Gen. Math. Notes, Vol. 26, No. 2, February 2015, pp. 104-118
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

On α-η-φ-Contraction in Fuzzy Metric Space and its Application

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(Received: 11-8-14 / Accepted: 3-11-14)

Abstract

The aim of this paper is to introduce new concepts of α-η-complete fuzzy metric space and α-η-continuous function and establish fixed point results for α-η-φ-contraction function in α-η-complete fuzzy metric space. As an application, we derive some Suzuki type fixed point theorems, fixed point in orbitally f-complete. Moreover, we introduce concept α-ψ-ϕ-contraction function and application on α-η-φ-contraction.

Keywords: *α-η-complete, α-η-continuous, α-η-φ-contraction, orbitally f-complete, Suzuki Type Fixed Point Result, α-ψ-ϕ-Contraction Function, Application on α-η-φ-Contraction.*

1 Introduction

The study of fixed points of functions in a fuzzy metric space satisfying certain contractive conditions has been at the center of vigorous research activity. In 1965, the concept of fuzzy sets was introduced by Zadeh [10]. With the concept of fuzzy sets, the fuzzy metric space was introduced by I. Kramosil and J. Michalek

[5] in 1975. Helpern [3] in 1981first proved a fixed point theorem for fuzzy functions. Also M.Grabiec [2] in 1988 proved the contraction principle in the setting of the fuzzy metric spaces. Moreover, A. George and P. Veeramani [1] in1994 modified the notion of fuzzy metric spaces with the help of t-norm.

This paper we introduce new concepts of α-η-complete fuzzy metric space and α-η-continuous function and establish fixed point results for α-η-φ-contraction function in α-η-complete fuzzy metric space. As an application, we derive some Suzuki type fixed point theorems, fixed point in orbitally f-complete. Moreover, we introduce concept α- ψ- ϕ-contraction function and application on α-η-φ-contraction.

2 Preliminaries

Definition 2.1 [4]: *Let $f: X \to X$ and $\alpha: X \times X \to [0, \infty)$ be two function, f is said to be α-admissible function if*

$$\alpha(x, y) \geq 1 \quad \Rightarrow \alpha\big(f(x), f(y)\big) \geq 1 \text{ for all } x, y \in X.$$

Definition 2.2 [6]: *Let $f: X \to X$ and $\alpha, \beta: X \times X \to [0, \infty)$ be two function,f is said to be (α, β)-admissible function if*

$$\begin{cases}\alpha(x, y) \geq 1 \\ \beta(x, y) \geq 1\end{cases} \Rightarrow \begin{cases}\alpha\big(f(x), f(y)\big) \geq 1 \\ \beta\big(f(x), f(y)\big) \geq 1\end{cases} \text{for all } x, y \in X.$$

Definition 2.3 [4]: *Let $f: X \to X$ and $\alpha, \eta: X \times X \to [0, \infty)$ be two function, f is said to be α-admissible function with respect η if*

$$\alpha(x, y) \geq \eta(x, y) \quad \Rightarrow \alpha(f(x), f(y)) \geq \eta(f(x), f(y)) \text{, for all } x, y \in X$$

Not that if we take $\eta(x, y) = 1$, then this definition reduces to definition (2.1).

Definition 2.4: *Let $(X, M, *)$ be a fuzzy metric space and*

$\alpha, \eta: X \times X \to [0, \infty)$, X is said to be α-η-complete iff every Cauchy sequence $\{x_n\}$ with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ converge in X

Note: X is said to be α-complete if $\eta(x, y) = 1$ for all $x, y \in X$.

Example 2.5: *Let $X = [0,1]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, if $t > 0$ with $a * b = min\,\{a, b\}$ for every $a, b \in [0,1]$, define $\alpha, \eta: X \times X \to [0, \infty)$*

By $$\alpha(x, y) = \begin{cases}(x+y)^2 & if\ x, y \in X \\ 0 & if\ o.w\end{cases}$$

$$\eta(x,y) = 2xy$$

Then $(X,M,*)$ is α-η-complete fuzzy metric space.

