New Finite Difference Derivation For Calculation of Natural Frequency of Sector Steel Plate

Dr. Sadjad A. Hemzah
Civil Engineering Department
College of Engineering, University of Al-Qadisiyah
Al-Qadisiyah, Iraq
E-mail: sah246@gmail.com
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Abstract

A free vibration analysis of isotropic thin circular plate with various edge conditions have been studied in the present work. This study involves the obtaining of natural frequencies by solving the mathematical model that governs the vibration behavior of the plate using finite difference method. The numerical results of natural frequencies of circular plate are presented for different cases such as aspect ratio, curvature effect, grid size and boundary conditions. A good results was obtained from finite difference procedure compared with that obtained from the finite element analysis using Abaqus Package program.

Keywords: Curved Plate, Free Vibration, Thin Plate, Finite Deference Analysis , Numerical Analysis.

اشتقاق جديد بطريقة الفروقات المحددة لحساب التردد الحر للصفائح الحديدية المنحنية

الخلاصة

تم في هذا البحث دراسة حساب الاهتزاز الحر باستخدام طريقة الفروقات المحددة في تحليل الصفائح الدائرية النحيفة. الدراسة اشتملت على ايجاد الاهتزاز الحر من خلال حل المعادلات الرياضية الخاصة للصفائح المقوسة باستخدام طريقة الفروقات. النتائج المتحصلة من التحليل باستخدام طريقة الفرقات المحدثة لهذه الصفائح ولحالات مختلفة مثل نسبة الطول الى العرض, درجة التقوس , حجم التقسيم للصفيحة و نوع المساند. أعطت النتائج المتحصلة من طريقة الفروقات المحددة توافقا جيدا مع النتائج المتحصلة من طريقة العناصر المحددة وباستخدام برنامج ABAQUS ولكافة أنواع المتغيرات المأخوذة ضمن هذه الدراسة.

كلمات البحث: الألواح المقوسة ، التردد الحر ، الألواح النحيفة ، طريقة الفروقات المحددة ,التحليل العددي.

1. Introduction

Plates, as structural elements, are extensively used in many fields of engineering including aerospace, civil structures, hydraulic structures, containers, ships, instruments, and machine parts. When in service, they are subjected to dynamic loadings the effect of which is very critical. Much research has been conducted into plate behavior, using a wide range of methods. An excellent monograph of the early literature relating to vibration analysis of plates was published by Leissa [1].

The small thickness makes the plate susceptible to various types of effective such as buckling modes and vibration modes. In engineering application, however, plate problems often involve consideration of dynamic disturbances, produced by time-dependent external forces or displacements. Dynamic loads may be created by moving vehicles, wind gusts, unbalanced machines, etc.

Most researchers [1][2][3], have used classical thin plate theory in their formulations to study the plate response; where the flexural vibration of the thin plate is characterized by a fourth-order partial differential equation. A direct solution of such equation might be difficult and most of the reported solutions are based on numerical methods such as finite difference method [4,9], and finite element method [4], [5].

A number of approaches were proposed by different researchers to solve the differential equation of plats. Finite Element and Finite Difference methods are the well-known approaches to be the most widely used numerical procedures to find the solution of the mentioned differential equation. Finite Element method advantageous is that it is very suitable for practical engineering problems of complex geometries. However, the computational complexity involved in this method constitutes the main disadvantage of this technique, especially in real-time application. On the other hand, the method is fast enough to analyze, relatively easy to program, and also seems to be more convenient for uniform structures such as plate system. The main serious impediment of this method is that it is not suitable for problems with complex and irregular. Moreover, since the finite difference method is difficult to vary the size of the difference in particular regions, it is not suitable for problems with rapidly changing variables such as stress concentration problems. However, because of the geometry uniformity of the thin plates, finite difference method seems to be more applicable and faster to calculate deformations, forces, stresses and strains and natural frequencies.

The objective of the present study is to develop an accurate and efficient method for determining the natural frequencies of isotropic curved thin plate with different boundary conditions, aspect ratios, curvature and grid size.

