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# Neighborhoods and Partial Sums of Subclass of *k*-Unifor -mly Convex Functions and Related Class of *k*-Starlike Functions with Negative Coefficients Based on Integral Operator

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**Abstract.** In this paper, we introduce a class $k-UCV(\lambda,\gamma,\beta,\delta)$ of analytic univalent functions with negative coefficients. This class is based on an integral operator $Q_\delta^\beta$ of the function $f(z)$. We obtain some neighbourhoods results. Partial sums $h_m(z)$ of the functions $f(z)$ in this class are considered. Also we study coefficient bounds, extreme points and radius of close-to-convexity, starlikeness and convexity of this class.

**Keywords:** Uniformly starlike; Uniformly convex; Integral operator; Neighbourhood; Partial sum; Extreme points; Radii of starlikeness; Coefficient bounds.

## 1. Introduction

Let $S$ be the class of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0, n \in I\!N) \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$. A function $f \in S$ is said to be in $k - ST(\gamma)$, the class of $k$-starlike functions of order $\gamma, 0 \leq \gamma < 1$, if $f$ satisfies the condition

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma, \quad k \geq 0. \tag{1.2}$$

Replacing $f$ in (1.2) by $zf'$, we obtain the condition

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right| + \gamma, \quad k \geq 0 \tag{1.3}$$

required for the function $f$ to be in the subclass $k - UCV(\gamma)$ of $k$-uniformly convex functions of order $\gamma$.

Uniformly starlike and convex functions were first introduced by Goodman [5] and then studied by various authors. Recently, the starlikeness and convexity of a certain integral operators have been studied by Raina and Bapna in [11]. Vector-valued analytic funtions of multiplication operators were studied in [10]. For the references of this topic, the reader is referred to [12].

**Lemma 1.1.** [2] *Let $w = u + iv$. Then*

$$Re\ w \geq \alpha \ \text{ if and only if } \ |w - (1 + \alpha)| \leq |w + (1 - \alpha)|.$$

**Lemma 1.2.** *Let $w = u + iv$ and $\alpha, \gamma$ are real numbers. Then*

$$Re\ w > \alpha|w - 1| + \gamma \ \text{ if and only if } \ Re\{w(1 + \alpha e^{i\theta}) - \alpha e^{i\theta}\} > \gamma.$$

**Definition 1.3.** *Suppose that $f \in S$ with the form (1.1) and $g(z) \in S$ is given by $g(z) = z - \sum_{n=2}^{\infty} b_n z^n, b_n \geq 0$. Then, we define the Hadamard product (or convolution) $f * g$ of $f$ and $g$ by*

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.4}$$

**Definition 1.4.** *For $0 \leq \lambda \leq 1, 0 \leq \gamma < 1, k \geq 0, \beta \geq 0$ and $\delta > -1$, we let $k - UCV(\lambda, \gamma, \beta, \delta)$ be the subclass of $S$ consisting of functions of the form (1.1)* *and satisfying the analytic criterion*

$$Re\left\{\frac{z(Q_\delta^\beta f(z))' + \lambda z^2 (Q_\delta^\beta f(z))''}{(1-\lambda)Q_\delta^\beta f(z) + \lambda z(Q_\delta^\beta f(z))'}\right\}$$

$$\geq k\left|\frac{z(Q_\delta^\beta f(z))' + \lambda z^2 (Q_\delta^\beta f(z))''}{(1-\lambda)Q_\delta^\beta f(z) + \lambda z(Q_\delta^\beta f(z))'} - 1\right| + \gamma, \tag{1.5}$$

*where $Q_\delta^\beta$ is the generalized Jung-Kim-Srivastava integral operator [6] defined by*

$$\begin{aligned} Q_\delta^\beta f(z) &= \frac{\Gamma(\beta+\delta+1)}{z\Gamma(\beta)\Gamma(\delta+1)}\int_0^z t^{\delta-1}(1-\frac{t}{z})^{\beta-1} f(t)dt, \ \beta \geq 0, \delta > -1 \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)} a_n z^n \end{aligned} \tag{1.6}$$

*and for $\beta = 0$, we have $Q_\delta^0 f(z) = f(z)$.*

We note that by specializing the parameters $\lambda, k, \gamma, \beta, \delta$, we obtain the following different subclasses as studied by various authors:

(i) If $k = 0, \beta = 0$, then the class $k - UCV(\lambda, \gamma, \beta, \delta)$ reduces to the class $0 - UCV(\lambda, \gamma, 0, \delta) \equiv P_1(1, \lambda, \gamma)$ was studied by Altintas and Owa [1] also by Kwon and Cho [8], and when $\beta = 0$ our class reduces to a class was studied by Aqlan [2].

