Pretopological Spaces

By

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Abstract:

In 1982, M.E., Abd El-Monsef introduced the notions of pre-open set and pre-continuity in topological spaces and obtained a number of their properties. J.Dontchev (1998) defined pre-open functions and investigated properties of such functions. In this paper we introduce the concept "pretopology" and the concept "P-pre-open sets". We give characterizations of P-pre-continuous functions, P-pre-irresolute functions, separation axioms and P-pre-closed graphs.

الفضاءات التبولوجية تقريباً

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ملخص:
في عام 1982 قدم (M.E.,Abd El-Monsef) مفهوم المجموعة المفتوحة تقريباً و الدوال المستمرة تقريباً في الفضاءات التبولوجية و حصل على عدد من خواصها. عرف (J.Dontchev 1998) الدوال المفتوحة تقريباً وتحقق من خواص هذه الدوال. في هذا البحث قدمنا مفهوم "التبولوجي تقريباً" و مفهوم "المجموعات المفتوحة تقريباً من النوع P". كذلك فقد قدمنا في هذا البحث خواص الدوال المستمرة تقريباً من النوع P و الدوال غير التحليلية تقريباً من النوع P وبديهيات الفصل و الرواسم المغلقة تقريباً من النوع P.

0. Introduction

In this paper, if X is anon-empty set then the topological space (X,τ), it is denoted by X if this can cause no confusion. Let X,Y and Z be topological spaces, let A be a subset of X. The closure (resp. interior, boundary) of A will be denoted by Cl(A) (resp. Int(A),Bd(A)). A subset A of X is called pre-open [4], if $A\subseteq Int(Cl(A))$, the complement of a pre-open is called pre-closed. The family of all pre-open sets in X will be denoted by P.O(X). The pre-interior of a set A is the largest pre-open set contained in A, denoted by pre-int(A).

(i.e. pre-int(A)$=\bigcup\{U: U\subseteq A$, U is pre-open$\}$. A function $f:X\to Y$ is called pre-continuous [4], if $f^{-1}(U)\in P.O(X)$ for each open set U of Y, f is pre-irresolute if $f^{-1}(U)\in P.O(X)$ for each $U\in P.O(Y)$. A function $f:X\to Y$ is called pre-open if $f(U)\in P.O(Y)$ for each open set U in X and it is called pre-closed if the image of each closed set in X is pre-closed in Y. If all sets of the family $\{U_\lambda\}_{\lambda\in\Lambda}$ are pre-open sets, then $\bigcup_{\lambda\in\Lambda} U_\lambda \subseteq \bigcup_{\lambda\in\Lambda}(Int(Cl(U_\lambda)) \subseteq Int(\bigcup_{\lambda\in\Lambda}(Cl(U_\lambda))) \subseteq Int(Cl(\bigcup_{\lambda\in\Lambda} U_\lambda))$, that is, P.O(X) is closed with respect to arbitrary unions.

1. Pretopologies

In this section, we introduce novel two definitions (to the best of our knowledge) of a topological spaces which are pretopological spaces and pretopology associated with a some topology. We give some results and example which are related with this subject.

1.1 Definition:

Let (X,τ) be a topological space. Let $\beta \subseteq P.O(X)$. Then we define the intersection of members of β, to be the pre-interior of $\cap\beta$ and is denoted by $\cap^p\beta$.

If $A_1, A_2,\ldots,A_n \in P.O(X)$, we will write:

$\cap^p\{A_1,A_2,\ldots,A_n\} = A_1 \cap^p A_2 \cap^p \ldots \cap^p A_n = \text{pre-}Int(A_1 \cap A_2 \cap \ldots \cap A_n)$.

1.2 Definition:

Let (X,τ) be a topological space. Let $\tau^p \subseteq P.O(X)$, we say that τ^p is a pretopology on X if the following conditions are satisfied:

i) $\phi, X \in \tau^p$

ii) For every $A, B \in \tau^p$, $A \cap^p B \in \tau^p$.

iii) For every $\{U_\lambda\}_{\lambda\in\Lambda} \subseteq \tau^p$, $\bigcup_{\lambda\in\Lambda} U_\lambda \in \tau^p$.

we will call the order pair (X, τ^p) a pretopological space.

