemma .

2.Preliminaries

Definition (2.1) : [6] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a t-norm if $*$ is satisfies the following conditions :

(i) $*$ is commutative and associative ;

(ii) $a * 1 = a$ for all $a \in [0,1]$;

(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all ($a, b, c, d \in [0,1]$). If $*$ is continuous then it is called continuous t-norm.

Definition (2.2) :[11] Let X be a non-empty set, $*$ be a continuous t-norm on I=[0,1]. A function $N : X \times (0,1) \longrightarrow [0,1]$ is called a fuzzy norm function on X if satisfies the following axioms for all $x, y \in X, t, s > 0$:

(N1) $N(x,t) > 0$;

(N2) $N(x,t) = 1 \iff x = 0$;

(N3) $N(\alpha x, t) = N\left(x, \frac{t}{|\alpha|}\right)$;

(N4) $N(x,t) * N(y,s) \leq N(x+y, t+s)$;

(N5) $N(x,.): (0,\infty) \to [0,1]$ is continuous;

(N6) $Lim_{t\to\infty} N(x,t) = 1$.

$(X, N,*)$ is said to be a fuzzy normed space.

Definition (2.3) : [2] Let $(X, N,*)$ be a fuzzy normed linear space .Let $\{x_n\}$ be a sequence in X .Then $\{x_n\}$ is said to be convergent if $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1 \ \forall t > 0$. In this case x is called the limit of the sequence $\{x_n\}$ and is denoted by $lim x_n$.

Definition (2.4) : [2] Let $(X, N,*)$ be a fuzzy normed linear space . A subset B of X is said to be closed if for any sequence $\{x_n\}$ in B converges to x i.e.$\lim_{n\to\infty} N(x_n - x, t) = 1 \ \forall t > 0$ implies that $x \in$ B.

Definition(2.5) : [9] Let $(X, N,*)$ and $(Y, N,*)$ be two fuzzy normed spaces. Then the function $f : X \to Y$ is said to be continuous at $x_\circ \in X$ if for all $\varepsilon \in (0,1)$ and all $t > 0$ there is exist $\delta \in (0,1)$ and $s > 0$ such that for all $x \in X$

$N(x - x_\circ, s) > 1 - \delta$ implies $N(f(x) - f(x_\circ), t) > 1 - \varepsilon$.

The function f is called continuous function if it continuous at every point of X.

Definition (2.6) : [3] Let $(X, N,*)$ be a fuzzy normed linear space. We define a set $B(x, \alpha, t)$ as $B(x, \alpha, t) = \{ y : N(x - y, t) > 1 - \alpha \}$.

Definition (2.7) : [10] Let $(X, N,*)$ be a fuzzy normed linear space and $B \subset X$. B is said to be fuzzy bounded if for each r , $0 < r < 1$, $\exists\ t > 0$ such that $N(x, t) > 1 - r \quad \forall\ x \in B$.

Theorem (2.8) : [1] Let f be linear functional of fuzzy normed linear space X in to another fuzzy normed linear space Y. Then the following statements are equivalent :

1- f is continuous .
2- f is continuous at origin .
3- f is bounded.

Theorem (2.9) : [8] Let X be linear space over a field F.

(1) If $x \epsilon X$, and a function $T_x : X' \longrightarrow F$ defined by $T_x(f) = f(x)$ for all $f \epsilon X'$, then T_x is linear function , i.e. $T_x \epsilon X''$, and it is called Evaluation Functional Induced by x.

(2) If the function $\psi : X \longrightarrow X''$ defined by $\psi(x) = T_x$ for all $x \epsilon$X,

then ψ injection linear function and ψ is called Canonical Function.

Definition (2.10) : Let X be a fuzzy normed linear space over a field F. We define X^{**} as :

$X^{**} = (X^*)^* = \{ f : X^* \to F ,\ f$ is bounded (continuous) linear functional$\}$

X^{**} is called the second dual space.