(ii) If $\lambda = 0, k = 0$ and $\beta = 0$, then the class $k - UCV(\lambda, \gamma, \beta, \delta)$ reduces to the class $0 - UCV(0, \gamma, 0, \delta) \equiv S^*(\gamma)$ was studied by Silverman [14].

(iii) If $\lambda = 1, k = 0$ and $\beta = 0$, then the class $k - UCV(\lambda, \gamma, \beta, \delta)$ reduces to the class $0 - UCV(1, \gamma, 0, \delta) \equiv C(\gamma)$ was studied by Silverman [14].

(iv) If $\lambda = 0, k = 1$ and $\beta = 0$, then the class $1 - UCV(0, \gamma, 0, \delta) \equiv UST(\gamma)$ was studied by Bharati and et. al. [3].

(v) If $\lambda = 0, \beta = 0$ and $\gamma = 0$, that is, $k - ST$ introduced by Kanas and Wiśniowska [7].

(vi) If $\lambda = 1, \gamma = 0$ and $\beta = 0$, then this is $k - UCV$ which is introduced and studied by Kanas and Wiśniowska [7].

*Remark 1.5.* $k - ST \subset k - UCV(\lambda, \gamma, \beta, \delta)$ when $\lambda = 0, \gamma = 0$ and $\beta = 0$
$k - UCV \subset k - UCV(\lambda, \gamma, \beta, \delta)$ when $\lambda = 1, \gamma = 0$ and $\beta = 0$.

The aim of this paper is to study the coefficient bounds and extreme points of the class $k - UCV(\lambda, \gamma, \beta, \delta)$. Furthermore, we obtain certain neighbourhood results for functions in $k - UCV(\lambda, \gamma, \beta, \delta)$. Partial sums $h_m(z)$ of functions $f(z)$ in the class $k - UCV(\lambda, \gamma, \beta, \delta)$ are considered. Also we obtain region of univalence of close-to-convexity, starlikeness and convexity.

## 2. Coefficient Bounds and Extreme Points

We give here a necessary and sufficient condition and extreme points for the functions $f(z)$ in the class $k-UCV(\lambda,\gamma,\beta,\delta)$.

**Theorem 2.1.** *The function $f(z)$ defined by (1.1) is in the class $k-UCV(\lambda,\gamma,\beta,\delta)$ if and only if*

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+k)-(k+\gamma)]\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n \leq 1-\gamma \tag{2.1}$$

*where $0\leq\gamma<1, k\geq 0, 0\leq\lambda\leq 1, \beta\geq 0$ and $\delta>-1$.*

*Proof.* By Definition 1.4, we get

$$Re\left\{\frac{z(Q_\delta^\beta f(z))'+\lambda z^2(Q_\delta^\beta f(z))''}{(1-\lambda)Q_\delta^\beta f(z)+\lambda z(Q_\delta^\beta f(z))'}\right\} \geq k\left|\frac{z(Q_\delta^\beta f(z))'+\lambda z^2(Q_\delta^\beta f(z))''}{(1-\lambda)Q_\delta^\beta f(z)+\lambda z(Q_\delta^\beta f(z))'}-1\right|+\gamma.$$

Then by Lemma 1.2, we have

$$Re\left\{\frac{z(Q_\delta^\beta f(z))'+\lambda z^2(Q_\delta^\beta f(z))''}{(1-\lambda)Q_\delta^\beta f(z)+\lambda z(Q_\delta^\beta f(z))'}(1+ke^{i\theta})-ke^{i\theta}\right\}\geq\gamma, \quad -\pi<\theta\leq\pi$$

or equivalently

$$Re\left\{\frac{(z(Q_\delta^\beta f(z))'+\lambda z^2(Q_\delta^\beta f(z))'')(1+ke^{i\theta})}{(1-\lambda)Q_\delta^\beta f(z)+\lambda z(Q_\delta^\beta f(z))'} - \frac{ke^{i\theta}((1-\lambda)Q_\delta^\beta f(z)+\lambda z(Q_\delta^\beta f(z))')}{(1-\lambda)Q_\delta^\beta f(z)+\lambda z(Q_\delta^\beta f(z))'}\right\}\geq\gamma. \tag{2.2}$$

