Journal of Basrah Researches ((Sciences)) Volume 37. Number 4. C ((2011))

The initial Fuzzy Semi-Normed (Normed) Vector Space I

Noori Farhan AL-Mayahi and Salah Mahdi Ali
Department of Mathematics, College of C. Sc. and Math.,
AL-Qadisiya University
nafm2005@yahoo.com & salah_x9@yahoo.com
ISSN –1817 –2695

Abstract

In this paper, we introduce the concepts of initial fuzzy vector topology , the initial fuzzy semi-normed (normed) vector space and we prove that the initial fuzzy vector topology has a base at zero.
2010 Mathematics Subject Classification : Primary 54A40, 46S40
Keywords: initial fuzzy topology, initial fuzzy semi-normed (normed) vector space.

1. Introduction

Fuzzy topological vector spaces were defined and studied by Katsaras [2, 3]. [2] contains a number of results leading to a characterization of the local base for the fuzzy vector topology. In [3] fuzzy semi-norms and fuzzy norms have been defined on fuzzy vector spaces. The initial fuzzy topology was defined and studied in [1]. The core of this paper is to find a local base for the initial fuzzy vector topology on vector spaces.

2. Preliminaries

Let X be a non-empty set. A fuzzy set in X is the element of the set I^X of all functions from X into the closed unit interval $I=[0,1]$. If $C_\alpha : X \to I$ is a function defined by $C_\alpha(x)=\alpha$ for all $x \in X$, $\alpha \in I$, then C_α is called a constant fuzzy set. Let f be a function from X into Y and $B \in I^Y$, then $f^{-1}(B)$ is a fuzzy set in X defined by $f^{-1}(B)(x)=(B \circ f)(x)$ for all $x \in X$. Also, for $A \in I^X$, $f(A) \in I^Y$ which is defined by

$$f(A)(y)=\begin{cases} \sup\limits_{x \in f^{-1}(y)} A(x) & ,if\ \ f^{-1}(y) \neq \phi \\ 0 & ,otherwise \end{cases}.$$

Let X be a vector space over a field F , where F is the field of either the real or the complex numbers. If $A_1, A_2, \ldots, A_n$ are fuzzy sets in X , then the sum $A_1 + A_2 + \cdots + A_n$ (see [2]) is the fuzzy set A in X defined by $A(x)= \sup\limits_{x_1+x_2+\cdots+x_n=x} \min\{A_1(x_1), A_2(x_2),\ldots,A_n(x_n)\}$. Also, if A is a fuzzy set in X and $\alpha \in F$,

then αA is a fuzzy defined by $\alpha A(x)=\begin{cases} A(x/\alpha) & if\ \ \alpha \neq 0\ \ forall\, x \in X \\ 0 & if\ \ \alpha=0, x\neq 0 \\ \sup\limits_{y \in X} A(y) & if\ \ \alpha=0, x=0 \end{cases}$.

If A is a fuzzy set in X and $x \in X$, then the fuzzy set $x+A$ is defined by $(x+A)(y)=A(y-x)$. A fuzzy set A in X is called :

(1) a convex fuzzy set if $\alpha A+(1-\alpha)A \subseteq A$ or $A(\alpha x+(1-\alpha)y) \geq \min\{A(x),A(y)\}$ for all $x,y \in X$ and all $\alpha \in [0,1]$.

(2) A balanced fuzzy set if $\alpha A \subset A$ or $A(\alpha x) \geq A(x)$ for all α with $|\alpha| \leq 1$.

(3) An absorbing fuzzy set if $X = \bigcup_{\alpha > 0} \alpha A$ or $\sup_{\alpha > 0} A(\alpha x) = 1$. Clearly if A is an absorbing fuzzy set, then $A(0) = 1$.

For the definition of a fuzzy topology, we will use the one given by Lowen [1] that is a fuzzy topology on a set X we will mean a subset γ of I^X satisfying the following conditions :

(i) γ contains every constant fuzzy set in X ;

(ii) If $A_1, A_2 \in \gamma$, then $A_1 \cap A_2 \in \gamma$.