Definition 2.6: *Let $(X,M,*)$ be a fuzzy metric space and let $\alpha,\eta: X\times X \to [0,\infty)$ and $f: X \to X$, f is said to be α-η-continuous function on X if for $x\in X$ and $\{x_n\}$ be a sequence in X with $x_n \to x$ as $n\to\infty$, $\alpha(x_n,x_{n+1}) \geq \eta(x_n x_{n+1})$ for all $n\in\mathbb{N}$ implies $f(x_n)\to f(x)$.*

Definition 2.7 [9]:

(1) *Let f be a function of a fuzzy metric space $(X,M,*)$ into itself. $(X,M,*)$ is said to be f-orbitally complete if and only if every Cauchy sequence which is contained in $\{x, f(x), f^2(x), f^3(x),\cdots\}$ for some $x \in X$ converges in X.*

A f -orbitally complete fuzzy metric space may not be complete.

(2) *A function $f: X\to X$ is called orbitally continuous at $x\in X$ if $lim_{n\to\infty} f^n(x) = x$ implies $lim_{n\to\infty} f f^n(x) = f(x)$*

The function f is orbitally continuous on X if f is orbitally continuous for all $x\in X$.

Remark 2.8:

(1) *Let $(X,M,*)$ be a fuzzy metric space and $f: X\to X$ be a function and let X be an orbitally f-complete. Define $\alpha,\eta: X\times X\to[0,\infty)$ by*

$$\alpha(x,y) = \begin{cases} 3 & if\ x,y\in O(w) \\ 1 & o.w \end{cases}, \quad \eta(x,y) = 1$$

Where $O(w)$ is an orbit of a point $w\in X$. $(X,M,*)$ is an α-η-complete.

If $\{x_n\}$ be a Cauchy sequence with $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n\in\mathbb{N}$ then $\{x_n\}\in O(w)$

Now, since X is an orbitally f-complete fuzzy metric space, then $\{x_n\}$ converge in X. We can say that X is α-η-complete.

(2) *Let $(X,M,*)$ and $\alpha,\eta: X\times X\to[0,\infty)$ is in **(1)**, let $f: X\to X$ be an orbitally continuous function on $(X,M\ *)$. Then f is α-η-continuous function. Indeed if $x_n\to x$ as $n\to\infty$ and $\alpha(x_n,x_{n+1}) \geq \eta(x_n x_{n+1})$ for all $n\in\mathbb{N}$, so $x_n\in O(w)$ for all $n\in\mathbb{N}$, then there exist sequence $(k_i)_{i\in\mathbb{N}}$ of positive integer such that $x_n \to f^{k_i}w \to x$ as $i\to\infty$. Now since f is an orbitally continuous on $(X,M,*)$, then $f(x_n) = f(f^{k_i}w)) \to f(x)$ as $i\to\infty$.*

Denote with ϕ the set of all the function $\varphi: [0,1] \to [0,1]$ with the following properties:

(1) φ is nondecreasing and continuous

(2) $\varphi(\tau) = 0 \quad iff\ \tau = 1$

Definition 2.9: *Let* $(X, M,*)$ *be a fuzzy metric space and* $f: X \to X$, *let* $\mathbb{M}(x,y) = min\,\{M(x,y,t), M(x,f(x),t), M(y,f(y),t), M(x,f(y),t) * M(y,f(x),t)\}$

We say

(1) f is an α-η-φ-contraction function if
$\alpha(x,y) \geq \eta(x,y) \implies M(f(x),f(y),t) \geq \varphi(\mathbb{M}(x,y))$.

(2) f is α-φ-contraction function if $\eta(x,y) = 1$ for all $x, y \in X$.

Theorem 2.10: *Let* $(X, M,*)$ *be a fuzzy metric space and let* $f: X \to X$, *suppose that* $\alpha, \eta: X \times X \to [0,\infty)$ *are two function. Assume that the following assertions hold*

(i) $(X, M,*)$ is an α-η-complete fuzzy metric space .

(ii) f is an α-admissible function with respect to η .

(iii) f is α-η-φ-contraction function on X .

(iv) f is an α-η-continuous function .

(v) There exist $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq \eta(x_0, f(x_0))$.

Then f has a fixed point in X .

Proof: Let $x_0 \in X$ such that $\alpha\big(x_0, f(x_0)\big) \geq \eta\big(x_0, f(x_0)\big)$

Define a sequence $\{x_n\}$ such that $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$

If $x_n = x_{n+1}$ for some n , then $x = x_n$ is a fixed point of f.