1.3 Definition:

Let (X,τ) be a topological space and τ^p be a pretopology on X. A pre-open set A in X is called P-pre-open in X if $A \in \tau^p$.

1.4 Definition:

Let (X, τ^p) be a pretopological space, a pre-closed subset B of X is called P-pre-closed in X if its complement B^c in X is P-pre-open in X.

Many concepts, like the interior, closure, exterior and the derived set,...,etc., can be defined in pretopological spaces as well as defined in topological spaces. Many result of topological spaces remain valid in pretopological space, whereas some become false. The P-pre-derived set (resp. P-pre-closure, P-pre-interior,P-pre-boundary) of a subset A of a space X will be denoted by pDer(A)(resp. pCl(A),pInt(A),pBd(A)).

In the following theorem, we give the main properties of such operations which give the deviations between these operations and that in topological spaces.

1.5 Theorem:

Let (X,τ) be a topological space and let $\tau^p \subseteq P.O(X)$ such that τ^p is a pretopology on X. Let A and B be subsets of X. Then:

(i) $pDer(A) \cup pDer(B) \subseteq pDer(A \cup B)$.

(ii) $pCl(A) \cup pCl(B) \subseteq pCl(A \cup B)$.

(iii) $pInt(A \cap B) \subseteq pInt(A) \cap pInt(B)$.

Proof:

(i) Let $x \in pDer(A) \cup pDer(B)$, if $x \in pDer(A)$, then for each P-pre-open subset U of X such that $x \in U$ and $(U \cap A) / \{x\} \neq \phi$. Since $A \subseteq A \cup B$, thus $(U \cap (A \cup B)) / \{x\} \neq \phi$ for every P-pre-open subset U of X, $x \in U$, therefore $x \in pDer(A \cup B)$, hence, $pDer(A) \cup pDer(B) \subseteq pDer(A \cup B)$. Similarly, if $x \in pDer(B)$.

(ii) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, thus $pCl(A) \subseteq pCl(A \cup B)$ and $pCl(B) \subseteq pCl(A \cup B)$, that implies $pCl(A) \cup pCl(B) \subseteq pCl(A \cup B)$.

(iii) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, thus $pInt(A \cap B) \subseteq pInt(A)$ and $pInt(A \cap B) \subseteq pInt(B)$, hence $pInt(A \cap B) \subseteq pInt(A) \cap pInt(B)$.

The inequalities in (i),(ii) and (iii) in above theorem, cannot be replaced, in general, by equalities as in case of topological spaces, as the following example.

1.6 Example:

Let $X=\{a,b,c\}$ and $\tau=\{\phi,X,\{a,b\}\}$. Thus
$P.O(X)=\{\phi,X,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\}\}$. Now, let $\tau^p=\{\phi,X,\{a,c\},\{b,c\}\}$.

(i) If $A=\{a\}, B=\{b\}$, then $pDer(A)=\phi, pDer(B)=\phi$ and $pDer(A \cup B)=\{c\}$, hence $pDer(A \cup B) \not\subseteq pDer(A) \cup pDer(B)$.

(ii) If $A=\{a\}, B=\{b\}$, then $pCl(A)=\{a\}$, $pCl(B)=\{b\}$ and $pCl(A \cup B)=X$, hence $pCl(A \cup B) \not\subseteq pCl(A) \cup pCl(B)$.

(iii) If $A=\{a,c\}, B=\{b,c\}$, then $pInt(A)=A$, $pInt(B)=B$ and $pInt(A \cap B)=\phi$, hence $pInt(A) \cap pInt(B) \not\subseteq pInt(A \cap B)$.

1.7 Definition:

Let (X,τ) be a topological space and $\tau^p \subseteq P.O(X)$ such that τ^p is a pretopology on X. We call τ^p a pretopology associated with τ if $\tau \subseteq \tau^p$.

1.8 Example:

Let (X,τ) be any topological space. Then $\tau^p=P.O(X)$ is a pretopology associated with τ.

1.9 Lemma [1]:

Let (X,τ) be a topological space. Let A and B be subsets of X such that $A \in \tau$, then $A \cap Cl(B) \subseteq Cl(A \cap B)$.