Let

$$F(z)=[z(Q_\delta^\beta f(z))'+\lambda z^2(Q_\delta^\beta f(z))''](1+ke^{i\theta})-ke^{i\theta}[(1-\lambda)Q_\delta^\beta f(z)+\lambda z(Q_\delta^\beta f(z))'],$$

and

$$E(z)=(1-\lambda)Q_\delta^\beta f(z)+\lambda z(Q_\delta^\beta f(z))'.$$

By Lemma 1.1, (2.2) is equivalent to

$$|F(z)+(1-\gamma)E(z)|\geq|F(z)-(1+\gamma)E(z)| \quad \text{for} \quad 0\leq\gamma<1.$$

But

$$\begin{aligned}
&|F(z)+(1-\gamma)E(z)| \\
=&\left|\left[z-\sum_{n=2}^{\infty} n\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n z^n\right.\right. \\
&\left.-\lambda\sum_{n=2}^{\infty} n(n-1)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n z^n\right](1+ke^{i\theta}) \\
&-ke^{i\theta}\left[(1-\lambda)\left(z-\sum_{n=2}^{\infty}\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n z^n\right)\right. \\
&\left.+\lambda z-\lambda\sum_{n=2}^{\infty} n\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n z^n\right] \\
&\left.+(1-\gamma)\left[z-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n z^n\right]\right| \\
=&\left|(2-\gamma)z-\sum_{n=2}^{\infty}[(n+\lambda n(n-1))+(1-\gamma)(1-\lambda+n\lambda)]\right. \\
&\times\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n z^n \\
&\left.-ke^{i\theta}\sum_{n=2}^{\infty}[n+\lambda n(n-1)-(1-\lambda+\lambda n)]\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n z^n\right| \\
&\geq(2-\gamma)|z|-\sum_{n=2}^{\infty}[(n+\lambda n(n-1))+(1-\gamma)(1-\lambda+\lambda n)] \\
&\quad\times\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n|z|^n \\
&\quad-k\sum_{n=2}^{\infty}[n+\lambda n(n-2)-1+\lambda]\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n|z|^n.
\end{aligned}$$

Also

$$\begin{aligned}
|F(z)-(1+\gamma)E(z)|=&\left|\left[z-\sum_{n=2}^{\infty} n\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n z^n\right.\right. \\
&\left.-\lambda\sum_{n=2}^{\infty} n(n-1)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n z^n\right](1+ke^{i\theta}) \\
&-ke^{i\theta}\left[z-(1-\lambda)\sum_{n=2}^{\infty}\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n z^n\right. \\
&\left.-\lambda\sum_{n=2}^{\infty} n\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n z^n\right]
\end{aligned}$$

$$-(1+\gamma)\left[z-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_nz^n\right]\Bigg|$$

$$=\Bigg|-\gamma z-\sum_{n=2}^{\infty}[(n+\lambda n(n-1))-(1+\gamma)(1-\lambda+n\lambda)]\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_nz^n$$

$$-ke^{i\theta}\sum_{n=2}^{\infty}[n+n\lambda(n-1)-(1-\lambda+n\lambda)]\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_nz^n\Bigg|$$

$$\leq\gamma|z|+\sum_{n=2}^{\infty}[(n+n\lambda(n-1))-(1+\gamma)(1-\lambda+n\lambda)]\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n|z|^n$$

$$+k\sum_{n=2}^{\infty}[n+n\lambda(n-1)-(1-\lambda+n\lambda)]\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n|z|^n$$

and so

$$|F(z)+(1-\gamma)E(z)|-|F(z)-(1+\gamma)E(z)|\geq 2(1-\gamma)|z|-$$

$$\sum_{n=2}^{\infty}[(2n+2n\lambda(n-1)-2\gamma(1-\lambda+n\lambda))-k(2n+2n\lambda(n-1)-2(1-\lambda+n\lambda))]$$

$$\times\frac{\Gamma(\beta+\delta+1)(\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n|z|^n\geq 0$$

or

$$\sum_{n=2}^{\infty}[n(1+k)+n\lambda(n-1)(1+k)-(1-\lambda+n\lambda)(\gamma+k)]\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n$$

$$\leq(1-\gamma)$$

This is equivalent to

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+k)-(k+\gamma)]\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n$$

$$\leq(1-\gamma).$$

Conversely, suppose that (2.1) holds. Then we must show

$$Re\left\{\frac{[z(Q_\delta^\beta f(z))'+\lambda z^2(Q_\delta^\beta f(z))''](1+ke^{i\theta})-ke^{i\theta}[(1-\lambda)Q_\delta^\beta f(z)+\lambda z(Q_\delta^\beta f(z))']}{(1-\lambda)Q_\delta^\beta f(z)+\lambda z(Q_\delta^\beta f(z))'}\right\}\geq\gamma.$$