2. Description of The Problem

The curved plate considered in this study is an isotropic plate with constant thickness (h) and has an inner radius R_{in} and outer radius R_{out} as shown in Fig. (1). The circular plate divided by a polar mesh. The interval of mesh in the direction of the radius is (β), while (φ) is the angle of each interval of mesh in the direction of central angle, the interval between the nodes in the perpendicular direction on the radius is denoted as (γ_{i-1} , γ_i , γ_{i+1}).

3. Governing Equations and Finite Difference Formulation and Solution

The governing equation of motion in polar coordinate for a curved isotropic plate of uniform thickness and without in plane load is [7]:

$$\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = \frac{q}{D} \qquad (1)$$

where:

q= is distributed load applied on the plate surface

ρ= density of the material

h= thickness of the plate

D= flexural rigidity

$\nabla^4 = \nabla^2 \nabla^2$

These variables could be expressed as follows:

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \emptyset^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) \qquad (2)$$

$$D = \frac{E\ h^3}{12(1-\upsilon^2)} \qquad (3)$$

$$\nabla^4 w = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \emptyset^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \emptyset^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) w \qquad (4)$$

therefore, the final governing equation of motion of isotropic circular plate will be as follows:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \emptyset^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \emptyset^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) w + \rho h \frac{\partial^2 w}{\partial t^2} = \frac{q}{D} \qquad (5)$$

The solution of equation (4) may be accomplish by finite difference method, as shown in Fig. (2) :

where:

$\lambda = \omega^2 . \rho.h$

$$C1 = \frac{1}{\beta^4} + \frac{1}{r\beta^3} + \frac{1}{4r^2\beta^2}$$

$$C2 = \frac{2}{r^2\varphi^2\beta^2} + \frac{1}{r^3\varphi^2\beta}$$

$$C3 = \frac{-4}{\beta^4} - \frac{2}{r\beta^3} - \frac{4}{r^2\varphi^2\beta^2} - \frac{2}{r^3\varphi^2\beta}$$

$$C4 = \frac{1}{r^4\varphi^4}$$

$$C5 = \frac{-4}{r^2\varphi^2\beta^2} - \frac{4}{r^4\varphi^4}$$

$$C6 = \frac{6}{\beta^4} + \frac{8}{r^2\varphi^2\beta^2} + \frac{6}{r^4\varphi^4}$$

$$C7 = \frac{2}{r^2\varphi^2\beta^2} - \frac{1}{r^3\varphi^2\beta}$$

$$C8 = \frac{-4}{\beta^4} + \frac{2}{r\beta^3} - \frac{4}{r^2\varphi^2\beta^2} + \frac{2}{r^3\varphi^2\beta}$$

$$C9 = \frac{1}{\beta^4} - \frac{1}{r\beta^3} - \frac{1}{4r^2\beta^2}$$

By applying the finite difference scheme at the interior nodes of the divided plate, the following system of simultaneous linear equations in matrices will be obtained[6]

$$[K]\{w\} + \lambda[B]\{w\} = 0 \qquad (7)$$

Where{w} is column matrix whose elements, ωi, represent the amplitude of the free vibration, [K] is a square matrix obtained from the finite difference expression of the biharmonic operator ∇^4, while[B] is a diagonal matrix representing the constants in the term of Eq. (7), and λ= ω2.ρ. Notice that Eq (7) is an Eigen-value problem. For a given thickness (h) and plate–aspect ratio a b (b= radius* substantial angle(rad)), the Eigen-value (ω) can be determined numerically by using any relevant technique. The smallest Eigen-value gives the most (fundamental) free vibration factor.

4. The Eigen-value Problem Solution

The Eigen-value problem may be solved by various numerical or analytical techniques. Numerical methods for nonlinear problems usually depend on iterative procedure. The procedure of the adopted numerical method that is used in this study is outlined, as follows(8):

1. Compute the stiffness matrix [K] from applying the coefficient patterns for finite difference operators Fig.(3) at the interior nodes of the plate.

2. Compute the geometry matrix [B] from applying the coefficient patterns for finite difference operators Fig.(3) at the interior nodes of the plate.