1.10 Proposition:

Let (X,τ) be a topological space and $\tau^p \subseteq P.O(X)$ such that τ^p is a pretopology associated with τ. If $U \in \tau$ and $V \in \tau^p$ then $U \cap V \in \tau^p$.

Proof:

Since $U\in\tau$ and $V\in\tau^p$ then $U\cap V\subseteq U\cap Int(Cl(V))=Int(U\cap Cl(V))\subseteq Int(Cl(U\cap V))$ [By Lemma (1.9)]. Hence $U\cap V$ pre-open set in X.
Therefore $U\cap V$=pre-Int$(U\cap V)$ ....(1)
Now, since τ^p is a pretopology associated with τ, thus $\tau\subseteq\tau^p$, hence $U\subseteq\tau^p$, then $U\cap V\in\tau^p$, but $U\cap^p V$=pre-Int$(U\cap V)$, by (1) we have $U\cap V\in\tau^p$.

The intersection of two P-pre-open sets need not be P-pre-open, as the following example.

1.11 Example:
In Example (1.6), if $A=\{a,c\}$, $B=\{b,c\}$ then $A,B\in\tau^p$, but $A\cap B=\{c\}\notin\tau^p$.

2. P-pre-continuous functions

In this section, we recall the definition of P-pre-continuous function and we give some properties and corollaries about this subject.

2.1 Definition:
Let (X,τ_1),(Y,τ_2) be topological spaces and τ_1^p be an associated pretopology with τ_1. We say that a function $f:X\to Y$ is an P-pre-continuous function if the inverse image of each open set in Y is P-pre-open in X.

2.2 Theorem:
Let $f:(X,\tau_1)\to(Y,\tau_2)$ be a function. Then the following statements are equivalent:
(i) f is P-pre-continuous.
(ii) The inverse image of each closed set in Y is P-pre-closed in X.
(iii) $pCl(f^{-1}(V))\subseteq f^{-1}(Cl(V))$, for every $V\subseteq Y$.
(iv) $f^{-1}(Int(U))\subseteq pInt(f^{-1}(U))$, for every $U\subseteq Y$.

Proof:
(i)→(ii):
Let $F\subseteq Y$ be closed, since f is P-pre-continuous then $f^{-1}(F^c)=(f^{-1}(F))^c$ is P-pre-open. Therefore $f^{-1}(F)$ is P-pre-closed in X.
(ii)→(iii):
Since $Cl(V)$ is closed for every $V\subseteq X$, then $f^{-1}(Cl(V))$ is P-pre-closed in X. Therefore $f^{-1}(Cl(V))=pCl(f^{-1}(Cl(V)))\supseteq pCl(f^{-1}(V))$.
(iii)→(iv):
Let $U\subseteq Y$, since $int(U)=(CL(U^c))^c$ then:
$f^{-1}(int(U))=f^{-1}((CL(U^c))^c)$
$=(f^{-1}((CL(U^c))))^c\subseteq(pCL((f^{-1}(U^c)))^c=(pCL((f^{-1}(U))^c))^c=pint(f^{-1}(U))$
Thus $f^{-1}(int(U))\subseteq pint(f^{-1}(U))$.
(iv)→(i):
Let $U\subseteq Y$ be an open set. Then $f^{-1}(U)=f^{-1}(Int(U))\subseteq pInt(f^{-1}(U))$, but $pint(f^{-1}(U))\subseteq f^{-1}(U)$, hence $f^{-1}(U)=pInt(f^{-1}(U))$. Therefore $f^{-1}(U)$ is P-pre-open in X, thus f is P-pre-continuous.

2.3 Remark:
Every continuous function is P-pre-continuous.

The converse of the Remark (2.3) is not true, in general, as the following example.

2.4 Example:

Let $X=\{a,b,c\}$, $\tau_1=\{\phi,X,\{a\},\{b,c\}\}$, thus $\tau_1^P=P.O(X)$, thus τ_1^P is associated topology with τ_1. Now, let $Y=\{1,2,3\}$, $\tau_2=\{\phi,Y,\{1\},\{1,2\}\}$ and $f:X\to Y$ be a function such that $f(a)=f(b)=1$, $f(c)=2$. We note that, f is P-pre-continuous, but it not continuous function, since $\{1\}\in\tau_2$, but $f^{-1}(\{1,2\})=\{a,b\}\notin\tau_1$.