Upon choosing the values of $z$ on the positive real axis where $0\leq z=r<1$, the above inequality reduces to

$$Re\left\{\frac{(1-\gamma)-\sum_{n=2}^{\infty}[n(1+ke^{i\theta})(1+n\lambda-\lambda)}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_nr^{n-1}}\right.$$

$$\left.-\frac{(\gamma+ke^{i\theta})(1-\lambda+n\lambda)]\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_nr^{n-1}}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_nr^{n-1}}\right\}\geq 0.$$

Since $Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to

$$Re\left\{\frac{(1-\gamma) - \sum_{n=2}^{\infty}[n(1+k)(1+n\lambda-\lambda)}{1 - \sum_{n=2}^{\infty}(1-\lambda+n\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n r^{n-1}} - \frac{(\gamma+k)(1-\lambda+n\lambda)]\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n r^{n-1}}{1 - \sum_{n=2}^{\infty}(1-\lambda+n\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n r^{n-1}}\right\} \geq 0.$$

Letting $r \to 1^-$, we get desired conclusion. ∎

**Corollary 2.2.** *If $f(z) \in k - UCV(\lambda, \gamma, \beta, \delta)$, then*

$$a_n \leq \frac{(1-\gamma)\Gamma(\beta+\delta+n)\Gamma(\delta+1)}{(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}$$

*where $0 \leq \gamma < 1, k \geq 0, 0 \leq \lambda \leq 1, \beta \geq 0$ and $\delta > -1$.*

Next we obtain the extreme points for $k - UCV(\lambda, \gamma, \beta, \delta)$.

**Theorem 2.3.** *Let $f_1(z) = z$ and*

$$f_n(z) = z - \frac{(1-\gamma)\Gamma(\beta+\delta+n)\Gamma(\delta+1)}{(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}z^n$$

*where $(n \geq 2, n \in \mathbb{N}, 0 \leq \gamma < 1, k \geq 0, 0 \leq \lambda \leq 1, \beta \geq 0, \delta > -1)$. Then $f(z)$ is in the class $k - UCV(\lambda, \gamma, \beta, \delta)$ if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \sigma_n f_n(z)$$

*where $(\sigma_n \geq 0$ and $\sum_{n=1}^{\infty} \sigma_n = 1$ or $1 = \sigma_1 + \sum_{n=2}^{\infty} \sigma_n)$.*

*Proof.* Let $f(z) = \sum_{n=1}^{\infty} \sigma_n f_n(z)$ where $(\sigma_n \geq 0$ and $\sum_{n=1}^{\infty} \sigma_n = 1)$. Then,

$$f(z) = z - \sum_{n=2}^{\infty} \frac{(1-\gamma)\Gamma(\delta+1)\Gamma(\beta+\delta+n)}{(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}\sigma_n z^n$$

and we get

$$\sum_{n=2}^{\infty}\left(\frac{(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{(1-\gamma)\Gamma(\delta+1)\Gamma(\beta+\delta+n)}\right)\times$$
$$\times\sigma_n\frac{(1-\gamma)\Gamma(\delta+1)\Gamma(\beta+\delta+n)}{(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}$$
$$=\sum_{n=2}^{\infty}\sigma_n=1-\sigma_1\leq 1 \quad \text{(by Theorem 2.1)}$$

By virtue of Theorem 2.1, we can show that $f(z)\in k-UCV(\lambda,\gamma,\beta,\delta)$.

Conversely, assume that $f(z)$ of the form (1.1) belongs to $k-UCV(\lambda,\gamma,\beta,\delta)$. Then

$$a_n\leq\frac{(1-\gamma)\Gamma(\delta+1)\Gamma(\beta+\delta+n)}{(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)},\quad (n\in I\!N, n\geq 2).$$

Setting

$$\sigma_n=\frac{(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{(1-\gamma)\Gamma(\delta+1)\Gamma(\beta+\delta+n)}a_n$$

and

$$\sigma_1=1-\sum_{n=2}^{\infty}\sigma_n.$$

we obtain

$$f(z)=\sum_{n=1}^{\infty}\sigma_n f_n(z)=\sigma_1 f_1(z)+\sum_{n=2}^{\infty}\sigma_n f_n(z).$$

This completes the proof. ■

## 3. Neighbourhoods and Partial Sums

We now extend the familiar concept of neighbourhoods of analytic functions for the family $k-UCV(\lambda,\gamma,\beta,\delta)$. The concept of neighbourhoods analytic function was first introduced by Goodman [4]. Later, Ruscheweyh [13] investigated this concept for the elements of several famous subclasses of analytic functions and Altintas and Owa [1] considered for a certain family of analytic functions with negative coefficients, also Liu and Srivastava [9] extended this concept to a certain subclass of meromorphically multivalent functions.

