Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume **5** (2010), 35 – 47

SOME APPLICATIONS OF GENERALIZED RUSCHEWEYH DERIVATIVES INVOLVING A GENERAL FRACTIONAL DERIVATIVE OPERATOR TO A CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS I

Waggas Galib Atshan and S. R. Kulkarni

Abstract. For certain univalent function f, we study a class of functions f as defined by making use of the generalized Ruscheweyh derivatives involving a general fractional derivative operator, satisfying

$$Re\left\{\frac{z(\mathcal{J}_1^{\lambda,\mu}f(z))'}{(1-\gamma)\mathcal{J}_1^{\lambda,\mu}f(z)+\gamma z^2(\mathcal{J}_1^{\lambda,\mu}f(z))''}\right\}>\beta.$$

A necessary and sufficient condition for a function to be in the class $A_\gamma^{\lambda,\mu,\nu}(n,\beta)$ is obtained. In addition, our paper includes distortion theorem, radii of starlikeness, convexity and close-to-convexity, extreme points. Also, we get some results in this paper.

1 Introduction

Let Ω denote the class of functions which are analytic in the unit disk $U=\{z\in \boldsymbol{C}: |z|<1\}$ and let $A(n)$ denote the subclass of Ω consisting of functions of the form:

$$f(z)=z-\sum_{k=n+1}^{\infty}a_k z^k, \qquad (a_k\geq 0, n\in I\!N), \tag{1.1}$$

where $f(z)$ is analytic and univalent in the unit disk U. Then the function $f(z)\in A(n)$ is said to be in the class $S(n,\alpha)$, if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\ , \quad (z\in U, 0\leq\alpha<1). \tag{1.2}$$

A function $f(z)\in S(n,\alpha)$ is called starlike function of order α. A function $f(z)\in A(n)$ is said to be in the class $C(n,\alpha)$ if and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha, \quad (z\in U, 0\leq\alpha<1). \tag{1.3}$$

2000 Mathematics Subject Classification: 30C45.
Keywords: Distortion theorem; Radii of starlikeness; Extreme points.

A function $f(z) \in C(n,\alpha)$ is called convex function of order α. It is observed that

$$f(z) \in C(n,\alpha) \ \text{ if and only if } zf'(z) \in S(n,\alpha) \ \ \forall\ n \in \mathbb{N} \ \ [2]. \tag{1.4}$$

A function $f(z) \in A(n)$ is said to be in the class $K(n,\alpha)$ if there is a convex function $g(z)$ such that

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha, \quad (\forall\ z \in U, 0 \le \alpha < 1). \tag{1.5}$$

A function $f(z) \in K(n,\alpha)$ is called close-to-convex of order α.

We shall need the fractional derivative operator ([10], [11]) in this paper.

Let $a, b, c \in \boldsymbol{C}$ with $C \neq 0, -1, -2, \cdots$. The Gaussian hypergeometric function ${}_2F_1$ is defined by

$${}_2F_1(z) \equiv \ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{1.6}$$

where $(\lambda)_n$ is the pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases}$$

Definition 1. *Let* $0 \le \lambda < 1$ *and* $\mu, \nu \in \mathbb{R}$. *Then, in terms of familiar (Gauss's) hypergeometric function* ${}_2F_1$, *the generalized fractional derivative operator* $J_{0,z}^{\lambda,\mu,\nu}$ *of a function* $f(z)$ *is defined by:*

$$J_{0,z}^{\lambda,\mu,\nu} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \left\{ z^{\lambda-\mu} \int_0^z (z-\mathcal{E})^{-\lambda} f(\mathcal{E}) \cdot \ {}_2F_1(\mu-\lambda, 1-\nu; 1-\lambda; 1-\frac{\mathcal{E}}{z}) d\mathcal{E} \right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad (0 \le \lambda < 1) \\ \frac{d^n}{dz^n} J_{0,z}^{\lambda-n,\mu,\nu} f(z), \qquad\qquad\qquad (n \le \lambda < n+1, n \in \mathbb{N}) \end{cases} \tag{1.7}$$

where the function $f(z)$ *is analytic in a simply-connected region of the* z*-plane containing the origin, with the order*

$$f(z) = O(|z|^{\epsilon}), (z \to 0), \tag{1.8}$$

for $\epsilon > \max\{0, \mu-\nu\} - 1$, *and the multiplicity of* $(z-\mathcal{E})^{-\lambda}$ *is removed by requiring* $\log(z-\mathcal{E})$ *to be real, when* $z - \mathcal{E} > 0$.