Suppose $x_n \neq x_{n+1}$

Since f is α-admissible function with respect η and

$\alpha(x_0, f(x_0)) \geq \eta(x_0, f(x_0))$

Then $\alpha(x_1, x_2) = \alpha\big(f(x_0), f(x_1)\big) \geq \eta\big(f(x_0), f(x_1)\big) = \eta(x_1, x_2)$

By continuing this process, we get

$$\alpha(x_n, f(x_n)) = \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) = \eta(x_n, f(x_n))$$

By (1) in definition (2.9)

$$M(x_n, x_{n+1}, t) = M(f(x_{n-1}), f(x_n), t) \geq \alpha\big(\mathbb{M}(x_{n-1}, x_n)\big)$$

Where

$$\mathbb{M}(x_{n-1}, x_n) = min\,\{M(x_{n-1}, x_n, t), M(x_{n-1}, f(x_{n-1}), t), M(x_n, f(x_n), t), M(x_{n-1}, f(x_n), t) * M(x_n, f(x_{n-1}), t)\}$$

$$= min\,\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), M(x_n, x_{n+1}, t), M(x_{n-1}x_{n+1}, t) * M(x_n, x_n, t)\}$$

$$\geq min\,\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), M(x_{n-1}, x_n, t) * M(x_n, x_{n+1}, t)\}$$

$$= min\,\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}$$

Since φ is nondecreasing and continuous, we have

$$M(x_n, x_{n+1}, t) \geq \varphi(min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\})$$

If $min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\} = M(x_n, x_{n+1}, t)$

Then

$$M(x_n, x_{n+1}, t) \geq \varphi(min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}) \geq \varphi\big(M(x_n, x_{n+1}, t)\big) > M(x_n, x_{n+1}, t)$$

Which is contradiction.

Therefore $min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\} = M(x_{n-1}, x_n, t)$

Hence for all $n \in \mathbb{N}$ we have

$$M(x_n, x_{n+1}, t) \geq \varphi\big(M(x_{n-1}, x_n, t)\big) \geq \varphi^2\big(M(x_{n-2}, x_{n-1}, t)\big) \geq \cdots \geq \varphi^n\big(M(x_0, x_1, t)\big)$$

Let $n, m \in \mathbb{N}$ with $n > m$, then

$$M(x_n, x_m, t) \geq \varphi\big(M(x_{n-1}, x_n, t)\big) \geq \cdots \geq \varphi^n\big(M(x_0, x_1, t)\big)$$

Therefore $lim_{n\to\infty} M(x_n, x_m, t) = 1$

Hence $\{x_n\}$ is a Cauchy sequence

Since X is an α-η-complete fuzzy metric space there is $x \in X$ such that $x_n \to x$ as $n \to \infty$.

Since f is an α-η-continuous function so $x_{n+1} = f(x_n) \to f(x)$ as $n \to \infty$

Hence $f(x) = x$

Suppose y is a fixed point of f such that $f(y) = y$

$M(x, y, t) = M(f(x), f(y), t) \geq \varphi(\mathbb{M}(x, y))$

Hence $x = y$

Example 2.11: *Let* $X = [0,3]$ *be equipped with the ordinary metric*

$d(x, y) = |x - y|$, $\varphi(\tau) = \sqrt{\tau}$ for all $\tau \in [0,1]$. Define $M(x, y, t) = e^{-\frac{2|x-y|}{t}}$ for all $x, y \in X$ and $t > 0$, and $\alpha, \eta: X \times X \to [0, \infty)$

By $\alpha(x, y) = \begin{cases} (x + y)^2 & if\ x, y \in X \\ 0 & if\ o.w \end{cases}$

$\eta(x, y) = 2xy,$

Clearly, $(X, M,*)$ is an α-η- complete fuzzy metric space with respect to t −norm $a * b = ab$. (by [8])

Let $f: X \to X$ be defined as

$$f(x) = \begin{cases} 1 & x \in [0,1] \\ \frac{3-x}{2} & x \in (1,3] \end{cases}$$

$$d\big(f(x), f(y)\big) = |f(x) - f(y)| = \frac{1}{2}|x - y| \leq |x - y| = d(x, y)$$

It follows that $M(f(x), f(y), t) = e^{-\frac{2|f(x)-f(y))|}{t}} \geq e^{-\frac{|x-y|}{t}} = \varphi(\mathbb{M}(x, y))$

Thus f is α-η-φ-contraction function in fuzzy metric space $(X, M,*)$.