(iii) If $A_i \in \gamma$ for all $i \in \Lambda$ (Λ any index), then $\bigcup_{i \in \Lambda} A_i \in \gamma$.

The pair (X, γ) is called a fuzzy topological space. Let $(X_j, \gamma_j)_{j \in \Lambda}$ be fuzzy topological spaces and $f_j : X \to (X_j, \gamma_j)$ are functions. If $f_j^{-1}(\gamma_j) = \{ f_j^{-1}(A) : A \in \gamma_j \}$, then the coarsest fuzzy topology on X making every f_j is fuzzy continuous is called the initial fuzzy topology on X with respect to $\{f_j\}_{j \in \Lambda}$ and denoted by $\langle \bigcup_{j \in \Lambda} f_j^{-1}(\gamma_j) \rangle_X$. If ρ is a (convex, balanced and absorbing) fuzzy set in a set X, then (see [3]) ρ is called a fuzzy semi-norm on X. If in addition $\inf_{\alpha > 0} (\alpha \rho)(x) = 0$ for $x \neq 0$, then ρ is called a fuzzy norm.

3. Main Results

Theorem 3.1:[3]

Let Φ be a family of balanced fuzzy sets in a vector space X over F. Then Φ is a base at zero for a fuzzy vector topology if, and only if, Φ satisfies the following conditions :

(1) $A(0) > 0$, for each $A \in \Phi$;

(2) for each non-zero constant fuzzy set C_β and any $\alpha \in (0, \beta)$ there exists $A \in \Phi$ with $A \subseteq C_\beta$ and $A(0) > \alpha$;

(3) If $A, B \in \Phi$ and $\alpha \in (0, \min\{A(0), B(0)\})$, then there exists $D \in \Phi$ with $D \subseteq A \cap B$ and $D(0) > \alpha$;

(4) If $A \in \Phi$ and t a non-zero scalar, then for each $\alpha \in (0, A(0))$ there exists $B \in \Phi$ with $B \subseteq tA$ and $B(0) > \alpha$;

(5) Let $A \in \Phi$ and $\alpha \in (0, A(0))$. Then, there exists $B \in \Phi$ such that $B(0) > \alpha$ and $B + B \subseteq A$;

(6) Let $A \in \Phi$ and $x_\circ \in X$. If $\alpha \in (0, A(0))$, then there exists a positive number s such that $A(tx_\circ) > \alpha$, for all scalar $t \in R$ with $|t| \leq s$;

(7) For each $A \in \Phi$ there exists a fuzzy set B in X with $B \subseteq A$, $B(0) = A(0)$ and such that for each $x \in X$ for which $B(x) > 0$ and each $n \in (0, B(x))$ there exists $D \in \Phi$ with $D \subseteq -x + B$ and $D(0) > n$.

Definition 3.2.

Let X be a vector space, $(X_j, \rho_j)_{j \in J}$ be fuzzy semi-norm (norm) vector spaces, let $f_j : X \to (X_j, \rho_j)$ be linear functions. Set $f_j^{-1}(\rho_j) = \{ f_j^{-1}(A) : A \in \delta_{\rho_j} \}$, where δ_{ρ_j} is a fuzzy topology generated by a fuzzy semi-norm (norm) ρ_j, ($j \in J$).

(1) We define the initial fuzzy vector topology on X with respect to $\{f_j\}_{j\in J}$ (in symbol $\langle\bigcup_{j\in J} f_j^{-1}(\rho_j)\rangle_X$) is the coarsest fuzzy vector topology on X containing $\bigcup_{j\in J} f_j^{-1}(\rho_j)$. It is clear that $\langle\bigcup_{j\in J} f_j^{-1}(\rho_j)\rangle_X$ making all f_j's are fuzzy continuous.

(2) The pair $(X,\langle\bigcup_{j\in J} f_j^{-1}(\rho_j)\rangle_X)$ is called the initial fuzzy semi-normed (normed) vector space.

Theorem 3.3.

The initial fuzzy vector topology on X has a base at zero.