The fractional derivative of order λ *of a function* $f(z)$ *is defined by*

$$D_z^{\lambda}\{f(z)\} = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\mathcal{E})}{(z-\mathcal{E})^{\lambda}} d\mathcal{E}, \quad 0 \le \lambda < 1, \tag{1.9}$$

where $f(z)$ *it is chosen as in (1.7), and the multiplicity of* $(z-\mathcal{E})^{-\lambda}$ *is removed by requiring* $\log(z-\mathcal{E})$ *to be real, when* $z - \mathcal{E} > 0$.

By comparing (e1.7) with (1.9), we find

$$J_{0,z}^{\lambda,\lambda,\nu} f(z) = D_z^{\lambda}\{f(z)\}, \quad (0 \leq \lambda < 1). \tag{1.10}$$

In terms of gamma function, we have

$$J_{0,z}^{\lambda,\mu,\nu} z^k = \frac{\Gamma(k+1)\Gamma(1-\mu+\nu+k)}{\Gamma(1-\mu+k)\Gamma(1-\lambda+\nu+k)} z^{k-\mu}, \tag{1.11}$$

$$(0 \leq \lambda < 1, \mu, \nu \in \mathbb{R} \quad \textit{and} \quad k > \max\{0, \mu-\nu\} - 1).$$

Definition 2. *Let $f(z) \in A(n)$ be given by (1.1). Then the class $A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$ is defined by*

$$A_{\gamma}^{\lambda,\mu,\nu}(n,\beta) = \left\{ f \in A(n) : Re\left\{ \frac{z(\mathcal{J}_1^{\lambda,\mu} f(z))'}{(1-\gamma)\mathcal{J}_1^{\lambda,\mu} f(z) + \gamma z^2(\mathcal{J}_1^{\lambda,\mu} f(z))''} \right\} > \beta, \right.$$

$$\left. (z \in U, 0 \leq \gamma < 1, n \in \mathbb{N}; 0 \leq \beta < 1; \lambda > -1) \right\}, \tag{1.12}$$

where $\mathcal{J}_1^{\lambda,\mu} f$ is a generalized Ruscheweyh derivative defined by Goyal and Goyal [3, p. 442] as

$$\mathcal{J}_1^{\lambda,\mu} f(z) = \frac{\Gamma(\mu-\lambda+\nu+2)}{\Gamma(\nu+2)\Gamma(\mu+1)} z J_{0,z}^{\lambda,\mu,\nu}(z^{\mu-1} f(z)) \tag{1.13}$$

$$= z - \sum_{k=n+1}^{\infty} a_k C_1^{\lambda,\mu}(k) z^k, \tag{1.14}$$

where

$$C_1^{\lambda,\mu}(k) = \frac{\Gamma(k+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu+1)}{\Gamma(k)\Gamma(k+\nu+1+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}. \tag{1.15}$$

For $\mu = \lambda = \alpha, \nu = 1$, the generalized Ruscheweyh derivatives reduces to ordinary Ruscheweyh derivatives of $f(z)$ of order α [7]:

$$D^{\alpha} f(z) = \frac{z}{\Gamma(\alpha+1)} D^{\alpha}(z^{\alpha-1} f(z)) = z - \sum_{k=n+1}^{\infty} a_k C_k(\alpha) z^k, \tag{1.16}$$

where

$$C_k(\alpha) = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)}{(k-1)!}. \tag{1.17}$$

The class $A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$ contains many well-known classes of analytic functions, for example:

(i) If $\mu = \lambda = \alpha, \nu = 1, n = 1$, we get the class $A_{\gamma}^{\lambda,\lambda,1}(1,\beta)$ was studied by Tehranchi and Kulkarni [12].

(ii) If $\mu=\lambda=0, \nu=1, \alpha=\beta, \gamma=0$, *we get the class of starlike function of order* $\alpha, (S(n,\alpha))$.

The same properties have been found for other classes in [4], [8] and [9].

Lemma 3. *Let* $w=u+i\nu$, *then*

$$Re \quad w \geq \beta \quad \text{if and only if} \quad |w-(1+\beta)| \leq |w+(1-\beta)|.$$

Definition 4. *Let* f, h *be analytic in* U. *Then* h *is said to be subordinate to* f, *written* $h \prec f$, *if there exist function* w *that is analytic in* U *with* $|w(z)|<1$ *in* U *and* $h(z)=f(w(z))$ *in* U *for some analytic function* w *with* $w(0)=0$ *and* $|w(z)| \leq |z|$ *in* U. *If* w *is not merely a rotation of the disk (that is, if* $|w(z)|<|z|$*), then* h *is said to be properly subordinate to* f.