Corollary 2.12: *Let* $(X, M,*)$ *be a fuzzy metric space and let* $f: X \to X$, *suppose that* $\alpha, \eta: X \times X \to [0, \infty)$ *are two function. Assume that the following assertions hold:*

(i) $(X, M,*)$ is an α-complete fuzzy metric space .
(ii) f is an α-admissible function .
(iii) f is α-φ-contraction function on X .
(iv) f is an α-continuous function .
(v) There exist $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$.

Then f has a fixed point in X .

Theorem 2.13: *Let $(X, M,*)$ be a fuzzy metric space and let $f: X \to X$, suppose that $\alpha, \eta: X \times X \to [0, \infty)$ are two function. Assume that the following assertions hold:*

(i) $(X, M,*)$ is an α-complete fuzzy metric space
(ii) f is an α-admissible function with respect to η .
(iii) f is α-η-φ-contraction function on X .
(iv) There exist $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq \eta(x_0, f(x_0))$.
(v) If $\{x_n\}$ is a sequence in X $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$

With $x_n \to x$ as $n \to \infty$, then

Either $\alpha(f(x_n), x) \geq \eta(f(x_n), f^2(x_n))$

Or $\alpha(f^2(x_n), x) \geq \eta(f^2(x_n), f^3(x_n))$ for all $n \in \mathbb{N}$

Then f has a fixed point in X .

Proof: Let $x_0 \in X$ such that $\alpha\big(x_0, f(x_0)\big) \geq \eta\big(x_0, f(x_0)\big)$

Define a sequence $\{x_n\}$ in X by $x_n = f^n(x_0) = f(x_{n-1})$ for all $n \in \mathbb{N}$.

Now is in the proof of theorem (2.10) we have $\alpha(x_{n+1}, x_n) \geq \eta(x_{n+1}, x_n)$

There exist $x \in X$ such that $x_n \to x$ as $n \to \infty$

Let $M(x, f(x), t) \neq 1$, from (v)

Either $\alpha(f(x_{n-1}), x) \geq \eta(f(x_{n-1}), f^2(x_{n-1}))$

Or $\alpha(f^2(x_{n-1}), x) \geq \eta(f^2(x_{n-1}), f^3(x_{n-1}))$

Then either $\alpha(x_n, x) \geq \eta(x_n, x_{n+1})$

Or $\alpha(x_{n+1}, x) \geq \eta(x_{n+1}, x_{n+2})$

Let $\alpha(x_n, x) \geq \eta(x_n, x_{n+1})$ from definition (2.9) condition (1), we get

$$M(x_{n+1}, f(x), t) = M(f(x_n), f(x), t) \geq \varphi\big(\mathbb{M}(x_n, x)\big)$$

Where

$$\mathbb{M}(x_n, x) = min\,\{M(x_n, x, t), M(x_n, f(x_n), t), M(x, f(x), t), M(x_n, f(x), t) * M(x, f(x_n), t)\}$$

$$= \varphi(\min\{M(x_n, x, t), M(x_n, x_{n+1}t), M(x, f(x), t), M(x_n, f(x), t) * M(x, x_{n+1}, t)\})$$

$$> min\{M(x_n, x, t), M(x_n, x_{n+1} t), M(x, f(x), t), M(x_n, f(x), t) * M(x, x_{n+1}, t)\}$$

By taking $n \to \infty$ in the above we get

$$M(x, f(x), t) = \varphi(min\,\{1, M(x, f(x), t)\}) > M(x, f(x), t)$$

Which is contradiction.

Hence $M(x, f(x), t) = 1 \implies f(x) = x$.

Corollary 2.14: *Let* $(X, M,*)$ *be a fuzzy metric space and let* $f: X \to X$ *, suppose that* $\alpha, \eta: X \times X \to [0, \infty)$ *are two function. Assume that the following assertions hold:*

(i) $(X, M,*)$ is an α-complete fuzzy metric space.
(ii) f is an α-admissible function.
(iii) f is α-φ-contraction function on X.
(iv) There exist $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$.
(v) If $\{x_n\}$ is a sequence in X $\alpha(x_n, x_{n+1}) \geq 1$

With $x_n \to x$ as $n \to \infty$, then

Either $\alpha(f(x_n), x) \geq 1$

Or $\alpha(f^2(x_n), x) \geq 1$ for all $n \in \mathbb{N}$

Then f has a fixed point in X.