Proof :

Let $I\Phi=\{C_{\alpha_j}\cap(\frac{1}{t}\rho_j\circ f_j), j\in J:\alpha_j\in(0,1], 0<t\in R\}$

Note, $C_{\alpha_j}\cap(\frac{1}{t}\rho_j\circ f_j)$ is balanced fuzzy set for all $t>0,\alpha_j\in(0,1]$, $j\in J$. one needs only show that $I\Phi$ satisfies conditions (1)-(7) of (Theorem 3.1.).

(1)

Let $A\in I\Phi$. Then $A=C_{\alpha_j}\cap(\frac{1}{t}\rho_j\circ f_j)$ for some $\alpha_j\in(0,1]$, $0<t\in R$, $j\in J$.

Now,

$A(0)=(C_{\alpha_j}\cap(\frac{1}{t}\rho_j\circ f_j))(0)=\min\{C_{\alpha_j}(0),(\frac{1}{t}\rho_j\circ f_j)(0)\}=\min\{\alpha_j,1\}=\alpha_j>0$.

{ $(\frac{1}{t}\rho_j\circ f_j)(0)=1$ since ρ is an absorbing fuzzy set , f is linear function}.

(2)

Let C_{β_j} be a non-zero constant fuzzy set in X and let $\alpha_j\in(0,\beta_j)$, $j\in J$.

Let $A=C_{\beta_j}\cap(\rho_j\circ f_j)$. Now, $A=C_{\beta_j}\cap(\rho_j\circ f_j)\subseteq C_{\beta_j}$ and

$A(0)=(C_{\beta_j}\cap(\rho_j\circ f_j)(0)=\min\{C_{\beta_j}(0),(\rho_j\circ f_j)(0)\}=\min\{\beta_j,1\}=\beta_j>\alpha_j$.

(3)

Let $A,B\in I\Phi$ and let $\alpha_j\in(0,\min\{A(0),B(0)\})$, $j\in J$. Then $A=C_{m_j}\cap(\frac{1}{t}\rho_j\circ f_j)$ and $B=C_{n_j}\cap(\frac{1}{s}\rho_j\circ f_j)$ where $m_j,n_j\in(0,1]$ and $0<t,s\in R$. Choose $q_j\in(\alpha_j,\min\{A(0),B(0)\})$. Choose r such that $r\ge s,t.$ Now, let $D=C_{q_j}\cap(\frac{1}{r}\rho_j\circ f_j)$. We have now that $|s/r|\le 1$ and because ρ_j is a balanced fuzzy set , for all $j\in J$ we have $(s/r)\rho_j\subseteq\rho_j$, it follows $\frac{1}{s}(s/r)\rho_j\subseteq\frac{1}{s}\rho_j$. Thus, $\frac{1}{r}\rho_j\circ f_j\subseteq\frac{1}{s}\rho_j\circ f_j$. Similarly $\frac{1}{r}\rho_j\circ f_j\subseteq\frac{1}{t}\rho_j\circ f_j$. So

$$D=C_{q_j}\cap(\frac{1}{r}\rho_j\circ f_j)\subseteq(C_{m_j}\cap C_{n_j})\cap((\frac{1}{t}\rho_j\circ f_j)\cap(\frac{1}{s}\rho_j\circ f_j))$$

$$=(C_{m_j}\cap(\frac{1}{t}\rho_j\circ f_j))\cap(C_{n_j}\cap(\frac{1}{s}\rho_j\circ f_j))=A\cap B.$$

Also, $D(0)=(C_{q_j}\cap(\frac{1}{r}\rho_j\circ f_j))(0)=\min\{q_j,1\}=q_j>\alpha_j$.