2.5 Remarks:

(i) If a function $f:(X,\tau_1)\to(Y,\tau_2)$ is P-pre-continuous and a funcion $g:(Y,\tau_2)\to(Z,\tau_3)$ is P-pre-continuous, then $gof:(X,\tau_1)\to(Z,\tau_3)$ may not be P-pre-continuous.
(ii) It can be easily shown, if $f:(X,\tau_1)\to(Y,\tau_2)$ is P-pre-continuous and a function $g:(Y,\tau_2)\to(Z,\tau_3)$ is continuous, then $gof:(X,\tau_1)\to(Z,\tau_3)$ is P-pre-coninuous.

Recall that a function $f:X\to Y$ is continuous if and only if $f(\overline{A})\subseteq\overline{f(A)}$, for every $A\subseteq X$. [4]

2.6 Proposition:

Let (X,τ_1) and (Y,τ_2) be topological spaces, τ_1^P and τ_2^P be two associated pretopologies with τ_1 and τ_2, respectively. If $f:X\to Y$ is a function and one of the following holds:
(i) $f^{-1}(pInt(B))\subseteq Int(f^{-1}(B))$, for each $B\subseteq Y$.
(ii) $Cl(f^{-1}(B))\subseteq f^{-1}(pCl(B))$, for each $B\subseteq Y$.
(iii) $f(Cl(A))\subseteq pCl(f(A))$, for each $A\subseteq X$.
Then f is continuous.

Proof:

(1) If (i) holds. Let U be an open subset of Y, i.e. $U\in\tau_2\subseteq\tau_2^P$, then $pInt(U)=U$, thus $f^{-1}(U)=f^{-1}(pInt(U))\subseteq Int(f^{-1}(U))$, but $Int(f^{-1}(U))\subseteq f^{-1}(U)$, therefore $f^{-1}(U)=Int(f^{-1}(U))$. Hence $f^{-1}(U)$ is open in X. Therefore f is continuous.
(2) If (ii) holds. Let F be a closed subset of Y, i.e. $F^c\in\tau_2\subseteq\tau_2^P$, $pCl(F)=F$, thus $Cl(f^{-1}(F))\subseteq f^{-1}(pCl(F))=f^{-1}(F)$, but $f^{-1}(F)\subseteq Cl(f^{-1}(F))$. Therefore $f^{-1}(F)=Cl(f^{-1}(F))$. Hence $f^{-1}(F)$ is closed in X. Therefore f is a continuous function.
(3) If (iii) holds. Let $A\subseteq X$, $f(Cl(A))\subseteq pCl(f(A))\subseteq Cl(f(A))$. Hence f is continuous.

2.7 Remark:

Many properties of pre-continuous [4] can be easily deduced from the previous results of P-pre-continuous by setting $\tau^P=P.O(X,\tau)$.

3. P-irresolute functions

In this section, we introduce the definition of P-irresolute functions and we give the relationship between the concept "P-irresolute functions" and the concept "P-continuous functions".

3.1 Definition:

Let (X,τ_1) and (Y,τ_2) be topological spaces, τ_1^P and τ_2^P be two associated ...pologies with τ_1 and τ_2, respectively. If $f:X\to Y$ is a function. Then we say that ... P-irresolute if the inverse image of each P-pre-open set is P-pre-open.

...2 Remark:

every P-irresolute function is P-pre-continuous.

In the following two examples we show the independency of the concepts ...uity and P-irresolute.

...3 Example:

Let $X=\{1,2,3\}$, $\tau_1=\{\phi,X,\{1\},\{2,3\}\}$, $Y=\{a,b,c\}$, $\tau_2=\{\phi,Y,\{a\},\{b,c\}\}$. Thus ...(X)=$\{\phi,X,\{1\},\{2,3\},\{2\},\{3\},\{1,2\},\{1,3\}\}$, ...(Y)=$\{\phi,Y,\{a\},\{b,c\},\{b\},\{c\},\{a,b\},\{a,c\}\}$. ...let $\tau_1^P=P.O(X)$, $\tau_2^P=P.O(Y)$. Thus τ_1^P and τ_2^P are associated pretopologies with ... and τ_2 respectively. Let $f:X\to Y$ be a function such that: $f(1)=b, f(3)=c, f(2)=a$. We ... that f is P-irresolute but it is not continuous, since $\{b,c\}\in\tau_2$, but f^{-1}...$(\{b,c\})=\{1,3\}\notin\tau_1$.