Corollary 2.15: *Let* $(X, M,*)$ *be a complete fuzzy metric space and let* $f: X \to X$ *be a continuous function such that* f *is contraction function that is*

$$M(f(x), f(y), t) \geq \varphi\big(\mathbb{M}(x, y)\big) \text{ for all } x, y \in X$$

Then f has a fixed point in X.

2.2 Fixed Point in Orbitally f- Complete Fuzzy Metric Space

Theorem 2.2.16: *Let* $(X, M,*)$ *be a fuzzy metric space and* $f: X \to X$ *such that the following assertion hold:*

(i) $(X, M,*)$ is an orbitally f-complete fuzzy metric space.
(ii) there exist φ be a function such that $M(f(x), f(y), t) \geq \varphi(\mathbb{M}(x, y))$ for all $x, y \in O(w)$ for some $w \in X$.
(iii) If $\{x_n\}$ be a sequence .such that $\{x_n\} \subseteq O(w)$ with $x_n \to x$ as $n \to \infty$, $x \in O(w)$.
Then f has fixed point in X.

Proof: Define $\alpha: X \times X \to [0,\infty)$ from remark (2.8) , $(X, M,*)$ is an α-complete fuzzy metric space and f is an α-admissible function.

Let $\alpha(x,y) \geq 1$ for all $x, y \in O(w)$

Then from (ii)

$$M(f(x), f(y), t) \geq \varphi(\mathbb{M}(x,y))$$

That is f is an α-φ-contraction function

Let $\{x_n\}$ sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ with $x_n \to x$

So $\{x_n\} \subseteq O(w)$ from (iii) we have $x \in O(w)$

That is $\alpha(x_n, x) \geq 1$ by corollary (2.13)

Then f has unique fixed point in X.

Corollary 2.2.17: *Let* $(X, M,*)$ *be a fuzzy metric space and* $f: X \to X$ *such that the following assertion hold:*

(i) $(X, M,*)$ is an orbitally f-complete fuzzy metric space.
(ii) there exist $k \in (0,1)$ such that $M(f(x), f(y), t) \geq k\mathbb{M}(x, y)$ for all $x, y \in O(w)$ for some $w \in X$.
(iii) If $\{x_n\}$ be a sequence such that $\{x_n\} \subseteq O(w)$ with $x_n \to x$ as $n \to \infty$. Then $x \in O(w)$. Then f has fixed point in X.

2.3 Suzuki Type Fixed Point Result

From Theorem (2.10) we deduce the following Suzuki type fixed point result

Theorem 2.3.18: *Let* $(X, M,*)$ *be a complete fuzzy metric space and let* $f: X \to X$ *continuous function .Assume that there exist* $k \in (0,1)$ *such that*

$$M(x, f(x), t) \geq M(x, y, t) \Rightarrow M(f(x), f(y), t) \geq k\mathbb{M}(x, y) \quad \_\_\_\_(1)$$

For all $x, y \in X$, where

$$\mathbb{M}(x,y) = min\,\{M(x,y,t), M(x,f(x),t), M(y,f(y),t), M(x,f(y),t) * M(y,f(x),t)\}$$

Then f has a fixed point in X .

Proof: Define $\alpha, \eta: X \times X \to [0,\infty)$ and $\varphi: [0,1] \to [0,1]$ by

$\alpha(x,y) = M(x, f(x), t) \quad , \quad \eta(x,y) = M(x,y,t)$

Such that $\alpha(x,y) \geq \eta(x,y)$ for all $x, y \in X$

And let $\varphi(\tau) = k\tau \ , \tau \in [0,1]$

By condition (i) –(v) of theorem (2.10)

Let $\alpha\big(x, f(x)\big) \geq \eta(x,y)$

Then $M(x, f(x), t) \geq M(x,y,t)$

From (1) we have $M(f(x), f(y), t) \geq k\mathbb{M}(x,y) = \varphi(\mathbb{M}(x,y))$

Then f is α-η-φ-contraction function by condition of theorem (2.10) and f has fixed point, then the unique fixed point follows from (1).