**Definition 3.1.** *Let $k\geq 0, 0\leq\gamma<1, 0\leq\lambda\leq 1, \beta\geq 0, \delta>-1, \alpha\geq 0$. We define the $\alpha$-neighbourhood of a function $f\in S$ and denote by $N_\alpha(f)$ consisting of all functions $g(z)=z-\sum_{n=2}^{\infty}b_n z^n\in S$ satisfying*

$$\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)(n(1+k)-(k-\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{(1-\gamma)\Gamma(\delta+1)\Gamma(\beta+\delta+n)}|a_n-b_n|\leq\alpha.$$

**Theorem 3.2.** *Let $f \in k-UCV(\lambda,\gamma,\beta,\delta)$ and for all real $\theta$ we have $\gamma(e^{i\theta}-1)-2e^{i\theta}\neq 0$. For any complex number $\mu$ with $|\mu|<\alpha \quad (0\leq\alpha)$, if $f$ satisfies the following condition:*

$$\frac{f(z)+\mu z}{1+\mu}\in k-UCV(\lambda,\gamma,\beta,\delta),$$

*then $N_\alpha(f)\subset k-UCV(\lambda,\gamma,\beta,\delta)$.*

*Proof.* It is obvious that $f\in UCV(\lambda,\gamma,\beta,\delta)$ if and only if

$$\left|\frac{(z(Q_\delta^\beta f(z))'+\lambda z^2(Q_\delta^\beta f(z))'')(1+ke^{i\theta})-(ke^{i\theta}+1+\gamma)((1-\lambda)Q_\delta^\beta f(z)+\lambda z(Q_\delta^\beta f(z))')}{(z(Q_\delta^\beta f(z))'+\lambda z^2(Q_\delta^\beta f(z))'')(1+ke^{i\theta})+(1-ke^{i\theta}-\gamma)((1-\lambda)Q_\delta^\beta f(z)+\lambda z(Q_\delta^\beta f(z))')}\right|$$
$$<1 \qquad -\pi<\theta<\pi$$

for any complex number $\eta$ with $|\eta|=1$, we have

$$\frac{(z(Q_\delta^\beta f(z))'+\lambda z^2(Q_\delta^\beta f(z))'')(1+ke^{i\theta})-(ke^{i\theta}+1+\gamma)((1-\lambda)Q_\delta^\beta f(z)+\lambda z(Q_\delta^\beta f(z))')}{(z(Q_\delta^\beta f(z))'+\lambda z^2(Q_\delta^\beta f(z))'')(1+ke^{i\theta})+(1-ke^{i\theta}-\gamma)((1-\lambda)Q_\delta^\beta f(z)+\lambda z(Q_\delta^\beta f(z))')}$$
$$\neq\eta.$$

In other words, we must have

$$(1-\eta)((z(Q_\delta^\beta f(z))'+\lambda z^2(Q_\delta^\beta f(z))'')(1+ke^{i\theta}))$$
$$-[ke^{i\theta}+1+\gamma+\eta(ke^{i\theta}-1-\gamma)]((1-\lambda)Q_\delta^\beta f(z)+\lambda z(Q_\delta^\beta f(z))')\neq 0$$

which is equivalent to

$$z-\sum_{n=2}^{\infty}\frac{(n-1)(1+ke^{i\theta}-\eta ke^{i\theta})-\eta(n+1)-\gamma(1-\eta)}{\gamma(\eta-1)-2\eta}\left[\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}\right]\times$$
$$\times(1+n\lambda-\lambda)z^n\neq 0$$