Definition (2.11) : Let $(X, N,*)$ and $(Y, N,*)$ be fuzzy normed spaces over F and $f : X \to Y$ be linear function . We define

$N(f, t) = \inf \{ N(f(x), t) : x \in X \}$ $for\ all\ t > o$.

Theorem (2.12) : Let $(X, N,*)$ and $(Y, N,*)$ be fuzzy normed spaces . Then $N(f, t)$ is defined in Definition (2.11) is a norm.

Proof : We check the items in Definition (2.2) .It is easy to see that (N_1) , $(N_2), (N_3), (N_5)$ and (N_6) are true. We consider (N_4) :

$$
\begin{aligned}
N(f, t) * N(g, t) &= \{\inf \{N(f(x), t) : x \in X\} * \{\inf \{N(g(x), s) : x \in X\} \\
&= \inf \{ N(f(x), t) * N(g(x), s) : x \in X \} \\
&\le \inf \{ N((f + g)(x), t + s) : x \in X \} \\
&= N(f + g, t + s) .
\end{aligned}
$$

3. Main results

Theorem (3.1) : Let $(X, N,*)$ be a fuzzy normed space over a field F.

(1) $If\ x \in X\ and\ T_x : X^* \to F$ defined as $T_x(f) = f(x)$ for all $f \in X^*$, then $T_x \in X^{**}\ and\ N(T_x, t) = N(f, t)$.

(2) $If\ \psi : X \to X''$ defined as $\psi(x) = T_x$ for all $x \in X$, then ψ is one-to-one linear function.

Proof : (1) T_x is linear (see theorem (2.9)).

To prove T_x is continuous .

$X^* = \{ f : X \to F , f$ is bounded (continuous) linear function $\}$.

Since f is continuous at every point of X, hence f is continuous at $x_0 \in X$. Then for all $\varepsilon \in (0,1)$ for all $t > 0$ there exist $\delta \in (0,1)$ and $s > 0$ such that for all $x \in X$

$N(x - x_0, s) > 1 - \delta \Rightarrow N(f(x) - f(x_0), t) > 1 - \varepsilon$

$\Rightarrow N(T_x - T_{x_\circ}), t) > 1 - \varepsilon$.

Therefore T_x is a continuous at x_0. Since x_0 is an arbitrary point

Then T_x is a continuous function, hence $T_x \in X^{**}$.

$$N(T_x, t) = \inf\{ N(T_x(f), t) : x \in X\}$$
$$= \inf \{ N(f(x), t) : x \in X\}$$
$$= N(f, t).$$

(2) see theorem (2.9).

Theorem (3.2) : Let $\{x_1, x_2, \dots, x_n\}$ be a linear independent set of vectors in a fuzzy normed linear space $(X, N, *)$ with $*$ is t-norm at (1,1). Then there is $c > 0$ and $\delta \in (0,1)$ such that for any set of scalars $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$;

$$N\left(\sum_{i=1}^{n} \lambda_i x_i, c\sum_{i=1}^{n} |\lambda_i|\right) < 1 - \delta \dots\dots\dots\dots\dots (1)$$

Proof : Let $s = |\lambda_1|, |\lambda_2|, \dots, |\lambda_n|$. If $s = 0$ then $\lambda_i = 0\ \forall i = 1,2, \dots, n$ and the relation (1) holds for any $c > 0$ and $\delta \in (0,1)$.

Next we suppose that $s > 0$. Then (1) is equivalent to

$N(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, c) < 1 - \delta \dots\dots\dots\dots\dots (2)$

For some $c > 0$ and $\delta \in (0,1)$, and for all scalars $\alpha' s$ with $\sum_{i=1}^{n} |\alpha_i| = 1$.

If possible suppose that (2) does not hold. Thus for each $c > 0$ and $\delta \in (0,1)$, $\exists$ a set of scalars $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ with $\sum_{i=1}^{n} |\alpha_i| = 1$ for which $\{\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n, c) \geq 1 - \delta$.

