ON HIGHER HOMOMORPHISMS OF COMPLETELY PRIME GAMMA RINGS

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1-INTRODEUCTION

An additive mapping θ of a ring R into a ring Ŕ is called a Jordan homomorphism if θ(ab+ba)= θ(a) θ(b)+ θ(b) θ(a) for all a,b$\in$R .I.N.Herstein in [3] had proved that every Jordan homomorphism on to prime ring of characterstic not 2 and 3 either a homomorphism or an anti-homomorphism.While Smiley [8] had given a brief proof of this result and at the same time had removed the requirement that the characteristic of R be not 3.After this I.N.Herstein in [2] had proved that every Jordan homomorphism from R onto a 2-torsion free prime ring Ŕ is either a homomorphism or an anti-homomorphism.

Let M and Γ be additive abelian groups, M is called a Γ-ring if for any x,y,z $\in$M and α, β $\in$ Γ,the following conditions are satisfied

(1)x α y $\in$M

(2)(x+y) α z=x α z+ y α z

x($\alpha+\beta$)z=x α z+x β z

x α (y+z)=x α y+x α z

(3)(x α y) β z=x α (y β z)

The notion of Γ-ring was introduced by Nobusawa[5] and generalized by Barnes[1],many properties of Γ-ring were obtained by many researches.

M is called a 2-torsion free if 2x=0 implies x=0 for all x$\in$M.AΓ-ringM is called prime if aΓMΓb=0 implies a=0 or b=0 and M is called completely prime if a Γb=0 implies a=0 or b=0(a,b $\in$M),Since a Γ b Γ a Γ b $\subset$a Γ M Γ b,then every completely prime Γ-ring is prime.

In [7],section two, Rajaa C.Shaheen, defined the concept of Jordan homomorphism on Γ-ring and study it onto 2-torsion free completely prime Γ-ring.In [5] Rajaa C.Shaheen define **a** ***generalized Jordan homomorphism on ring***

and study this concept onto 2-torsion free semi- prime ring.In[7],Section three , Rajaa C.Shaheen defined this concept on Γ-rings and study it onto2-torsion free completely prime Γ- ring.

In [4] Anwar ,defined the concept of Jordan higher homomorphism and Generalized Jordan higher homomorphism on ring and study it on 2-torsion free prime ring .

Our study is to define the concept of Jordan higher homomorphism on Γ- ring and study it on 2-torsion free completely prime Γ- ring in section two also .

we define the concept of Generalized Jordan higher homomorphism on Γ- ring and study it on 2-torsion free completely prime Γ- ring in section three .

2-Jordan Higher Homomorphism on Gamma Rings

***Definition 2.1**:- Let $\theta = (\Phi_i)_{i\in N}$ be a family of additive mappings of a Γ-ring M into a Γ-ring N, we say that θ is a Higher Homomorphism on Gamma Ring(HH.for short)if for every $n\in N$,we have*

$$\phi_n(a\alpha b)=\sum_{i=1}^{n} \phi_i(a)\ \alpha\ \phi_i(b) \quad \forall\, a,b\in M,\ \alpha\in\Gamma.$$

***Definition 2.2**:- Let $\theta = (\Phi_i)_{i\in N}$ be a family of additive mappings of a Γ-ring M into a Γ-ring N, we say that θ is an anti- Higher Homomorphism on Gamma Ring(AHH.for short)if for every $n\in N$,we have*

$$\phi_n(a\alpha b)=\sum_{i=1}^{n} \phi_i(b)\ \alpha\ \phi_i(a) \quad \forall\, a,b\in M,\ \alpha\in\Gamma.$$

***Definition 2.3**:- Let $\theta = (\Phi_i)_{i\in N}$ be a family of additive mappings of a Γ-ring M into a Γ-ring N, we say that θ is a Jordan Higher Homomorphism on Gamma Ring(JHH.for short)if for every $n\in N$,we have*

$$\phi_n(a\alpha b+b\,\alpha a)=\sum_{i=1}^{n} (\phi_i(a)\ \alpha\ \phi_i(b)+\phi_i(b)\ \alpha\ \phi_i(a)) \quad \forall\, a,b\in M,\ \alpha\in\Gamma.$$

Note that,when N is 2-torsion free, we define JHH be merely insisting that

$$\phi_n(a\alpha a)=\sum_{i=1}^{n} \phi_i(a)\ \alpha\ \phi_i(a) \quad \forall\, a\in M,\ \alpha\in\Gamma.$$