However, $f\in k-UCV(\lambda,\gamma,\beta,\delta)$ if and only if $\frac{(f*\phi)(z)}{z}\neq 0, z\in U-\{0\}$ where $\phi(z)=z-\sum\limits_{n=2}^{\infty}e_nz^n$, and

$$e_n=\frac{(n-1)(1+ke^{i\theta}-\eta ke^{i\theta})-\eta(n+1)-\gamma(1-\eta)}{\gamma(\eta-1)-2\eta}\left[\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}\right]\times$$
$$\times(1+n\lambda-\lambda)$$

we note that

$$|e_n|\leq\frac{n(1+k)-(k-\gamma)}{(1-\gamma)}(1+n\lambda-\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)}$$

since $\dfrac{f(z)+\mu z}{1+\mu}\in k-UCV(\lambda,\gamma,\beta,\delta)$, therefore $z^{-1}\left(\dfrac{f(z)+\mu z}{1+\mu}*\phi(z)\right)\neq 0$, which is equivalent to

$$\frac{(f*\phi)(z)}{(1+\mu)z}+\frac{\mu}{1+\mu}\neq 0. \tag{3.1}$$

Now suppose that $\left|\frac{(f*\phi)(z)}{z}\right| < \alpha$. Then by (3.1), we must have

$$\left|\frac{(f*\phi)(z)}{z(1+\mu)} + \frac{\mu}{1+\mu}\right| \geq \frac{|\mu|}{|1+\mu|} - \frac{1}{|1+\mu|}\left|\frac{(f*\phi)(z)}{z}\right| > \frac{|\mu|-\alpha}{|1+\mu|} \geq 0,$$

this is a contradiction by $|\mu| < \alpha$ and however we have $\left|\frac{(f*\phi)(z)}{z}\right| \geq \alpha$. If $g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in N_\alpha(f)$, then

$$\begin{aligned}\alpha - \left|\frac{(g*\phi)(z)}{z}\right| &\leq \left|\frac{((f-g)*\phi)(z)}{z}\right| \leq \sum_{n=2}^{\infty} |a_n - b_n||e_n||z^n| \\ &< \sum_{n=2}^{\infty} \frac{n(1+k)-(k-\gamma)}{(1-\gamma)}|a_n - b_n|(1+n\lambda-\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)} \leq \alpha.\end{aligned}$$

We now conclude that $\frac{(g*\phi)(z)}{z} \neq 0$, which implies that $g \in UCV(\lambda,\gamma,\beta,\delta)$. ■

**Theorem 3.3.** *Let $f \in S$ be given by (1.1). Define $h_1(z) = z, h_m(z) = z - \sum_{n=2}^{m} a_n z^n, m = 2,3,\cdots$, also suppose that $\sum_{n=2}^{\infty} c_n a_n \leq 1$, where*

$$c_n = \frac{(1+n\lambda-\lambda)(n(1+k)-(k-\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{(1-\gamma)\Gamma(\delta+1)\Gamma(\beta+\delta+n)}.$$

*Then*

(i) $f \in k-UCV(\lambda,\gamma,\beta,\delta)$

(ii) $1 - \frac{1}{c_{m+1}} < Re\left\{\frac{f(z)}{h_m(z)}\right\} < 1 + \frac{1}{c_{m+1}}, Re\left\{\frac{h_m(z)}{f(z)}\right\} > \frac{c_{m+1}}{1+c_{m+1}}, z \in U,$ $m = 2,3,\cdots$.

*Proof.* (i) Since $\frac{z+\mu z}{1+\mu} = z \in k-UCV(\lambda,\gamma,\beta,\delta), |\mu| < 1$, by Theorem 3.2, we have $N_1(z) \subset k-UCV(\lambda,\gamma,\beta,\delta)$ ($N_1(z)$ denoting the 1-neighbourhood). Now since $\sum_{n=2}^{\infty} c_n a_n \leq 1, f \in N_1(z)$ and $f \in k-UCV(\lambda,\gamma,\beta,\delta)$.

(ii) We have

$$c_n = \left[\frac{(n-1)(1+k)+2\gamma}{1-\gamma} + 1\right](1+n\lambda-\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\delta+1)\Gamma(\beta+\delta+n)},$$

also $\{c_n\}$ is an increasing sequence. So we obtain

$$\sum_{n=2}^{m} a_n + c_{m+1}\sum_{n=m+1}^{\infty} a_n \leq 1. \tag{3.2}$$

Now by introducing $G_1(z)$ given by $G_1(z) = c_{m+1}\left[\frac{f(z)}{h_m(z)} - \left(1 - \frac{1}{c_{m+1}}\right)\right]$ and making use of (3.2), we get

$$\left|\frac{G_1(z)-1}{G_1(z)+1}\right| = \left|\frac{-c_{m+1}\sum\limits_{n=m+1}^{\infty} a_n z^n}{2 - c_{m+1}\sum\limits_{n=m+1}^{\infty} a_n z^n}\right| < \frac{c_{m+1}\sum\limits_{n=m+1}^{\infty} a_n}{2 - c_{m+1}\sum\limits_{n=m+1}^{\infty} a_n - \sum\limits_{n=1}^{m} a_n} < 1.$$