Corollary 2.3.19: *Let $(X, M,*)$ be a complete fuzzy metric space and let $f: X \to X$ continuous function. Assume that there exist $k \in (0,1)$ such that*

$$M(x, f(x), t) \geq M(x,y,t) \Longrightarrow M(f(x), f(y), t) \geq kM(x,y,t)$$

For all $x, y \in X$,

Then f has a fixed point in X .

2.4 An α- ψ- ϕ-Contraction Function

Definition 2.4.20: *Let $(X, M,*)$ be a complete fuzzy metric space. Let $f: X \to X$ be an α-admissible which satisfies the following*

$$\psi\big(M(f(x), f(y), t)\big) \geq \psi\big(\mathbb{M}(x,y)\big) - \phi(\mathbb{M}(x,y))$$

Such that

(i) ψ is continuous and decreasing with $\psi(\tau) = 0$ iff $\tau = 1$.
(ii) ϕ is continuous with $\phi(\tau) = 0$ iff $\tau = 1$
f is called α- ψ- ϕ- contraction function .

Theorem 2.4.21: *Let $(X, M,*)$ be a complete fuzzy metric space . Let $f: X \to X$ be an α-admissible which satisfies the following*

$$\psi\big(M(f(x), f(y), t)\big) \geq \psi\big(\mathbb{M}(x,y)\big) - \phi(\mathbb{M}(x,y))$$

Such that

(i) $(X, M,*)$ is an α-complete.
(ii) f is α-continuous function
(iv) There exist $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$.

Proof: Let $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$

Define a sequence $\{x_n\}$ such that $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$

If $x_n = x_{n+1}$ for some n , then $x = x_n$ is a fixed point of f.

Suppose $x_n \neq x_{n+1}$

Since f is α-admissible function with respect η and

$\alpha(x_0, f(x_0)) = \alpha(x_0, x_1) \geq 1$

Then $\alpha(x_1, x_2) = \alpha\big(f(x_0), f(x_1)\big) \geq \eta\big(f(x_0), f(x_1)\big) = \eta(x_1, x_2)$

By continuing this process, we get

$\alpha(x_n, f(x_n)) = \alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$

$$\psi\big(M(x_n, x_{n+1}, t)\big) = \psi(M(f(x_{n-1}), f(x_n), t))$$
$$\geq \psi\big(\mathbb{M}(x_{n-1}, x_n)\big) - \phi\big(\mathbb{M}(x_{n-1}, x_n)\big)$$

Where

$$\mathbb{M}(x_{n-1}, x_n) = min\,\{M(x_{n-1}, x_n, t), M(x_{n-1}, f(x_{n-1}), t), M(x_n, f(x_n), t), M(x_{n-1}, f(x_n), t) * M(x_n, f(x_{n-1}), t)\}$$

$$= min\,\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), M(x_n, x_{n+1}, t), M(x_{n-1}x_{n+1}, t) * M(x_n, x_n, t)\}$$

$$\geq min\,\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), M(x_{n-1}, x_n, t) * M(x_n, x_{n+1}, t)\}$$

$$= min\,\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}$$

Since ψ decreasing, then $M(x_{n-1}, x_n, t) < M(x_n, x_{n+1}, t)$ that is $\{M(x_n, x_{n+1}, t)\}$ is an increasing sequence thus there exist $l(t) \in (0,1]$ such that $lim_{n\to\infty} M(x_n, x_{n+1}, t) = l(t) < 1$

Now taking $n \to \infty$ we obtain for all $t > 0$

$\psi\big(l(t)\big) \geq \psi\big(l(t)\big) - \phi(l(t))$ which is contradiction $l(t) = 1$

Thus we conclude for all $t > 0$

$$lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$$

$\{x_n\}$ is Cauchy sequence .since X is α-complete ,then $x_n \to x$

$$\psi(M(x_{n+1}, f(x), t)) = \psi(M(f(x_n), f(x), t) \\ \geq \psi\big(\mathbb{M}(x_n, x)\big) - \phi\big(\mathbb{M}(x_n, x)\big)$$

Letting $n \to \infty$ in the above inequality using properties ψ, ϕ

$$\psi(lim_{n\to\infty} M(x_n, f(x), t)) \geq \psi(1) - \phi(1) = 0$$

Thus $lim_{n\to\infty} M(x_n, f(x), t) = 1$

Hence $x_n \to f(x) \Longrightarrow f(x) = x$.