2 Main Results

The following theorem gives a necessary and sufficient condition for function to be in $A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$.

Theorem 5. *Let* $f(z) \in A(n)$, *then* $f(z) \in A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$ *if and only if*

$$\sum_{k=n+1}^{\infty}(\gamma\beta(1+k-k^2)+k-\beta)C_1^{\lambda,\mu}(k)a_k < 1-\beta(1-\gamma), \tag{2.1}$$

where $0 \leq \gamma < 1, 0 \leq \beta < 1, \lambda > -1, n \in I\!N$ *and* $C_1^{\lambda,\mu}(k)$ *is given by (1.15).*

Proof. Assume that $f \in A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$ so we have

$$Re\left\{\frac{z(\mathcal{J}_1^{\lambda,\mu}f(z))'}{(1-\gamma)\mathcal{J}_1^{\lambda,\mu}f(z)+\gamma z^2(\mathcal{J}_1^{\lambda,\mu}f(z))''}\right\} > \beta$$

$$Re\left\{\frac{z-\sum\limits_{k=n+1}^{\infty}kC_1^{\lambda,\mu}(k)a_kz^k}{(1-\gamma)\left(z-\sum\limits_{k=n+1}^{\infty}C_1^{\lambda,\mu}(k)a_kz^k\right)+\gamma\left(-\sum\limits_{k=n+1}^{\infty}k(k-1)C_1^{\lambda,\mu}(k)a_kz^k\right)}\right\} > \beta.$$

Hence

$$Re\left\{\frac{1-\sum\limits_{k=n+1}^{\infty}kC_1^{\lambda,\mu}(k)a_kz^{k-1}}{(1-\gamma)-\sum\limits_{k=n+1}^{\infty}(1-\gamma+\gamma k(k-1))C_1^{\lambda,\mu}(k)a_kz^{k-1}}\right\} > \beta,$$

or equivalently

$$Re\left\{\frac{1-\sum\limits_{k=n+1}^{\infty}kC_1^{\lambda,\mu}(k)a_kz^{k-1}-\beta(1-\gamma)+\beta\sum\limits_{k=n+1}^{\infty}(1-\gamma+\gamma k(k-1))C_1^{\lambda,\mu}(k)a_kz^{k-1}}{(1-\gamma)-\sum\limits_{k=n+1}^{\infty}(1-\gamma+\gamma k(k-1))C_1^{\lambda,\mu}(k)a_kz^{k-1}}\right\}>0.$$

This inequality is correct for all $z \in U$. Letting $z \longrightarrow 1^-$ yields

$$Re\left\{\frac{1-\beta(1-\gamma)-\sum\limits_{k=n+1}^{\infty}(k-\beta(1-\gamma+\gamma k(k-1)))C_1^{\lambda,\mu}(k)a_k}{(1-\gamma)-\sum\limits_{k=n+1}^{\infty}(1-\gamma+\gamma k(k-1))C_1^{\lambda,\mu}(k)a_k}\right\}>0$$

and so by the mean value theorem, we have

$$Re\left\{1-\beta(1-\gamma)-\sum_{k=n+1}^{\infty}(k-\beta(1-\gamma+\gamma k^2-\gamma k))C_1^{\lambda,\mu}(k)a_k\right\}>0,$$

so we have

$$\sum_{k=n+1}^{\infty}(\gamma\beta(1+k-k^2)+k-\beta)C_1^{\lambda,\mu}(k)a_k<1-\beta(1-\gamma).$$

Conversely, let (2.1) hold. We will prove that (1.12) is correct and then $f \in A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$.

By Lemma 3 it is enough to prove that $|w-(1+\beta)|<|w+(1-\beta)|$, where

$$w=\frac{z(\mathcal{J}_1^{\lambda,\mu}f(z))'}{(1-\gamma)\mathcal{J}_1^{\lambda,\mu}f(z)+\gamma z^2(\mathcal{J}_1^{\lambda,\mu}f(z))''}$$