Therefore $Re\ G_1(z) > 0$ and we obtain $Re\left\{\frac{f(z)}{h_m(z)}\right\} > 1 - \frac{1}{c_{m+1}}$. Now let

$$G_2(z) = c_{m+1}\left[\frac{f(z)}{h_m(z)} - \left(1 + \frac{1}{c_{m+1}}\right)\right],$$

then we have

$$\left|\frac{G_2(z)+1}{G_2(z)-1}\right| = \left|\frac{-c_{m+1}\sum\limits_{n=m+1}^{\infty} a_n z^n}{-c_{m+1}\sum\limits_{n=m+1}^{\infty} a_n z^n - 2}\right| < \frac{c_{m+1}\sum\limits_{n=m+1}^{\infty} a_n}{2 - c_{m+1}\sum\limits_{n=m+1}^{\infty} a_n - \sum\limits_{n=1}^{m} a_n} < 1.$$

Therefore $Re\ G_2(z) < 0$ and we get $Re\left\{\frac{f(z)}{h_m(z)}\right\} < 1 + \frac{1}{c_{m+1}}$. For the second inequality, we define

$$F(z) = (1 + c_{m+1})\left[\frac{h_m(z)}{f(z)} - \frac{c_{m+1}}{1 + c_{m+1}}\right],$$

then by using (3.2), we obtain

$$\begin{aligned}\left|\frac{F(z)-1}{F(z)+1}\right| &= \left|\frac{(1+c_{m+1})(h_m(z) - f(z))}{(1+c_{m+1})h_m(z) - (c_{m+1}-1)f(z)}\right| \\ &= \left|\frac{(1+c_{m+1})\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{2 + (c_{m+1}-1)\sum\limits_{n=m+1}^{\infty} a_n z^{n-1} - 2\sum\limits_{n=2}^{m} a_n z^{n-1}}\right| \\ &\le \frac{(1+c_{m+1})\sum\limits_{n=m+1}^{\infty} a_n}{2 - 2\sum\limits_{n=2}^{m} a_n + (1 - c_{m+1})\sum\limits_{n=m+1}^{\infty} a_n} \le 1.\end{aligned}$$

This shows that $Re\ F(z) > 0$ and finally $Re\left\{\frac{h_m(z)}{f(z)}\right\} > \frac{c_{m+1}}{1 + c_{m+1}}$ ■

## 4. Radius of Close-to-convexity, Starlikeness and Convexity

We concentrate upon getting the radius of close-to-convexity, starlikeness and convexity.

**Theorem 4.1.** *Let the function $f(z)$ defined by (1.1) be in the class $k-UCV(\lambda,\gamma,\beta,\delta)$. Then $f(z)$ is close-to-convex of order $\mathcal{E}$ $(0 \le \mathcal{E} < 1)$ in $|z| < r_1(\lambda,\gamma,k,\beta,\delta,\mathcal{E})$ where*

$$r_1(\lambda,\gamma,k,\beta,\delta,\mathcal{E}) = \inf_n \left\{ \frac{(1-\mathcal{E})(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{n(1-\gamma)\Gamma(\delta+1)\Gamma(\beta+\delta+n)} \right\}^{\frac{1}{n-1}}, \quad n \ge 2. \tag{4.1}$$

*Proof.* We must show that $|f'(z)-1| \le 1-\mathcal{E}$ for $|z| < r_1(\lambda,\gamma,k,\beta,\delta,\mathcal{E})$. we have

$$|f'(z)-1| \le \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

Thus $|f'(z)-1| \le 1-\mathcal{E}$ if

$$\sum_{n=2}^{\infty} \left(\frac{n}{1-\mathcal{E}}\right) a_n |z|^{n-1} \le 1. \tag{4.2}$$

By Theorem 2.1, we have

$$\sum_{n=2}^{\infty} \frac{(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{(1-\gamma)\Gamma(\beta+\delta+n)\Gamma(\delta+1)} a_n \le 1. \tag{4.3}$$

Hence (4.2) will be true if

$$\frac{n|z|^{n-1}}{1-\mathcal{E}} \le \frac{(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{(1-\gamma)\Gamma(\delta+1)\Gamma(\beta+\delta+n)}$$

equivalently if

$$|z| \le \left\{ \frac{(1-\mathcal{E})(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{n(1-\gamma)\Gamma(\delta+1)\Gamma(\beta+\delta+n)} \right\}^{\frac{1}{n-1}}, \quad n \ge 2. \tag{4.4}$$

■

The theorem follows from (4.4).