(4)

Let $A\in I\Phi$ and $0\neq t\in R$. For $j\in J$, choose $\alpha_j\in(0,A(0))$. We have that

$A=C_{m_j}\cap(\frac{1}{s}\rho_j\circ f_j)$ for some $m_j\in(0,1]$, $0<s\in R$,

(Note, $A(0)=(C_{m_j}\cap(\frac{1}{s}\rho_j\circ f_j))(0)=\min\{m_j,1\}=m_j$)

Now, for $x\in X$

$$tA(x)=t(C_{m_j}\cap(\frac{1}{s}\rho_j\circ f_j))(x)=(C_{m_j}\cap(\frac{1}{s}\rho_j\circ f_j))(x/t)$$

$$=\min\{C_{m_j}(x/t),(\frac{1}{s}\rho_j\circ f_j)(x/t))\}$$

$$=\min\{m_j,(\frac{t}{s}\rho_j\circ f_j)(x)\}$$

$$=(C_{m_j}\cap(\frac{t}{s}\rho_j\circ f_j))(x).$$

Thus, $B=tA=C_{m_j}\cap(\frac{t}{s}\rho_j\circ f_j)\in I\Phi$ and

$B(0)=tA(0)=(C_{m_j}\cap(\frac{t}{s}\rho_j\circ f_j))(0)=\min\{m_j,1\}=m_j>\alpha_j$.

(5)

Let $A\in I\Phi$. We have that $A=C_{m_j}\cap(\frac{1}{s}\rho_j\circ f_j)$ for some $m_j\in(0,1]$, $0<s\in R$, $j\in J$. Let $\alpha_j\in(0,A(0))$.

Now, let $t=2s$ and let $B=C_{m_j}\cap(\frac{1}{t}\rho_j\circ f_j)$. Choose any $x\in X$. Then, we have :

$$(B+B)(x)=\sup_{x_1+x_2=x}\min\{B(x_1),B(x_2)\}$$

$$=\sup_{x_1+x_2=x}\min\{(C_{m_j}\cap(\frac{1}{t}\rho_j\circ f_j))(x_1),(C_{m_j}\cap(\frac{1}{t}\rho_j\circ f_j))(x_2)\}$$

$$=\sup_{y\in X}\min\{C_{m_j},\min\{(\frac{1}{t}\rho_j\circ f_j)(y),(\frac{1}{t}\rho_j\circ f_j)(x-y)\}\}$$

$$=\sup_{y\in X}\min\{C_{m_j},\min\{(\frac{1}{2s}\rho_j\circ f_j)(y),(\frac{1}{2s}\rho_j\circ f_j)(x-y)\}\}$$

$$=\sup_{y\in X}\min\{C_{m_j},\min\{\rho_j(2sf_j(y)),\rho_j(2sf_j(x-y))\}\}$$

$$=\sup_{y\in X}\min\{C_{m_j},\min\{\rho_j(f_j(2sy)),\rho_j(f_j(2sx-2sy))\}\}$$

$$\leq\sup_{y\in X}\min\{C_{m_j},\rho_j[f_j((\frac{1}{2})(2sy))+f_j((\frac{1}{2})(2sx-2sy))]\}$$

$$=\sup_{y\in X}\min\{C_{m_j},\rho_j(sf_j(y+x-y))\}$$

$$=\sup_{y\in X}\min\{C_{m_j},(\frac{1}{s}\rho_j\circ f_j)(x)\}=\min\{C_{m_j},(\frac{1}{s}\rho_j\circ f_j)(x)\}=A(x).$$

Also, $B(0) = (C_{m_j} \cap (\frac{1}{t}\rho_j \circ f_j)(0) = \min\{m_j, 1\} = m_j = A(0) > \alpha_j$. (6)

Let $A \in I\Phi$. We have that $A = C_{m_j} \cap (\frac{1}{r}\rho_j \circ f_j)$ for some $m_j \in (0,1]$, $0 < r \in R$, $j \in J$. Let $x_\circ \in X$ and $\alpha_j \in (0, A(0))$. Since ρ_j is absorbing, then $\sup_{t>0}(\frac{1}{t}\rho_j \circ f_j)(x_\circ) = \sup_{t>0}\rho_j(t\, f_j(x_\circ)) = 1$. We thus have that there exists $s \in R, s > 0$ such that $(\frac{1}{s}\rho_j \circ f_j)(x_\circ) = \rho_j(f_j(sx_\circ)) > \alpha_j$. Choose $t \in R$ such that $|t| \le s$, then $\left|\frac{t}{s}\right| \le 1$. Since ρ_j is a balanced fuzzy set, we have for $x \in X$