Now we assume y is a fixed point of f such that $f(y) = y$

$$\psi\big(M(x, y, t)\big) = \psi\big(M(f(x), f(y), t)\big) \geq \psi\big(\mathbb{M}(x, y)\big) - \phi\big(\mathbb{M}(x, y)\big) = 0$$
$$\Longrightarrow M(x, y, t) = 1 \quad \Longrightarrow x = y\,.$$

2.5 Application On α-η-φ-Contraction

Definition 2.5.22: *Let $(X, M, *)$ be a fuzzy metric space and $f: X \to X$ be an (α, β)-admissible function, f is said to be*

(a) (α, β)-contraction function of type (I) , if
$\alpha(x, y)\beta(x, y)M(f(x), f(y), t) \geq \varphi\big(\mathbb{M}(x, y)\big)$.

(b) (α, β)-contraction function of type (II) , if there exist $0 < \ell \leq 1$ such that
$(\alpha(x, y)\beta(x, y) + \ell)^{M(f(x), f(y), t)} \geq (1 + \ell)^{\varphi(\mathbb{M}(x,y))}$

Theorem 2.5.23: *Let $(X, M, *)$ be a complete fuzzy metric space and let $f: X \to X$ be an α-continuous and (α, β)-contraction function of type (I), (II), if there exist $\alpha(x_0, f(x_0)) \geq 1$ and $\beta(x_0, f(x_0)) \geq 1$, then f has a unique fixed point in X.*

Proof: Let $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$ and $\beta(x_0, f(x_0)) \geq 1$

Define a sequence $\{x_n\}$ such that $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$

Since f is (α, β)-admissible function and $\alpha(x_0, f(x_0)) \geq 1$

Then $\alpha(x_1, x_2) = \alpha\big(f(x_0), f(x_1)\big) \geq 1$

By continuing this process, we get

$\alpha(x_n, x_{n+1}) = \alpha(x_n, f(x_n)) \geq 1$

Similarly we have $\beta(x_n, x_{n+1}) = \beta(x_n, f(x_n)) \geq 1$

If $x_n = x_{n+1}$ for some n , then $x = x_n$ is a fixed point of f.

Suppose $x_n \neq x_{n+1}$

(a)

$$\begin{aligned}\alpha(x_n, x_{n+1})\beta(x_n, x_{n+1})M(x_n, x_{n+1}, t) &= \alpha(x_n, x_{n+1})\beta(x_n, x_{n+1})M(f(x_{n-1}), f(x_n), t)\\ &\geq \varphi(\mathbb{M}(x_{n-1}, x_n))\end{aligned}$$

Where

$$\mathbb{M}(x_{n-1}, x_n) = min\,\{M(x_{n-1}, x_n, t), M(x_{n-1}, f(x_{n-1}), t), M(x_n, f(x_n), t), M(x_{n-1}, f(x_n), t) * M(x_n, f(x_{n-1}), t)\}$$

$$= min\,\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), M(x_n, x_{n+1}, t), M(x_{n-1}x_{n+1}, t) * M(x_n, x_n, t)\}$$

$$\geq min\,\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), M(x_{n-1}, x_n, t) * M(x_n, x_{n+1}, t)\}$$

$$= min\,\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}$$

If $min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\} = M(x_n, x_{n+1}, t)$

Then

$$\begin{aligned}M(x_n, x_{n+1}, t) &\geq \varphi(min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}) \geq \varphi\big(M(x_n, x_{n+1}, t)\big)\\ &> M(x_n, x_{n+1}, t)\end{aligned}$$

Which is contradiction

Therefore $min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\} = M(x_{n-1}, x_n, t)$

Hence for all $n \in \mathbb{N}$ we have

$$\begin{aligned}M(x_n, x_{n+1}, t) &\geq \varphi\big(M(x_{n-1}, x_n, t)\big) \geq \varphi^2\big(M(x_{n-2}, x_{n-1}, t)\big)\\ &\geq \cdots \geq \varphi^n\big(M(x_0, x_1, t)\big)\end{aligned}$$

Let $n, m \in \mathbb{N}$ with $n > m$, then

$$\begin{aligned}M(x_n, x_m, t) &= \alpha(x_n, x_{n+1})\beta(x_n, x_{n+1})M(f(x_{n-1}), f(x_n), t) \geq \varphi\big(\mathbb{M}(x_{n-1}, x_n)\big)\\ &\geq \cdots \geq \varphi^n\big(M(x_0, x_1, t)\big)\end{aligned}$$