**Theorem 4.2.** *Let $f(z)$ defined by (1.1) be in the class $k-UCV(\lambda,\gamma,\beta,\delta)$. Then $f(z)$ is starlike of order $\mathcal{E}$ $(0 \le \mathcal{E} < 1)$ in $|z| < r_2(\lambda,\gamma,k,\beta,\delta,\mathcal{E})$, where*

$$\begin{aligned} & r_2(\lambda,\gamma,k,\beta,\delta,\mathcal{E}) \\ &= \inf_n \left\{ \frac{(1-\mathcal{E})(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{(n-\mathcal{E})(1-\gamma)\Gamma(\delta+1)\Gamma(\beta+\delta+n)} \right\}^{\frac{1}{n-1}}, \\ & n \ge 2. \end{aligned} \tag{4.5}$$

*Proof.* It suffices to show that $\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1 - \mathcal{E}$ for $|z| < r_2(\lambda,\gamma,k,\beta,\delta,\mathcal{E})$. We have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1-\mathcal{E} \quad \text{if} \quad \sum_{n=2}^{\infty}\frac{(n-\mathcal{E})a_n|z|^{n-1}}{(1-\mathcal{E})} \le 1 \tag{4.6}$$

by using (4.3), (4.6) will be true if

$$\frac{n-\mathcal{E}}{1-\mathcal{E}}|z|^{n-1} \le \frac{(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{(1-\gamma)\Gamma(\delta+1)\Gamma(\beta+\delta+n)}$$

or equivalently

$$\begin{aligned} |z| \le & \left\{ \frac{(1-\mathcal{E})(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{(n-\mathcal{E})(1-\gamma)\Gamma(\delta+1)\Gamma(\beta+\delta+n)} \right\}^{\frac{1}{n-1}}, \\ & n \ge 2. \end{aligned}$$

∎

The theorem follows easily from last expression.

**Theorem 4.3.** *Let $f$ be in $k-UCV(\lambda,\gamma,\beta,\delta)$, then $f(z)$ is convex of order $\mathcal{E}$ for $0 \le \mathcal{E} < 1$ in $|z| < r_3(\lambda,\gamma,k,\beta,\delta,\mathcal{E})$, where*

$$\begin{aligned} & r_3(\lambda,\gamma,k,\beta,\delta,\mathcal{E}) \\ &= \inf_{n\ge 2} \left\{ \frac{(1-\mathcal{E})(1+n\lambda-\lambda)[n(1+k)-(k+\gamma)]\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{n(n-\mathcal{E})(1-\gamma)\Gamma(\delta+1)\Gamma(\beta+\delta+n)} \right\}^{\frac{1}{n-1}}. \end{aligned}$$

*Proof.* To establish the required result, it sufficies to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1-\mathcal{E} \quad \text{for} \quad |z| < r_3 \tag{4.7}$$

Substituting the series expansions of $f''(z)$ and $f'(z)$ in the left hand of (4.7), we have

$$\left|\frac{-\sum\limits_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty} n a_n z^{n-1}}\right| \le \frac{\sum\limits_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1-\sum\limits_{n=2}^{\infty} n a_n |z|^{n-1}}.$$

The last expression above is bounded by $(1-\mathcal{E})$ if

$$\sum_{n=2}^{\infty} \frac{n(n-\mathcal{E})}{1-\mathcal{E}} a_n |z|^{n-1} \le 1. \tag{4.8}$$

In view of (4.7), it follows that (4.8) is true if

$$\frac{n(n-\mathcal{E})}{1-\mathcal{E}}|z|^{n-1} < \left(\frac{(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)(\Gamma(\delta+n)}{(1-\gamma)\Gamma(\delta+1)\Gamma(\beta+\delta+n)}\right)$$

or

$$|z| < \left\{\frac{(1-\mathcal{E})(1+n\lambda-\lambda)(n(1+k)-(k+\gamma))\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{n(n-\mathcal{E})(1-\gamma)\Gamma(\beta+\delta+n)\Gamma(\delta+1)}\right\}^{\frac{1}{n-1}}$$

and this completes the proof. ∎

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