3. Compute the inverse matrix for the stiffness matrix [K] (by Gauss-Jordan or other suitable technique)

4. Make [C] matrix by multiplying the inverse matrix $[K]^{-1}$ by the geometry matrix, as follows:

$$[C]=[K]^{-1}[B] \qquad (8)$$

$$[C]\{w\}=\lambda^{*}[I]\{w\} \tag{9}$$

$$(C-\lambda^{*}[I])\{w\}=0 \tag{10}$$

where

$$\lambda^{*} = \frac{1}{\lambda} = \frac{D}{\omega^{2}\rho h}$$

where the largest Eigen-value $(\frac{1}{\lambda})$ of the matrix [K]-1is the reciprocal of the smallest value of the Eigen-value(λ) of the matrix [K].

By applying the following steps (a-d) to Eq. (10), the minimum Eigen-value can be obtained:

a- Assume initial trial vector $\{\omega_0\}$ which may be taken equal to 1.0

b- Substitute the vector$\{\omega_0\}$ at Eq.(10)

c- Approximate value of $(\frac{1}{\lambda})$ is obtained by dividing the first element of the column matrix[C] $\{\omega_0\}$ by w1,

$$(\frac{1}{\lambda}) = \text{First row of } [C]\{\omega_0\} / w_1$$

where w1 is the first element of the matrix $\{\omega_0\}$

d- The second approximate value of the characteristic vector $\{\omega_0\}$is obtained by:

$$\omega_1 = \frac{[C].\omega_0}{\lambda *}$$

These steps can be continued until the errors become sufficiently small where the used error criteria is the average of the sum of the absolute differences:

$$\varepsilon r = \frac{\sum|\omega_i - \omega_{i-1}|}{n}$$

Thus; the natural frequency of isotropic curved thin plate with constant thickness will be:

$$\omega = \sqrt{\frac{D}{\lambda^{*}.\rho}}$$

5. Boundary Conditions

The situation of the boundary condition of the circular plate should be characterized to find the solution. Thus the boundary conditions of a circular plate with a radius (r) may be defined as:

1. Fixed edge (clamped edge) :

$$\omega = 0: \frac{\partial w}{\partial r} = 0\ (External\ immaginary\ deflection = Internal\ deflection)$$

2. Simply supported edge:

$$\omega = 0: \frac{\partial w}{\partial r} \neq 0\ (External\ immaginary\ deflection = -Internal\ deflection)$$

6. Numerical Results

The derived numerical schemes for determining the natural frequency (w) were programmed in order to analyze different cases for an isotropic curved plate. Results of the numerical analysis for the present investigation was compared with those obtained from the finite element analysis program Abaqus. Abaqus provides both triangular and quadrilateral shell elements with linear interpolation and user-choice of larger- and small-strains formulations. For most of applications large-strain shell elements (S4R, S3R, and SAX1) are appropriate with these considerations. In the following study the S4R (4 node shell element) is used in the analysis.

The present study was performed for a steel plate with the following properties:

- Inner radius = 1 m and outer radius = 2 m.
- modulus of elasticity E = 200 GPa.
- poisons ratio $\upsilon = 0.3$.
- thickness t = 10 mm .

Several parameters were concerned in this study. These parameters are :

1. Mesh Size
2. Aspect Ratio and boundary conditions
3. Curvature effect

- **Mesh size**

In order to investigate the effect of mesh used in the presented finite difference method, a curved plate of angle 38.197° and with the previous properties was analyzed with different mesh sizes and for two types of boundary conditions, simply support and Clamped support. The results of convergence as a function of mesh size for both types of boundary conditions were shown in table(1) and table (2). Also Fig. (3) and (4) show the convergence of the calculated natural frequency with those of Abaqus F.E. Program, and Fig. (5) shows a sample of the deflection values of SSSS plate for both finite difference and finite element results. It can be noticed that the natural frequency value obtained from both finite difference and finite element methods goes to a value of approximately 19.81 and 35.84 for both simply and Clamped supports, also, the (15×15) mesh gives a difference of about 0.35% and 0.93% of final predicted values of both two types of supports. However, a mesh of 10×10 is very useful to be used in the analysis procedure because its saving time and it has a difference around 1% of predicted values of natural frequencies.