Now we shall give the following lemma which is basic to prove the main result in this section

We should mentioned the reader that $\phi_i(a)\ \alpha\ \phi_j(b)=0 \quad \forall\ a,b\in M,\ \alpha\in\Gamma$ *and* $a\alpha b\ \beta c=a\ \beta b\alpha c\quad \forall\ a,b,c\in M(resp.,N),\ \alpha,\beta\in\Gamma$ *will be freely used in this section .*

<u>**Lemma 2.4**</u> *:-Let M ,N be a* Γ*-ring and* $\theta=(\Phi_i)_{i\in N}$ *be a Jordan Higher Homomorphism then the following statements are holds:*

(i) $\phi_n(a\alpha b\beta a)=\sum_{i=1}^{n}\ \phi_i(a)\ \alpha\ \phi_i(b)\ \beta\ \phi_i(a)$

(ii) $\phi_n(a\alpha b\beta c+c\alpha b\beta a)=\sum_{i=1}^{n}\ \phi_i(a)\alpha\ \phi_i(b)\beta\ \phi_i(c)+\phi_i(c)\alpha\ \phi_i(b)\ \beta\ \phi_i(a)$

<u>**Proof**</u>*:-(i)Since* $\phi_n(a\alpha b+b\ \alpha a)=\sum_{i=1}^{n}\ (\phi_i(a)\ \alpha\ \phi_i(b)+\phi_i(b)\ \alpha\ \phi_i(a))$

replace b by $a\beta b+b\ \beta a$

$W=\phi_n(a\alpha(a\beta b+b\ \beta a)+(a\beta b+b\ \beta a)\ \alpha a)$

$=\sum_{i=1}^{n}\ \phi_i(a)\ \alpha\ \phi_i(a\beta b+b\ \beta a)+\sum_{i=1}^{n}\ \phi_i(a\beta b+b\ \beta a)\ \alpha\ \phi_i(a)$

$=\sum_{i=1}^{n}\ \phi_i(a)\ \alpha\ (\sum_{k=1}^{i}\ \phi_k(a)\ \beta\ \phi_k(b)+\ \phi_k(b)\ \beta\ \phi_k(a))+\sum_{i=1}^{n}\ (\sum_{k=1}^{i}\ \phi_k(a)\ \beta\ \phi_k(b)+$
$\phi_k(b)\ \beta\ \phi_k(a))\ \alpha\ \phi_i(a)$

since $\phi_i(a)\ \alpha\ \phi_j(b)=0 \quad \forall\ a,b\in M,\ \alpha\in\Gamma$.

$W=\sum_{i=1}^{n}\ \phi_i(a)\ \alpha\ \phi_i(a)\ \beta\ \phi_i(b)+2\sum_{i=1}^{n}\ \phi_i(a)\ \alpha\ \phi_i(b)\ \beta\ \phi_i(a)$

$+\sum_{i=1}^{n}\ \phi_i(b)\ \beta\ \phi_i(a)\ \alpha\ \phi_i(a)$

on the other hand,

$W=\phi_n(a\alpha(a\beta b+b\ \beta a)+(a\beta b+b\ \beta a)\ \alpha a)$

$=\phi_n(a\alpha a\beta b+b\beta a\alpha a)+2\phi_n(a\alpha b\beta a)$

$$=\sum_{i=1}^{n} \phi_i(a\alpha a)\ \beta\ \phi_i(b)+\sum_{i=1}^{n} \phi_i(b)\ \beta\ \phi_i(a\alpha a)+2\phi_n(a\alpha b\beta a)$$

$$=\sum_{i=1}^{n} \phi_i(a)\alpha\ \phi_i(a)\beta\ \phi_i(b)+\sum_{i=1}^{n} \phi_i(b)\ \beta\ \phi_i(a)\alpha\ \phi_i(a)+2\phi_n(a\alpha b\beta a)$$

then by comparing these two expression of W, we get

$$2\phi_n(a\alpha b\beta a)=2\sum_{i=1}^{n} \phi_i(a)\ \alpha\ \phi_i(b)\ \beta\ \phi_i(a)$$

Since N is 2-torsion free,then

$$\phi_n(a\alpha b\beta a)=\sum_{i=1}^{n} \phi_i(a)\ \alpha\ \phi_i(b)\ \beta\ \phi_i(a)$$