or show that

$$\begin{aligned}T &= \frac{1}{|N(z)|}|z(\mathcal{J}_1^{\lambda,\mu}f(z))'-(1+\beta)(1-\gamma)\mathcal{J}_1^{\lambda,\mu}f(z)-(1+\beta)\gamma z^2(\mathcal{J}_1^{\lambda,\mu}f(z))''|\\ &< \frac{1}{|N(z)|}|z(\mathcal{J}_1^{\lambda,\mu}f(z))'+(1-\beta)(1-\gamma)\mathcal{J}_1^{\lambda,\mu}f(z)+(1-\beta)\gamma z^2(\mathcal{J}_1^{\lambda,\mu}f(z))''|\\ &= Q\ ,\end{aligned}$$

where $N(z)=(1-\gamma)\mathcal{J}_1^{\lambda,\mu}f(z)+\gamma z^2(\mathcal{J}_1^{\lambda,\mu}f(z))''$ and it is easy to verify that $Q-T>0$ and so the proof is complete. □

Corollary 6. *Let $f \in A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$, then*

$$a_k<\frac{1-\beta(1-\gamma)}{C_1^{\lambda,\mu}(k)|\gamma\beta(1+k-k^2)+k-\beta|},\quad k=n+1,n+2,\cdots.$$

Theorem 7. *The class* $A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$ *is convex set.*

Proof. Let $f(z), g(z)$ be the arbitrary elements of $A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$, then for every t $(0 < t < 1)$, we show that $(1-t)f(z)+tg(z) \in A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$ thus, we have

$$(1-t)f(z)+tg(z) = z - \sum_{k=n+1}^{\infty} [(1-t)a_k + tb_k]z^k$$

and

$$\begin{aligned}&\sum_{k=n+1}^{\infty}\left[\frac{\gamma\beta(1+k-k^2)+k-\beta}{1-\beta(1-\gamma)}\right][(1-t)a_k+tb_k]C_1^{\lambda,\mu}(k)\\ &=(1-t)\sum_{k=n+1}^{\infty}\frac{\gamma\beta(1+k-k^2)+k-\beta}{1-\beta(1-\gamma)}a_kC_1^{\lambda,\mu}(k)\\ &+t\sum_{k=n+1}^{\infty}\frac{\gamma\beta(1+k-k^2)+k-\beta}{1-\beta(1-\gamma)}b_kC_1^{\lambda,\mu}(k) < 1.\end{aligned}$$

This completes the proof. □

Corollary 8. *Assume that* $f(z)$ *and* $g(z)$ *belong to* $A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$, *then the function* $y(z)$ *defined by* $y(z)=\frac{1}{2}(f(z)+g(z))$ *also belong to* $A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$.

3 Distortion Theorem

In the next theorem, we will find distortion bound for $\mathcal{J}_1^{\lambda,\mu}f(z)$.

Theorem 9. *Let* $f(z) \in A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$, *then*

$$\begin{aligned}&|z| - \frac{1-\beta(1-\gamma)}{\gamma\beta(2+n-(1+n)^2)+(n+1)-\beta}|z|^{n+1} \le |\mathcal{J}_1^{\lambda,\mu}f(z)|\\ &\le |z| + \frac{1-\beta(1-\gamma)}{\gamma\beta(2+n-(1+n)^2)+(n+1)-\beta}|z|^{n+1}.\end{aligned}$$

Proof. Let $f(z) \in A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$. By Theorem 5, we have

$$\sum_{k=n+1}^{\infty} C_1^{\lambda,\mu}(k)a_k < \frac{1-\beta(1-\gamma)}{\gamma\beta(2+n-(1+n)^2)+(n+1)-\beta}, \quad n \in I\!N = \{1,2,\cdots\}.$$

Therefore

$$|\mathcal{J}_1^{\lambda,\mu}f(z)| \le |z|+|z|^{n+1}\sum_{k=n+1}^{\infty} C_1^{\lambda,\mu}(k)a_k < |z|+\frac{1-\beta(1-\gamma)}{\gamma\beta(2+n-(1+n)^2)+(n+1)-\beta}|z|^{n+1}$$

and

$$|\mathcal{J}_1^{\lambda,\mu} f(z)| \geq |z| - |z|^{n+1} \sum_{k=n+1}^{\infty} C_1^{\lambda,\mu}(k) a_k > |z| - \frac{1-\beta(1-\gamma)}{\gamma\beta(2+n-(n+1)^2)+(n+1)-\beta} |z|^{n+1}.$$

This completes the proof. □

4 Radii of Starlikeness, Convexity and Close-to-convexity

In the next theorems, we will find the radii of starlikeness, convexity and close-to-convexity for the class $A_\gamma^{\lambda,\mu,\nu}(n,\beta)$.