Then for $c = \delta = \frac{1}{m}$, $m = 1,2, \dots$, $\exists$ a set of scalars $\left\{\alpha_1^{(m)}, \alpha_2^{(m)}, \dots, \alpha_n^{(m)}\right\}$ with $\sum_{i=1}^{n} \left|\alpha_i^{(m)}\right| = 1$ such that $N\left(y_m, \frac{1}{m}\right) \geq 1 - \frac{1}{m}$ where $y_m = \alpha_1^{(m)} x_1 + \alpha_2^{(m)} x_2 + \cdots + \alpha_n^{(m)} x_n$.

Since $\sum_{i=1}^{n} \left|\alpha_i^{(m)}\right| = 1$, we have $0 \leq |\alpha_i^{(m)}| \leq 1$ for $i = 1,2, \dots, n$.

So for each fixed j the sequence $\{\alpha_i^{(m)}\}$ is bounded and hence $\{\alpha_1^{(m)}\}$ has convergent subsequence . Let α_1 denote the limit of the subsequence and let $\{y_1, m\}$ denote the corresponding subsequence of $\{y_m\}$. By the same argument $\{y_1, m\}$ has a subsequence $\{y_2, m\}$ for which the corresponding subsequence of scalars $\{\alpha_2^{(m)}\}$ converges to α_2.continuing in this way , after n steps we obtain a subsequence $\{y_n, m\}$ where $y_{n,m} = \sum_{i=1}^{n} \gamma_i^{(m)} x_i$ with $\sum_{i=1}^{n} |\gamma_i^{(m)}| = 1$ and $\gamma_i^{(m)} \to \gamma_i$ as $m \to \infty$.

Let $y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$. Thus we have

$\lim_{m\to\infty} N(y_n, m - y, t) = 1 \quad \forall t > 0 \dots\dots\dots\dots\dots (3)$

Now for $k > 0$, choose m such that $\frac{1}{m} < k$.

We have $\left(1 - \frac{1}{m}\right) * N\left(0, k - \frac{1}{m}\right) \leq N\left(y_{n,m}, \frac{1}{m}\right) * N\left(0, k - \frac{1}{m}\right) \leq$

$N\left(y_{n,m} + 0, \frac{1}{m} + k - \frac{1}{m}\right) = N\left(y_{n,m}, k\right)$.

i.e. $\left(1 - \frac{1}{m}\right) * N\left(0, k - \frac{1}{m}\right) \leq N\left(y_{n,m}, k\right)$

i.e. $\lim_{m\to\infty} N\left(y_{n,m}, k\right) \leq 1$

i.e. $\lim_{m\to\infty} N\left(y_{n,m}, k\right) = 1 \dots\dots\dots\dots\dots (4)$

Now $N\left(y - y_{n,m}, k\right) * N\left(y_{n,m} k\right) \leq N\left(y - y_{n,m} + y_{n,m}, k + k\right) = N(y, 2k)$

$\Rightarrow \lim_{m\to\infty} N\left(y - y_{n,m}, k\right) * \lim_{m\to\infty} N\left(y_{n,m} k\right) \leq N(y \leq, 2k)$

(by continuity of t-norm at (1,1)) $\Rightarrow 1 * 1 \leq N(y, 2k)$ by $(3)\&(4)$

$\Rightarrow 1 = 1 * 1 = N(y, 2k)$.

Since $k > 0$ is arbitrary , by (N2) it follows that $y = 0$.

Again since $\sum_{i=1}^{n} \left|\alpha_i^{(m)}\right| = 1$ and $\{x_1, x_2, \dots, x_n\}$ are linear independent set of vectors , so $y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$.

Thus we arrive at a contradiction and the lemma is proved.

Theorem (3.3) : (**Riesz Lemma**) Let M be closed proper subspace of a fuzzy normed linear space $(X, N,*)$ and let λ be a real number such that $0 < \lambda < 1$. Then there exists a vector $x_\lambda \in X$ such that $N(x_\lambda, 1) > 0$ and $N(x_\lambda - x, \lambda) = 0$ for all $x \in M$.

Proof :Since M is proper subspace of X, $\exists\, v \in X - M$.

Denote $d = \wedge_{x\in M} \wedge \{ t > 0 : N(v - x, t) > 0 \}$.