- **Aspect Ratio and boundary conditions**

The effect of both aspect ratio and boundary condition was investigated. The aspect ratio for the curved plate was taken as (a/b) where a represent the length of the plate in radial direction and b represents the average width of the plate which can be obtained by multiplying the average radius by the radian value of the substantial angle. The studied aspect ratios were starts from (0.5 to 3) and for four types of boundary conditions surrounding the plate. These boundary conditions are :

1- SSSS (simply supported all around)

2- CCCC (all edges are Clamped)

3- SCSC (both bottom and top edges are simple and both left and right edges are Clamped)

4- CSCS (both bottom and top edges are Clamped and both left and right edges are simple)

Table (3) and Fig. (6) show the natural frequencies results of the present finite difference procedure with different types of support conditions and aspect ratios. Note that S and C symbols represent simple and Clamped support.

From these results it can be seen that values of the natural frequency of a plate of boundary conditions (SCSC) are closed to a plate of Clamped supports in lower values of aspect ratios and they are closed to a simply support plate in a higher values of aspect ratios. While , a plate of boundaries (CSCS) has an adverse behavior within the used range of aspects ratios.

- **Curvature effect**

The curvature effect was studied by increasing the average radius of the plate (i.e radius at the center of the plate) from 1 to ∞ so that the aspect ratio was kept to be equal to 1 with a length of 1m. So when R goes to ∞, the plate will be like a rectangular plate of dimensions (1x1)m . Also two types of boundary conditions was taken into account in studying the curvature effect, simply and Clamped supports. The results of curvature effect on both types of boundary conditions are listed in table (4). While Fig. (7) shows these results on a logarithmic x-axis represents the radius value and natural frequencies value represented on the y-axis. From the results shown below it can be seen clearly that when R goes to ∞, the plate will behave as a square plat of dimensions 1x1m which has a natural frequency of 19.739 and 35.841for both simple and Clamped supports respectively. In the current finite difference scheme the calculated natural frequency has a difference of round 0.7% from the analytical values of square plate.

5. Conclusions

The free vibration of thin curved circular isotropic plate was investigated herein. A finite difference approach was presented to analyze the plate. The validity of the adopted approach was compared with finite element approach which was done by using software package (ABAQUS). The investigation was carried out to simulate the natural frequency of thin curved plate with different types of boundary conditions, aspect ratios, curvature and different mesh size. The following are the main points concluded after studying the results obtained from the present study:

1. The results of the adopted finite difference approach of analysis showed good agreement with those obtained by finite element analysis (ABAQUS).

2. The schemes of finite difference approach which were derived in the present study can be used to analyze curved circular thin plate with a good agreement results.
3. The present study shows the validity of the derived finite difference schemes with a very small error and for any suitable mesh size if compared with finite element analysis program package results
4. Finite Difference Method is more efficient for such problems than Finite Elements Method (software package), since F.D.M gives good agreement of results with less time of calculations.
5. The finite difference method gives an under estimations value for natural frequency which it is more safe than finite element method which give an over estimation values for such structures .

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Table (1): Finite difference and F.E. Results for a simply supported curved plate

Mesh Size	ω (F.D) (rad/sec)	ω (F.E) (rad/sec)
4x4	18.693	21.087
5x5	19.067	20.655
6x6	19.277	20.416
7x7	19.408	20.271
8x8	19.497	20.178
9x9	19.560	20.113
10x10	19.607	20.066
11x11	19.644	20.031
12x12	19.674	20.003
13x13	19.699	19.981
14x14	19.721	19.964
15x15	19.739	19.950

Table (2): Finite difference and F.E. Results for a Clamped supported curved plate

Mesh Size	ω (F.D) (rad/sec)	ω (F.E) (rad/sec)
4x4	28.949	42.198
5x5	31.019	39.822
6x6	32.341	38.684
7x7	33.230	38.035
8x8	33.853	37.628
9x9	34.306	37.354
10x10	34.644	37.162
11x11	34.903	37.021
12x12	35.106	36.915
13x13	35.268	36.833
14x14	35.399	36.768
15x15	35.507	36.716

Table (3): Finite difference natural frequencies in terms of $\sqrt{\frac{D}{h.\rho}}$ of curved plate with different boundary conditions and aspect ratios