...4 Example:

Let $X=\{1,2,3\}$, $\tau_1=\{\phi,X,\{1\},\{1,2\}\}$, $Y=\{a,b,c\}$, $\tau_2=\{\phi,Y,\{a\},\{b,c\}\}$. Thus ...(X)=$\{\phi,X,\{1\},\{1,2\},\{1,3\}\}$, $P.O(Y)=\{\phi,Y,\{a\},\{b,c\},\{b\},\{c\},\{a,b\},\{a,c\}\}$. ...let $\tau_1^P=P.O(X)$, $\tau_2^P=P.O(Y)$. Thus τ_1^P and τ_2^P are associated pretopologies with ... and τ_2 respectively. Let $f:X\to Y$ be a function such that: $f(1)=f(2)=b, f(3)=c$. We ... that f is continuous, but it is not P-irresolute, since $\{c\}\in\tau_2^P$, but f^{-1}...$(\{c\})=\{3\}\notin\tau_1^P$.

Now, we give some properties of the concepts P-irresolute function.

...5 Proposition:

Let $f:(X,\tau_1)\to(Y,\tau_2)$ be an injective P-irresolute function. Then ...$(f(A))\subseteq pInt(f(A))\subseteq f(pInt(A))$.

Proof:

First: Let $y\in Int(f(A))$, then there exists an open neighborhood U of y in Y such that $y\in U\subseteq f(A)$, since $U\in\tau_1\subseteq\tau_1^P$, thus $y\in pInt(f(A))$, hence $Int(f(A))\subseteq pInt(f(A))$.

Second: Let $y\in pInt(f(A))$, then there exists an P-pre-open U in Y such that $y\in U\subseteq f(A)$, thus $y\in f(A)$, hence there exists $x\in A$ such that $y=f(x)$, then $x\in f^{-1}(U)$... $f^{-1}(f(A))=A$ (since f is injective function). Thus:

$$x\in f^{-1}(U)\subseteq f^{-1}(f(A))=A$$

Since f is P-irresolute function, then $f^{-1}(U)$ is P-pre-open in X. Therefore $x\in pInt(A)$, then $y=f(x)\in f(pInt(A))$, thus $pInt(f(A))\subseteq f(pInt(A))$.

By First and Second we have $In(f(A))\subseteq pInt(f(A))\subseteq f(pInt(A))$.

3.6 Proposition:

Let $f:X\to Y$ and $g:Y\to Z$ be two functions. If f is P-irresolute and g is P-pre-continuous, then $gof:X\to Z$ is P-pre-continuous.

Proof:

Let B be an open set in Z, since g is P-pre-continuous, then $g^{-1}(B)$ is P-pre-open in Y, since f is P-irresolute then $f^{-1}(g^{-1}(B))=(gof)^{-1}(B)$ is P-pre-open in X. Thus $gof:X\to Z$ is P-pre-continuous.

3.7 Proposition:

Let $f:X\to Y$ be a function. Then the following statements are equivalent:

(i) f is P-irresolute.

(ii) The inverse image of each P-pre-closed set is P-pre-closed.

(iii) For each $B\subseteq Y$, $pCl(f^{-1}(B))\subseteq f^{-1}(pCl(B))\subseteq f^{-1}(Cl(B))$.

(iv) For each $B\subseteq Y$, $f^{-1}(Int(B))\subseteq f^{-1}(pInt(B))\subseteq pInt(f^{-1}(B))$.

Proof:

(i)→(ii): Let F be an P-pre-cosed set in Y, then F^c be an P-pre-open set in Y. Since f is P-irresolute, thus $f^{-1}(F^c)$ is P-pre-open in X, hence $(f^{-1}(F^c))^c=f^{-1}(F)$ is P-pre-closed in X.