Theorem 10. *Let* $f(z) \in A_\gamma^{\lambda,\mu,\nu}(n,\beta)$. *Then* $f(z)$ *is a starlike of order* $\alpha (0 \leq \alpha < 1)$ *in* $|z| < r = r_1(\lambda,\mu,\nu,\beta,\gamma,\alpha)$*,where*

$$r_1(\lambda,\mu,\nu,\beta,\gamma,\alpha) = \inf_k \left\{ \frac{(1-\alpha)[\gamma\beta(1+k-k^2)+k-\beta] C_1^{\lambda,\mu}(k)}{(k-\alpha)(1-\beta(1-\gamma))} \right\}^{\frac{1}{k-1}} \tag{4.1}$$

and $C_1^{\lambda,\mu}(k)$ *is given by (1.15).*

Proof. Let $f(z) \in A_\gamma^{\lambda,\mu,\nu}(n,\beta)$. Then by Theorem 5

$$\sum_{k=n+1}^{\infty} \frac{(\gamma\beta(1+k-k^2)+k-\beta) C_1^{\lambda,\mu}(k)}{1-\beta(1-\gamma)} a_k < 1.$$

For $0 \leq \alpha < 1$, we need to show that $\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1-\alpha$, we have to show that

$$\left| \frac{zf'(z)-f(z)}{f(z)} \right| = \left| \frac{-\sum_{k=n+1}^{\infty} (k-1) a_k z^{k-1}}{1-\sum_{k=n+1}^{\infty} a_k z^{k-1}} \right| \leq \frac{\sum_{k=n+1}^{\infty} (k-1) a_k |z|^{k-1}}{1-\sum_{k=n+1}^{\infty} a_k |z|^{k-1}} < 1-\alpha.$$

Hence $\sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha}\right) a_k |z|^{k-1} \leq 1$. This is enough to consider

$$|z|^{k-1} \leq \frac{(1-\alpha)[\gamma\beta(1+k-k^2)+k-\beta] C_1^{\lambda,\mu}(k)}{(k-\alpha)(1-\beta(1-\gamma))},$$

therefore,

$$|z| \leq \left\{ \frac{(1-\alpha)[\gamma\beta(1+k-k^2)+k-\beta] C_1^{\lambda,\mu}(k)}{(k-\alpha)(1-\beta(1-\gamma))} \right\}^{\frac{1}{k-1}}. \tag{4.2}$$

Setting $|z| = r_1(\lambda,\mu,\nu,\beta,\gamma,\alpha)$ in (4.2), we get the radii of starlikeness, which completes the proof of the Theorem 10. □

By using (1.4) (Alexander's theorem [2] : f is convex if and only if zf' is starlike), we obtain:

Theorem 11. *Let* $f(z) \in A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$*. Then* $f(z)$ *is convex of order* $\alpha (0 \leq \alpha < 1)$ *in* $|z| < r = r_2(\lambda,\mu,\nu,\beta,\gamma,\alpha)$*, where*

$$r_2(\lambda,\mu,\nu,\beta,\gamma,\alpha) = \inf_k \left\{ \frac{(1-\alpha)[\beta\gamma(1+k-k^2)+k-\beta]}{k(k-\alpha)(1-\beta(1-\gamma))} C_1^{\lambda,\mu}(k) \right\}^{\frac{1}{k-1}} \tag{4.3}$$

and $C_1^{\lambda,\mu}(k)$ *is given by (1.15).*

Proof. Let $f(z) \in A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$. Then by Theorem 5

$$\sum_{k=n+1}^{\infty} \frac{(\gamma\beta(1+k-k^2)+k-\beta)C_1^{\lambda,\mu}(k)}{1-\beta(1-\gamma)} a_k < 1.$$

For $0 \leq \alpha < 1$, we show that $\left|\frac{zf''(z)}{f'(z)}\right| < 1-\alpha$,that is

$$\left| \frac{-\sum\limits_{k=n+1}^{\infty} k(k-1)a_k z^{k-1}}{1-\sum\limits_{k=n+1}^{\infty} k a_k z^{k-1}} \right| \leq \frac{\sum\limits_{k=n+1}^{\infty} k(k-1)a_k |z|^{k-1}}{1-\sum\limits_{k=n+1}^{\infty} k a_k |z|^{k-1}} < 1-\alpha,$$

or equivalently $\sum\limits_{k=n+1}^{\infty} k\left(\frac{k-\alpha}{1-\alpha}\right) a_k |z|^{k-1} \leq 1$. It is enough letting

$$|z|^{k-1} \leq \frac{(1-\alpha)[\gamma\beta(1+k-k^2)+k-\beta]}{k(k-\alpha)(1-\beta(1-\gamma))} C_1^{\lambda,\mu}(k).$$