Support Type	SSSS	CCCC	SCSC	CSCS
Aspect Ratio	ω (rad/sec)	ω (rad/sec)	ω (rad/sec)	ω (rad/sec)
0.5	47.45008	88.66515	81.68419	53.33016
0.75	27.08775	48.76832	42.69226	34.12574
1	19.60723	34.64397	27.87902	27.88447
1.25	16.08721	28.66401	20.82557	25.24501
1.5	14.16622	25.79708	17.04361	23.91319
1.75	13.00771	24.27092	14.84224	23.1532
2	12.25685	23.38445	13.47623	22.67953
2.25	11.74301	22.83161	12.58302	22.36439
2.5	11.37613	22.46648	11.97287	22.14402
2.75	11.10513	22.21385	11.54042	21.98377
3	10.89932	22.03229	11.22416	21.86353

Table (4): Finite difference natural frequencies in terms of $\sqrt{\frac{D}{h.\rho}}$ of curved plate with different boundary conditions and different radiuses

R(m)	Theta (deg)	CCCC	SSSS
		ω (rad/sec)	ω (rad/sec)
1	57.29578	35.06038	19.99886
1.5	38.19719	34.64397	19.73921
2.5	22.91831	34.45288	19.66049
3.5	16.37022	34.40288	19.64656
4.5	12.7324	34.38265	19.6421
5.5	10.41741	34.37249	19.6402
10.5	5.456741	34.35769	19.63795
15.5	3.696502	34.35468	19.6376
20.5	2.794916	34.35359	19.63749
30.5	1.87855	34.35279	19.63742
50.5	1.13457	34.35237	19.63738
100.5	0.570107	34.35219	19.63737
∞	0.057267	34.35213	19.63736

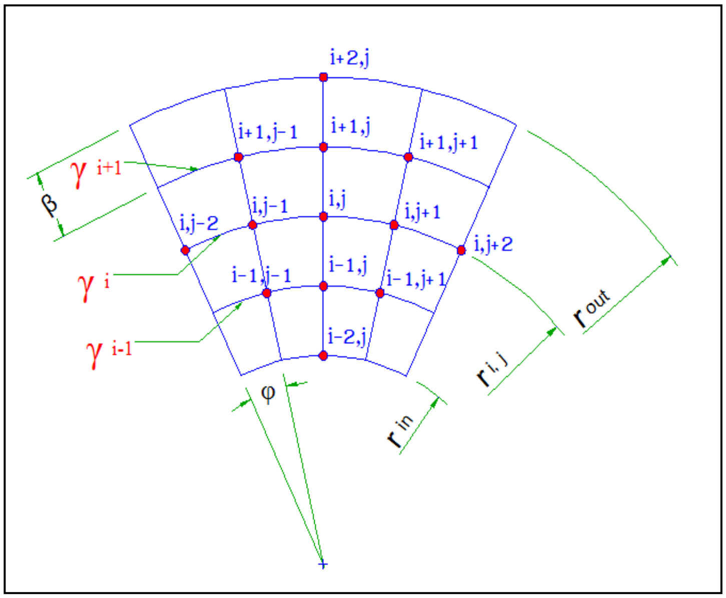
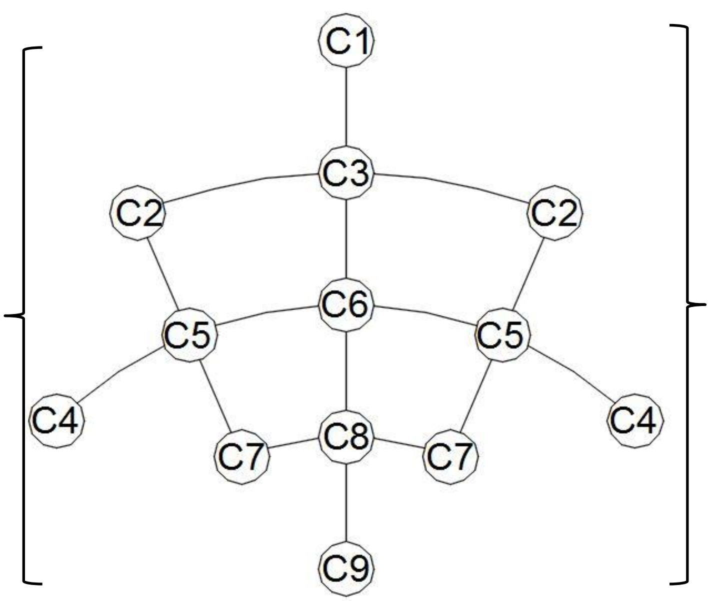
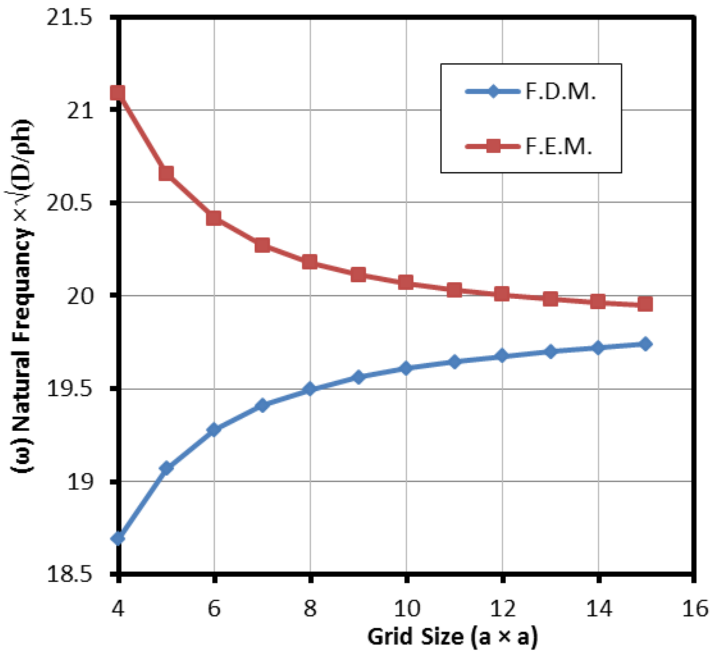