$(\rho_j \circ f_j)(\frac{t}{s}x) \ge (\rho_j \circ f_j)(x)$. Thus, $(\frac{1}{t}\rho_j \circ f_j)(x) \ge (\frac{1}{s}\rho_j \circ f_j)(x) > \alpha_j$. Hence,

$$A(tx_\circ) = \min\{C_{m_j}(tx_\circ), (\frac{1}{r}\rho_j \circ f_j)(tx_\circ)\} = \min\{m_j, (\frac{1}{t}\rho_j \circ f_j)(rx_\circ)\} \ge \alpha_j \qquad (7)$$

Let $A = C_{m_j} \cap (\frac{1}{t}\rho_j \circ f_j) \in I\Phi$, $m_j \in (0,1]$, $j \in J$ and $0 < t \in R$. For $x_j \in X_j$ define $\hat{\rho}_j : X_j \to I$ by $\hat{\rho}_j(x_j) = \sup_{v>1}\rho_j(vx_j)$. Then, $\hat{\rho}_j \circ f_j : X \to I$ is a fuzzy set defined by : $(\hat{\rho}_j \circ f_j)(x) = \hat{\rho}_j(f_j(x)) = \sup_{v>1}\rho_j(vf_j(x)) = \sup_{v>1}(\rho_j \circ f_j)(vx)$, for all $x \in X$. Moreover, for each $\alpha_j \in (0,1)$ we have $\alpha_j\rho_j \subseteq \hat{\rho}_j \subseteq \rho_j$. Take, $B = C_{m_j} \cap (\frac{1}{t}\hat{\rho}_j \circ f_j)$. Note, $B \subseteq A$ and $B(0) = m_j = A(0)$. Choose $x_\circ \in X$ with $B(x_\circ) > 0$ and let $n, n_1 \in R$ such that $0 < n < n_1 < B(x_\circ)$. Since $(\hat{\rho}_j \circ f_j)(tx_\circ)) = (\frac{1}{t}\hat{\rho}_j \circ f_j)(x_\circ) \ge \min\{m_j, (\frac{1}{t}\hat{\rho}_j \circ f_j)(x_\circ)\} = B(x_\circ) > n_1$, then there exists $s_\circ > 1$ such that $(\rho_j \circ f_j)(ts_\circ x_\circ) > n_1$. Choose $s \in R$ such that $1 < s < s_\circ$, then $(\hat{\rho}_j \circ f_j)(tsx_\circ) \ge (\rho_j \circ f_j)(ts_\circ x_\circ) > n_1$ and so $B(sx_\circ) > n_1$. Since B is a convex fuzzy set, taking $q = \frac{1}{s}$ we have $B(x + x_\circ) = B(q(sx_\circ) + (1-q)\frac{x}{(1-q)})$

$$\ge \min\{B(sx_\circ), B(\frac{x}{1-q})\}$$

$$\ge \min\{n_1, m_j, (\hat{\rho}_j \circ f_j)(\frac{tx}{(1-q)})\}$$

$\ge \min\{n_1, (\rho_j \circ f_j)(\alpha x)\}$ if $0 < \alpha < \frac{t}{1-q}$. Therefore, $D = C_{n_1} \cap (\frac{1}{\alpha}\rho_j \circ f_j) \subseteq -x_\circ + B$ and $D(0) = n_1 > n$.

Thus, $I\Phi$ is a base at zero for the initial fuzzy vector topology on X.

References

[1] R. Lowen , Fuzzy topological spaces and fuzzy compactness, J. Math. Appl, **56** (1976) 621-633.
[2] A. K. Katsaras and D. B. Liu, Fuzzy vector spaces and fuzzy topological vector spaces, J. Math. Anal. Appl. **58** (1977) 135-146.
[3] A. K. Katsaras, Fuzzy topological vector spaces I, Fuzzy Sets and Systems,**6** (1981) 85-95.