Therefore,

$$|z| \leq \left\{ \frac{(1-\alpha)[\gamma\beta(1+k-k^2)+k-\beta]}{k(k-\alpha)(1-\beta(1-\gamma))} C_1^{\lambda,\mu}(k) \right\}^{\frac{1}{k-1}}. \tag{4.4}$$

Setting $|z| = r_2(\lambda,\mu,\nu,\beta,\gamma,\alpha)$ in (4.4), we get the radii of convexity, which completes the proof of Theorem 11. □

Theorem 12. *If* $f(z) = z - \sum\limits_{k=n+1}^{\infty} a_k z^k \in A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$*, then* $f(z)$ *is close-to-convex of order* $\delta, 0 \leq \delta < 1$ *in* $|z| < r = r_3(\lambda,\mu,\nu,\beta,\gamma,\delta)$*,where*

$$r_3(\lambda,\mu,\nu,\beta,\gamma,\delta) = \inf_k \left\{ \frac{(1-\delta)[\gamma\beta(1+k-k^2)+k-\beta]}{k(1-\beta(1-\gamma))} C_1^{\lambda,\mu}(k) \right\}^{\frac{1}{k-1}} \tag{4.5}$$

and $C_1^{\lambda,\mu}(k)$ *is given by (1.15).*

Proof. Let $f(z) \in A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$. Then by Theorem 5

$$\sum_{k=n+1}^{\infty} \frac{(\gamma\beta(1+k-k^2)+k-\beta)C_1^{\lambda,\mu}(k)}{1-\beta(1-\gamma)} a_k < 1,$$

for $0 \le \delta < 1$, we need to show that $|f'(z)-1| \le 1-\delta$ for $|z| < r = r_3(\lambda,\mu,\nu,\beta,\gamma,\delta)$,when $r_3(\lambda,\mu,\nu,\beta,\gamma,\delta)$ is given by (4.5). Now

$$|f'(z)-1| = \left| \sum_{k=n+1}^{\infty} k a_k z^{k-1} \right| \le \sum_{k=n+1}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z)-1| < 1-\delta$ if $\sum\limits_{k=n+1}^{\infty} \frac{k a_k}{(1-\delta)} |z|^{k-1} \le 1$ but, by Theorem 5 above inequality holds true if

$$|z|^{k-1} \le \frac{(1-\delta)[\beta\gamma(1+k-k^2)+k-\beta]}{k(1-\beta(1-\gamma))} C_1^{\lambda,\mu}(k).$$

This completes the proof. □

5 Extreme Points

In the next theorem, we will find extreme points for the class $A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$.

Theorem 13. *Let* $f_n(z) = z$ *and*

$$f_k(z) = z - \frac{1-\beta(1-\gamma)}{[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)} z^k, (k = n+1, n+2, \cdots, n \in I\!N = \{1,2,\cdots\}),$$

where $C_1^{\lambda,\mu}(k)$ *is given by (1.15). Then* $f \in A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$ *if and only if it can be expressed in the form* $f(z) = \sum\limits_{k=n}^{\infty} \sigma_k f_k(z)$,*where* $\sigma_k \ge 0$ *and* $\sum\limits_{k=n}^{\infty} \sigma_k = 1$. *In particular, the extreme points of* $A_{\gamma}^{\lambda,\mu,\nu}(n,\beta)$ *are the functions* $f_n(z) = z$ *and*

$$f_k(z) = z - \frac{(1-\beta(1-\gamma))}{[\beta\gamma(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)} z^k, \quad k = n+1, n+2, \cdots.$$

Proof. Firstly, let us express f as in the above theorem, therefore we can write

$$\begin{aligned} f(z) &= \sum_{k=n}^{\infty} \sigma_k f_k(z) = \sigma_n z + \sum_{k=n+1}^{\infty} \sigma_k \left[z - \frac{1-\beta(1-\gamma)}{[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)} z^k \right] \\ &= z(\sigma_n + \sum_{k=n+1}^{\infty} \sigma_k) - \sum_{k=n+1}^{\infty} \frac{1-\beta(1-\gamma)}{[\beta\gamma(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)} \sigma_k z^k \\ &= z - \sum_{k=n+1}^{\infty} g_k z^k, \end{aligned}$$

where

$$g_k = \frac{1-\beta(1-\gamma)}{[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)}\sigma_k.$$