Figure (1): Geometry of the curved plate and node distribution

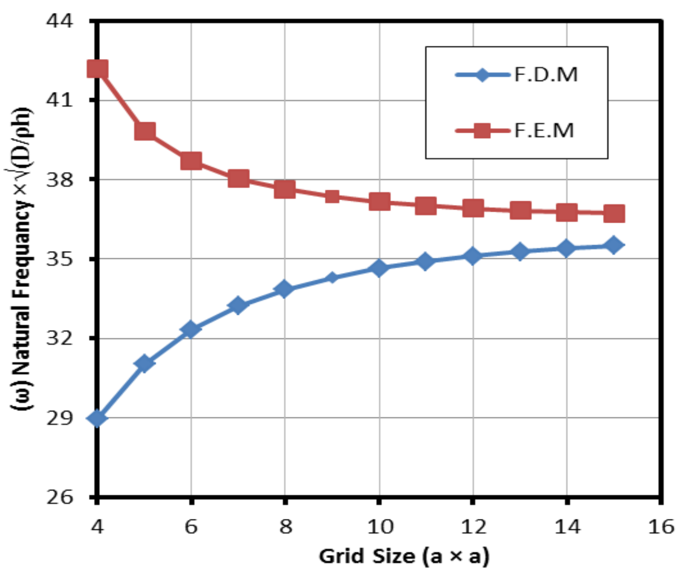


$$W_i + \left\{\frac{1}{D}\right\} \qquad W_i = 0 \qquad (6)$$

Figure (2): Finite difference scheme of curved plate differential equation



Figure(3): Finite difference versus Finite Element Results for a simply supported curved plate



Figure(4): Finite difference versus Finite Element Results for a Clamped supported curved plate