(ii)→(iii): Let $B\subseteq Y$, since $B\subseteq pCl(B)$ then $f^{-1}(B)\subseteq f^{-1}(pCl(B))$ ....(*)
Since pCl(B) is P-pre-closed in Y then $f^{-1}(pCl(B))$ is P-pre-closed in X, hence $pCl(f^{-1}(pCl(B)))=f^{-1}(pCl(B))$, thus by (*), $pCl(f^{-1}(B))\subseteq f^{-1}(pCl(B))$.

(iii)→(i): Let V be an P-pre-open set in Y, then V^c be P-pre-closed in Y. Thus $pCl(f^{-1}(V^c))\subseteq f^{-1}(pCl(V^c))\subseteq f^{-1}(V^c)$. Hence $f^{-1}(V^c)=(f^{-1}(V))^c$ is P-pre-closed in X, thus $f^{-1}(V)$ is P-pre-open in X. Therefore f is P-irresolute.

(iv)→(i): Let U be an P-pre-open set in Y, then $f^{-1}(pInt(U))\subseteq pInt(f^{-1}(U))$, thus $f^{-1}(U)\subseteq pInt(f^{-1}(U))$, Hence $f^{-1}(U)$ is P-pre-open in X. Therefore f is P-irresolute.

(i)→(iv): Let $B\subseteq Y$, clearly $Int(B)\subseteq pInt(B)$.
Hence $f^{-1}(Int(B))\subseteq f^{-1}(pInt(B))$ ....(*).
Now, since $pInt(B)\subseteq B$, then $f^{-1}(pInt(B))\subseteq f^{-1}(B)$ ......(**)
But pInt(B) is P-pre-open in Y and f is P-irresolute thus $f^{-1}(pInt(B))$ is P-pre-open in X, hence $pInt(f^{-1}(pInt(B)))=f^{-1}(pInt(B))$. Therefore by (**), we have $pInt(f^{-1}(pInt(B)))=f^{-1}(pInt(B))\subseteq pInt(f^{-1}(B))$ ....(***).
Thus by (*) and (***) we have $f^{-1}(Int(B))\subseteq f^{-1}(pInt(B))\subseteq pInt(f^{-1}(B))$, for each $B\subseteq Y$.

4. Separation axioms in pretopological spaces.

In this section, we study the separation axioms in pretoplogical spaces and we give some results related with this subject.

4.1 Definition:

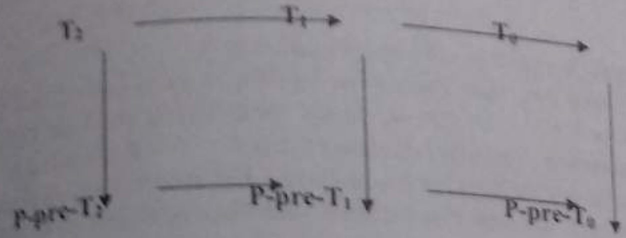
Let (X,τ) be a topological space and τ^P be a pretopology associated with τ. Then we say that (X,τ) is:

(i) P-pre-T_0, if for every two disjoint points of X, there exists an P-pre-open neighborhood of one of them to which the other does not belong.

(ii) P-pre-T_1, if for every two disjoint points x and y in X, there exist two P-pre-open sets U and V such that $x\in U, y\notin U$ and $y\in V, x\notin V$.

(iii) P-pre-T_2, if for every two disjoint points x and y in X, there exist two disjoint P-pre-open sets U and V such that $x\in U$ and $y\in V$.

The following diagram shows the relationships between the separation axioms in associated pretopological spaces and the separation axioms in a topological spaces.



The inverse directions in above diagram are not true, in general, as the following examples.

4.2 Examples:

(i) Let $X=\{a,b\}$, $\tau=\{\phi,X,\{a\}\}$, we note that the topological space (X,τ) is T_0, but it is not T_1, thus it is not T_2.

(ii) Let $X=\{a,b\}$, $\tau=\{\phi,X\}$, thus $P.O(X)=\{\phi,X,\{a\},\{b\}\}$. Let $\tau^P=\{\phi,X,\{a\}\}$, we note that (X,τ) is P-pre-T_0 space, but it is not P-pre-T_1 space and thus it is not P-pre-T_2 space.