Therefore, $f \in A_\gamma^{\lambda,\mu,\nu}(n,\beta)$, since

$$\sum_{k=n+1}^{\infty} \frac{g_k[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)}{1-\beta(1-\gamma)} = \sum_{k=n+1}^{\infty} \sigma_k = 1-\sigma_n < 1.$$

Conversely, assume that $f \in A_\gamma^{\lambda,\mu,\nu}(n,\beta)$, then by (2.1) we may set

$$\sigma_k = \frac{[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)}{1-\beta(1-\gamma)}a_k, k \geq n+1 \text{ and } 1-\sum_{k=n+1}^{\infty}\sigma_k = \sigma_n.$$

Then

$$\begin{aligned} f(z) &= z-\sum_{k=n+1}^{\infty} a_k z^k = z-\sum_{k=n+1}^{\infty} \frac{(1-\beta(1-\gamma))\sigma_k}{[\gamma\beta(1+k-k^2)+k-\beta]C_1^{\lambda,\mu}(k)} z^k \\ &= z-\sum_{k=n+1}^{\infty}\sigma_k(z-f_k(z)) = z\left(1-\sum_{k=n+1}^{\infty}\sigma_k\right)+\sum_{k=n+1}^{\infty}\sigma_k f_k(z) \\ &= \sigma_n z+\sum_{k=n+1}^{\infty}\sigma_k f_k(z) = \sum_{k=n}^{\infty}\sigma_k f_k(z). \end{aligned}$$

This completes the proof. □

6 Subordination

Theorem 14. *For $n=1$, let $f(z) \in A_\gamma^{\lambda,\mu,\nu}(1,\beta)$ and $h(z)$ be an arbitrary element of $A(1)$ such that $h \prec f$, defined in Definition 4, and if*

$$h_k = \frac{1}{k!}\left[\frac{d^k(f(w(z)))}{dz^k}\right]_{z=0} \tag{6.1}$$

also if

$$\frac{\sum_{k=2}^{\infty}[\beta(1-\gamma)+\beta\gamma k(k-1)-k]|h_k|}{|h_1|} < (1-\beta(1-\gamma)). \tag{6.2}$$

Then $h \in A_\gamma^{\lambda,\mu,\nu}(1,\beta)$.

Proof. Since $h \prec f$ by definition of subordination there is analytic function $w(z)$ such that $|w(z)| \leq |z|$ and $h(z) = f(w(z))$. But $h(z)$ is the composition of two analytic functions in the unit disk, therefore we can expand this function in terms of Taylor series at origin as below

$$h(z) = \sum_{k=0}^{\infty} h_k z^k,$$

where h_k is defined in (6.1). Hence

$$h_0 = \frac{f(w(0))}{0!} = 0, \quad h_1 = \frac{w'(0)f'(0)}{1!} = w'(0).$$

Therefore, we can write

$$h(z) = h_1 z + \sum_{k=2}^{\infty} h_k z^k$$

and

$$\mathcal{J}_1^{\lambda,\mu} h(z) = h_1 z + \sum_{k=2}^{\infty} C_1^{\lambda,\mu}(k) h_k z^k,$$

we must prove $h(z) \in A_{\gamma}^{\lambda,\mu,\nu}(1,\beta)$ in other words we show that

$$Re\left\{\frac{z(\mathcal{J}_1^{\lambda,\mu}h(z))' - \beta(1-\gamma)\mathcal{J}_1^{\lambda,\mu}h(z) - \beta\gamma z^2(\mathcal{J}_1^{\lambda,\mu}h(z))''}{(1-\gamma)\mathcal{J}_1^{\lambda,\mu}h(z) + \gamma z^2(\mathcal{J}_1^{\lambda,\mu}h(z))''}\right\} > 0$$

or

$$Re\{[h_1 z + \sum_{k=2}^{\infty} k h_k C_1^{\lambda,\mu}(k) z^k - \beta(1-\gamma)h_1 z - \beta(1-\gamma)\sum_{k=2}^{\infty} h_k C_1^{\lambda,\mu}(k) z^k$$
$$-\beta\gamma \sum_{k=2}^{\infty} k(k-1) h_k C_1^{\lambda,\mu}(k) z^k]/[(1-\gamma)h_1 z + (1-\gamma)\sum_{k=2}^{\infty} C_1^{\lambda,\mu}(k) h_k z^k$$
$$+\gamma \sum_{k=2}^{\infty} k(k-1) h_k C_1^{\lambda,\mu}(k) z^k]\} > 0,$$

or we prove

$$Re\left\{\frac{h_1(1-\beta(1-\gamma)) - \sum_{k=2}^{\infty} h_k C_1^{\lambda,\mu}(k) z^{k-1}(\beta(1-\gamma) + \beta\gamma k(k-1) - k)}{(1-\gamma)h_1 + \sum_{k=2}^{\infty} h_k C_1^{\lambda,\mu}(k)((1-\gamma) + \gamma k(k-1)) z^{k-1}}\right\} > 0.$$