Therefore $lim_{n\to\infty} M(x_n, x_m, t) = 1$

Hence $\{x_n\}$ is a Cauchy sequence

Since X is an α-η-complete fuzzy metric space there is $x \in X$ such that $x_n \to x$ as $n \to \infty$.

$$M(x_n, f(x), t) = \alpha(x_n, x)\beta(x_n, x)M(f(x_n), f(x), t) \geq \varphi\big(\mathbb{M}(x_n, x)\big)$$

Hence $f(x) = x$

Suppose y is a fixed point of f such that $f(y) = y$

$M(x, y, t) = \alpha(x, y)\beta(x, y)M(f(x), f(y), t) \geq \varphi(\mathbb{M}(x, y))$.

Hence $x = y$.

(b)

$$\begin{aligned}(\alpha(x_n, x_{n+1})\beta(x_n, x_{n+1}) + \ell)^{M(x_n, x_{n+1}, t)} & \\ &= (\alpha(x_n, x_{n+1})\beta(x_n, x_{n+1}) + \ell)^{M(f(x_{n-1}), f(x_n), t)} \\ &\geq (1 + \ell)^{\varphi(\mathbb{M}(x_{n-1}, x_n))}\end{aligned}$$

Where

$$\begin{aligned}\mathbb{M}(x_{n-1}, x_n) = min\,\{&M(x_{n-1}, x_n, t), M(x_{n-1}, f(x_{n-1}), t), M(x_n, f(x_n), t), \\ &M(x_{n-1}, f(x_n), t) * M(x_n, f(x_{n-1}), t)\}\end{aligned}$$

$$\begin{aligned}= min\,\{&M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), M(x_n, x_{n+1}, t), \\ &M(x_{n-1} x_{n+1}, t) * M(x_n, x_n, t)\}\end{aligned}$$

$$\geq min\,\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), M(x_{n-1}, x_n, t) * M(x_n, x_{n+1}, t)\}$$

$$= min\,\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}$$

If $min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\} = M(x_n, x_{n+1}, t)$

Then

$$\begin{aligned}M(x_n, x_{n+1}, t) \geq \varphi(min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}) &\geq \varphi\big(M(x_n, x_{n+1}, t)\big) \\ &> M(x_n, x_{n+1}, t)\end{aligned}$$

Which is contradiction

Therefore $min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\} = M(x_{n-1}, x_n, t)$

Hence for all $n \in \mathbb{N}$ we have

$$\begin{aligned}M(x_n, x_{n+1}, t) \geq \varphi\big(M(x_{n-1}, x_n, t)\big) &\geq \varphi^2\big(M(x_{n-2}, x_{n-1}, t)\big) \\ &\geq \cdots \geq \varphi^n\big(M(x_0, x_1, t)\big)\end{aligned}$$

Let $n, m \in \mathbb{N}$ with $n > m$, then

$$M(x_n, x_m, t) = (\alpha(x_n, x_{n+1})\beta(x_n, x_{n+1}) + 1)^{M(f(x_{n-1}), f(x_n), t)}$$
$$\geq (1+\ell)^{\varphi(\mathbb{M}(x_{n-1}, x_n))} \geq \cdots \geq (1+\ell)^{\varphi^n(M(x_0, x_1, t))}$$

Therefore $lim_{n\to\infty} M(x_n, x_m, t) = 1$

Hence $\{x_n\}$ is a Cauchy sequence

Since X is an α-η-complete fuzzy metric space there is $x \in X$ such that $x_n \to x$ as $n \to \infty$.

$$(\alpha(x_n, x)\beta(x_n, x) + \ell)^{M(x_n, f(x), t)} = (\alpha(x_n, x)\beta(x_n, x) + \ell)^{M(f(x_n), f(x), t)}$$
$$\geq (1+\ell)^{\varphi(\mathbb{M}(x_n, x))}$$

Hence $f(x) = x$

Suppose y is a fixed point of f such that $f(y) = y$

$(\alpha(x, y)\beta(x, y) + \ell)^{M(f(x), f(y), t)} \geq (1+\ell)^{\varphi(\mathbb{M}(x, y))}$.

Hence $x = y$.

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