(iii) Let $X=\{a,b,c\}$, $\tau=\{\phi,X\}$, thus $P.O(X)=\{\phi,X,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$. Let $\tau^P=P.O(X)$, we note that (X,τ) is P-pre-T_0 but it is not T_0 and thus it is not T_1 nor T_2, also this topological space is P-pre-T_1 and P-pre-T_2.

4.3 Proposition:

Let (X,τ) be a topological space and τ^P a pretopology on X associated with τ.

Then:

(i) (X,τ) is P-pre-T_0 space if and only if $pCl(\{x\})\neq pCl(\{y\})$, for each $x,y\in X$ such that $x\neq y$.

(ii) (X,τ) is P-pre-T_1 space if and only if for every $x\in X$, $\{x\}$ is P-pre-closed.

(iii) (X,τ) is P-pre-T_2 space if and only if for every disjoint points $x,y\in X$, there exists an P-pre-open U such that $x\in U$ and $y\notin pCl(U)$.

Proof:

(i)

(⇒) Suppose (X,τ) is P-pre-T_0 and let $x,y\in X$ such that $x\neq y$, thus there exists an P-pre-neighborhood U of one of them which the other does not belong, like x and $y\notin U$. Hence $\{x\}\cap U\neq\phi$ and $\{y\}\cap U=\phi$. Thus $x\notin pCl(\{y\})$, but $x\in pCl(\{x\})$, hence $pCl(\{x\})\neq pCl(\{y\})$.

(⇐) Suppose that for every two point $x,y\in X$ such that $x\neq y$, we have $pCl(\{x\})\neq pCl(\{y\})$. Since $pCl(\{x\})$ is P-pre-closed in X, then $(pCl(\{x\}))^c$ is P-pre-open in X. Now, let $U=(pCl(\{x\}))^c$, we note that $x\notin U$, we claim that $y\in U$, since

$y \in pCl(\{y\}) \neq pCl(\{x\})$, if $y \in pCl(\{x\})$ then for every P-pre-neighborhood V of y, we have $V \cap \{x\} \neq \phi$, hence $x \in V$, therefore every P-pre-neighborhood of y must contains x.

Now, let $z \in pCl(\{x\})$, then for every P-pre-neighborhood W of z, we have $W \cap \{x\} \neq \phi$ thus $x \in W$ and then $y \in W$, thus $\{y\} \cap W \neq \phi$, hence $z \in pCl(\{y\})$. Therefore $pCl(\{x\}) \subseteq pCl(\{y\})$. Similarly, we can prove $pCl(\{y\}) \subseteq pCl(\{x\})$ and this contradiction. Thus $y \notin pCl(\{x\})$, hence $y \in (pCl(\{x\}))^c = U$. Therefore (X,τ) is P-pre-T_0.

(ii)

($\rightarrow$): Suppose that (X,τ) is P-pre-T_1 space and let $x \in X$.

Now, let $y \notin \{x\}$, then $x \neq y$, thus there exist two P-pre-open sets U and V such that $x \in U, y \notin U$, and $x \notin V, y \in V$, $U \cap \{y\} = \phi$, hence $y \notin pCl(\{x\})$, thus $pCl(\{x\}) \subseteq \{x\}$, but $\{x\} \subseteq pCl(\{x\})$, therefore $pCl(\{x\}) = \{x\}$, hence $\{x\}$ is P-pre-closed.

($\leftarrow$): Suppose for every $x \in X$, $\{x\}$ is P-pre-closed. Let $x, y \in X$, such that $x \neq y$, we note that $U = (\{x\})^c, V = (\{y\})^c$ are two P-pre-closed such that $x \in V$ and $y \in U$. Thus (X,τ) is P-pre-T_1.

(iii)

($\rightarrow$): Suppose that (X,τ) is P-pre-T_2 space. Let $x, y \in X$ such that $x \neq y$, then there exist two disjoint P-pre-open sets U,V such that $x \in U$ and $y \in V$. Hence $U \cap V = \phi$, since $y \in V$, V is P-pre-open and $V \cap U = \phi$, then $y \notin pCl(U)$ and $x \in U$.

($\leftarrow$): Suppose for every disjoint two points $x, y \in X$, there exists an P-pre-open set U in x such that $x \in U$ and $y \notin pCl(U)$.