We claim that $d > 0$, i.e. $\wedge_{x\in M}\wedge\{ t > 0 : N(v - x, t) > 0 \} = 0 \Rightarrow$ for a given $\varepsilon > 0$, $\exists\, x(\varepsilon) \in Y$ such that $\wedge\{ t > 0 : N(v - x, t) > o \} < \varepsilon$ $\Rightarrow N(v - x, \varepsilon) > 0$.

Choose $\alpha \in (0,1)$ such that $N(v - x, \varepsilon) > 1 - \alpha$. i.e. $y \in B(v, 1 - \alpha, \varepsilon)$.

Since $\varepsilon > 0$ is arbitrary , it follows that v is in the closure of M.

Since M is closed , it implies that $v \in M$ which is a contradiction . Thus $d > 0$.

We now take $\lambda \in (0,1)$. So $\frac{d}{\lambda} > d$. Thus for some $x_\circ \in M$,

we have $d \le \wedge\{ t > 0 : N(v - x_\circ, t) > 0\} < K' < \frac{d}{\lambda}$ ……..(1)

Let $x_\lambda = \frac{v - x_\circ}{k'}$. Now $(x_\lambda, 1) = N(\frac{v-x_\circ}{k'}, 1)$.

i.e. $N(x_\lambda, 1) = N(v - x_\circ, k')$ ……..(2)

Now $\wedge\{ t > 0 : N(v - x_\circ, t) > 0 \} < k' \Rightarrow N(v - x_\circ, k') > 0$.

From (2) we have $N(x_\lambda, 1) > 0$.

Now for $x \in M$, $\wedge\{ t > 0 : N(x_\lambda - x, t) > 0\} =$
$\wedge\{t > 0: N(v - x_\circ - k'x, k't) > 0 \} =$
$\frac{1}{k'}\wedge\{s > 0: N(v - x_\circ - k'x, s) > 0 \}$.

i.e. $\wedge\{t > 0 : N(x_\lambda - x, t) > 0 \} \ge \frac{d}{k'}$ ($since \quad x_\circ + k'x \in M$)

$\Rightarrow \wedge\{ t > 0 : N(x_\lambda - x, t) > 0 \} > \lambda$ by (1)

i.e. $N(x_\lambda - x, \lambda) \le 0 \Rightarrow N(x_\lambda - x, \lambda) = 0 \quad \forall\, x \in M$.

Definition (3.4) : [2] Let $(X, N,*)$ be a fuzzy normed linear space . A subset B of X is said to be compact if any sequence $\{x_n\}$ in B has a subsequence converging to an element of B .

Theorem (3.5) : Let $(X, N,*)$ be a fuzzy normed linear space and $x \ne 0$. If suppose that $A = \{ x \in X : N(x, 1) > 0\}$ is compact , then X is finite dimensional.

Proof : If possible suppose that dim $X = \infty$. Take $x_1 \in X$ such that $N(x_1, 1) > 0$. Suppose X_1 is the subspace of X generated by x_1 . Since dim $X_1 = 1$, it is aclosed and proper subset of X .Thus by the Lemma (3.3)

$\exists\, x_2 \in X$ $such\ that$ $N(x_2, 1) > 0$ and $N(x_2 - x_1, \frac{1}{2}) = 0$.

The elements x_1, x_2 generate a two dimensional proper closed subspace of X.

By the Lemma (3.3) , $\exists\, x_3 \in X$ with $N(x_3, 1) > 0$ such that

$N\left(x_3 - x_1, \frac{1}{2}\right) = 0$, $N\left(x_3 - x_2, \frac{1}{2}\right) = 0$.

Proceeding in the same way , we obtain a sequence $\{x_n\}$ of elements $x_n \in A$ such that $N(x_n, 1) > 0$ and $N\left(x_n - x_m, \frac{1}{2}\right) = 0$ $(m \ne n)$. It follows that neither the sequence $\{x_n\}$ nor its any subsequence converges. This contradicts the compactness of A . Hence dim X is finite .

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