Letting $z \to 1^-$, and using the mean value theorem, we have to prove

$$Re\left\{h_1(1-\beta(1-\gamma)) - \sum_{k=2}^{\infty}(\beta(1-\gamma) + \beta\gamma k(k-1) - k) h_k C_1^{\lambda,\mu}(k)\right\} > 0$$

by (6.2) the last inequality is true and the result can be obtained. $\square$

References

[1] E. S. Aqlan, *Some problems connected with geometric function theory*, Ph.D. Thesis (2004), Pune University, Pune.

[2] P. L. Duren, *Univalent Functions*, Grundelheren der Mathematischen Wissenchaften **259**, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983. MR708494 (85j:30034). Zbl 0514.30001.

[3] S. P. Goyal and Ritu Goyal, *On a class of multivalent functions defined by generalized Ruscheweyh derivatives involving a general fractional derivative operator*, Journal of Indian Acad. Math. **27(2)** (2005), 439-456. MR2259538 (2007d:30005). Zbl 1128.30008.

[4] S. Kanas and A. Wisniowska, *Conic regions and k-uniformly convexity II*, Folia Sci. Tech. Reso. **170** (1998), 65-78. MR1693661 (2000e:30017). Zbl 0995.30013.

[5] R. K. Raina and T. S. Nahar, *Characterization properties for starlikeness and convexity of some subclasses of analytic functions involving a class of fractional derivative operators*, Acta Math. Univ. Comen., New Ser. **69**, No.1, 1-8 (2000). ISSN 0862-9544. MR1796782 (2001h:30014). Zbl 0952.30011.

[6] V. Ravichandran, N. Sreenivasagan, and H. M. Srivastava, *Some inequalities associated with a linear operator defined for a class of multivalent functions*, JIPAM, J. Inequal. Pure Appl. Math. 4, No. **4**, Paper No. 70, 12 p., electronic only (2003). ISSN 1443-5756. MR2051571 (2004m:30022). Zbl 1054.30013.

[7] T. Rosy, K. G. Subramanian and G. Murugusundaramoorthy, *Neighbourhoods and partial sums of starlike functions based on Ruscheweyh derivatives*, JIPAM, J. Inequal. Pure Appl. Math. 4, No. 4, Paper No. 64, 8 p., electronic only (2003). ISSN 1443-5756. MR2051565. Zbl 1054.30014.

[8] S. Shams and S. R. Kulkarni, *Certain properties of the class of univalent functions defined by Ruscheweyh derivative*, Bull. Calcutta Math. Soc. **97** (2005), 223-234. MR2191072. Zbl 1093.30012.

[9] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109-116. MR0369678 (51 #5910). Zbl 0311.30007.

[10] H. M. Srivastava, *Distortion inequalities for analytic and univalent functions associated with certain fractional calculus and other linear operators* (In Analytic and Geometric Inequalities and Applications eds. T. M. Rassias and H. M. Srivastava), Kluwar Academic Publishers, **478** (1999), 349-374. MR1785879 (2001h:30016). Zbl 0991.30007.

****

[11] H. M. Srivastava and R. K. Saxena, *Operators of fractional integration and their applications*, Applied Mathematics and Computation, **118** (2001), 1-52. MR1805158 (2001m:26016). Zbl 1022.26012.

[12] A. Tehranchi and S. R. Kulkarni, *Study of the class of univalent functions with negative coefficients defined by Ruscheweyh derivatives (II)*, J. Rajasthan Acad. Phy. Sci., **5(1)** (2006), 105-118. MR2214020. Zbl 1138.30015.

Waggas Galib Atshan
Department of Mathematics,
College of Computer Science and Mathematics,
University of AL-QADISIYA, Diwaniya, Iraq.
e-mail: waggashnd@yahoo.com

S. R. Kulkarni
Department of Mathematics,
HFergusson College, Pune - 411004,
India.
e-mail: kulkarni_ferg@yahoo.com