Now, let $x, y \in X$ such that $x \neq y$, then there exists an P-pre-open set U in X such that $x \in U$ and $y \notin pCl(U)$. We have $pCl(U)$ is P-pre-closed in X, thus $(pCl(U))^c$ is P-pre-open in X and we note that $y \in (pCl(U))^c$. But $U \cap (pCl(U))^c = \phi$, therefore (X,τ) is P-pre-T_2.

4.4 Remarks:

(1) The property of being an P-pre-T_i space is P-pre-topological property, i=0,1,2.

(2) Since the intersection of an open set and an P-pre-open set is always P-pre-open. Hence every open subspace of P-pre-T_i space is P-pre-T_i space, i=0,1,2.

5. P-pre-closed graph.

In this section, we introduce a new concept, namely P-pre-closed graph as generalized to the concept "closed graph" and we give the relationship between this subject and P-irresolute functions.

5.1 Definition:

A subset B of the product space $X \times Y$ is P-pre-closed in $X \times Y$ if for each $(x,y) \in X \times Y / B$, there exist two P-pre-open neighborhoods U and V of x and y, respectively, such that $(U \times V) \cap B = \phi$. A function $f: X \rightarrow Y$ has an P-pre-closed graph, if the graph $G(f) = \{(x, f(x)) : x \in X\}$ is P-pre-closed in $X \times Y$.

5.2 Proposition:

A function $f:X\to Y$ has an P-pre-closed graph if and only if for each $x\in X$, $y\in Y$ such that $y\neq f(x)$, there exist two P-pre-open sets U and V containing x and y, respectively, such that $f(U)\cap V=\phi$.

Proof:

(⇒) Suppose that f has an P-pre-closed graph, let $x\in X$ and $y\in Y$ such that $y\neq f(x)$, then there exist two P-pre-open sets U and V containing x and y, respectively, such that $(U\times V)\cap G(f)=\phi$. Suppose $f(U)\cap V\neq\phi$, then there exists $z\in f(U)\cap V$, thus there exists $x_1\in U$ such that $z\in V$ and $z=f(x_1)$, we note that $(x_1,z)\in U\times V$ and $(x_1,z)=(x_1,f(x_1))\in G(f)$, hence $(U\times V)\cap G(f)\neq\phi$ and this contradiction. This implies that for every $x\in U$ and $y\in V$, $f(x)\neq y$. So $f(U)\cap V=\phi$.

(⇐) Let $(x,y)\in X\times Y/G(f)$, then $(x,y)\notin G(f)$. Thus there exist two P-pre-open sets U and V containing x and y, respectively, such that $f(U)\cap V=\phi$. This implies that for each $x\in U$ and $y\in V$, we have $f(x)\neq y$. So $(U\times V)\cap G(f)=\phi$, thus G(f) is P-pre-closed. Hence f has an P-pre-closed graph

5.3 Proposition:

If $f:X\to Y$ is P-irresolute and Y is an P-pre-T_2 space, then f has an P-pre-closed graph.

Proof:

Let $(x,y)\in X\times Y/G(f)$. Then $y\neq f(x)$ and since Y is P-pre-T_2, then there exist two P-pre-open sets U and V in Y such that $f(x)\in U$ and $y\in V$, $U\cap V=\phi$. Since f is P-irresolute the there exists an P-pre-open neighborhood W of x such that $f(W)\subseteq U$, hence $f(W)\cap V=\phi$. This implies that $(W\times V)\cap G(f)=\phi$. Hence f has an P-pre-closed graph.

5.4 Proposition:

If $f:X\to Y$ is P-irresolute injective function with an P-pre-closed graph, then X is P-pre-T_2.

Proof:

Let $x,y\in X$ such that $x\neq y$. Then $f(x)\neq f(y)$, this implies that $(x,f(y))\in (X\times Y)/G(f)$. Since f has an P-pre-closed graph then there exist two P-pre-open neighborhoods U and V of x and f(y), respectively, such that $(U\times V)\cap G(f)=\phi$. This gives $f(U)\cap V=\phi$, since f is P-irresolute then there exists an P-pre-open set W containing y such that $f(W)\subseteq V$. Hence $f(W)\cap f(U)=\phi$, therefore $W\cap U=\phi$ and X is an P-pre-T_2 space.

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