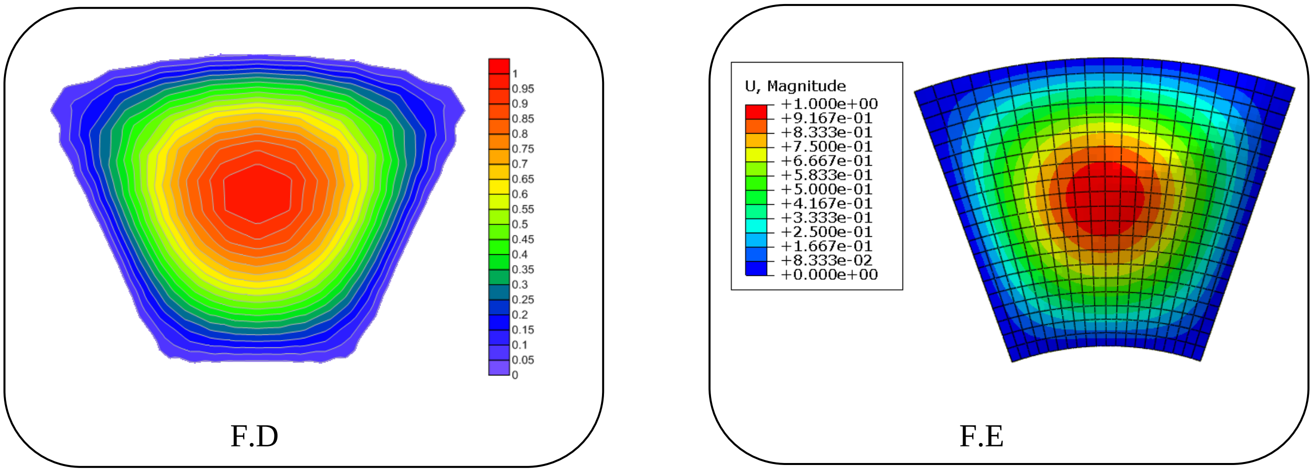


Figure (5): Deflection values for first mode of SSSS plate for finite difference and finite element

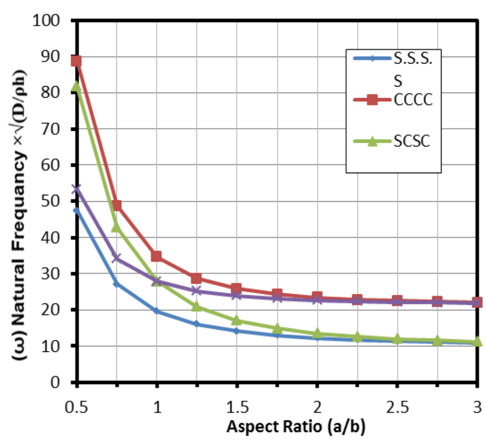


Figure (6): Finite difference natural frequencies of curved plate with different boundary conditions and aspect ratios

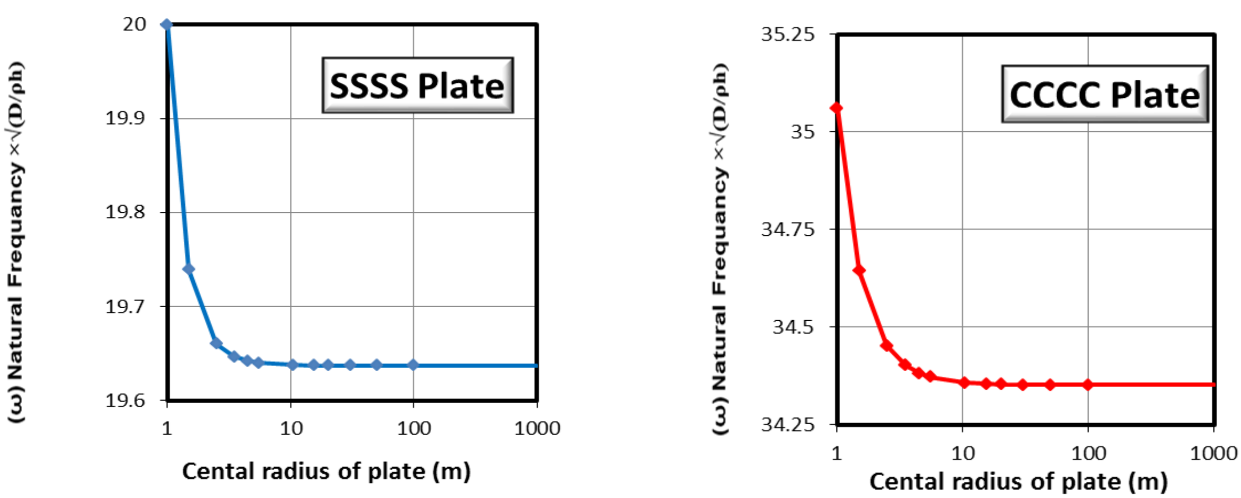


Figure (7): Finite difference natural frequencies of curved plate with different boundary conditions and different radiuses