(ii) by replacing a by a+c in (i)

$$W=\phi_n((a+c)\ \alpha\ b\ \beta\ (a+c))$$

$$=\sum_{i=1}^{n} \phi_i(a+c)\ \alpha\ \phi_i(b)\ \beta\ \phi_i(a+c)$$

$$=\sum_{i=1}^{n} \phi_i(a)\ \alpha\ \phi_i(b)\ \beta\ \phi_i(a)+\sum_{i=1}^{n} \phi_i(a)\ \alpha\ \phi_i(b)\ \beta\ \phi_i(c)+\sum_{i=1}^{n} \phi_i(c)\ \alpha\ \phi_i(b)$$

$$\beta\ \phi_i(a)+\sum_{i=1}^{n} \phi_i(c)\ \alpha\ \phi_i(b)\ \beta\ \phi_i(c)$$

On the other hand

$$W=\phi_n((a+c)\ \alpha\ b\ \beta\ (a+c))$$

$$=\phi_n(a\ \alpha\ b\ \beta\ a+a\ \alpha\ b\ \beta\ c+c\ \alpha\ b\ \beta\ a+c\ \alpha\ b\ \beta\ c)$$

$$=\phi_n(a\ \alpha\ b\ \beta\ a)+\phi_n(a\ \alpha\ b\ \beta\ c+c\ \alpha\ b\ \beta\ a)+\phi_n(c\ \alpha\ b\ \beta\ c)$$

by comparing these two expression of W,we get

$$\phi_n(a\ \alpha\ b\ \beta\ c+c\ \alpha\ b\ \beta\ a)=\sum_{i=1}^{n} \phi_i(a)\ \alpha\ \phi_i(b)\ \beta\ \phi_i(c)+\sum_{i=1}^{n} \phi_i(c)\ \alpha\ \phi_i(b)\ \beta\ \phi_i(a)$$

Theorem2.5 *:-Let $\theta=(\Phi_i)_{i\in N}$ be a Jordan Higher Homomorphism from a Gamma ring M into a 2-torsion free completely prime Gamma ring N then θ is either a higher homomorphism or an anti-higher homomorphism*

Proof:-*Since*

$$\phi_n(a\alpha b\beta c+c\alpha b\beta a)=\sum_{i=1}^{n}\phi_i(a)\alpha\phi_i(b)\beta\phi_i(c)+\sum_{i=1}^{n}\phi_i(c)\alpha\phi_i(b)\beta\phi_i(a)$$

Replace c by $a\alpha b$

$$W=\phi_n(a\alpha b\beta(a\alpha b)+(a\alpha b)\alpha b\beta a)$$

$$=\sum_{i=1}^{n}\phi_i(a)\alpha\phi_i(b)\beta\phi_i(a\alpha b)+\sum_{i=1}^{n}\phi_i(a\alpha b)\alpha\phi_i(b)\beta\phi_i(a)$$

on the other hand

$$W=\phi_n(a\alpha b\beta(a\alpha b)+(a\alpha b)\alpha b\beta a)$$

$$=\phi_n((a\alpha b)\beta(a\alpha b))+\phi_n(a\alpha(b\beta b)\alpha a)$$

$$=\sum_{i=1}^{n}\phi_i(a\alpha b)\beta\phi_i(a\alpha b)+\sum_{i=1}^{n}\phi_i(a)\alpha\phi_i(b\beta b)\beta\phi_i(a)$$

By comparing these two expression of W,we get

$$\sum_{i=1}^{n}(\phi_i(a\alpha b)-\phi_i(a)\alpha\phi_i(b))\beta\phi_i(a\alpha b)-$$

$$\sum_{i=1}^{n}(\phi_i(a\alpha b)-\phi_i(a)\alpha\phi_i(b))\beta\phi_i(b)\alpha\phi_i(a)=0$$

Then

$$\sum_{i=1}^{n}(\phi_i(a\alpha b)-\phi_i(a)\alpha\phi_i(b))\beta(\sum_{i=1}^{n}(\phi_i(a\alpha b)-\phi_i(b)\alpha\phi_i(a))=0$$

Since N is completely prime Gamma ring

then either $\sum_{i=1}^{n}(\phi_i(a\alpha b)-\phi_i(a)\alpha\phi_i(b))=0$ *or*

$$\sum_{i=1}^{n}(\phi_i(a\alpha b)-\phi_i(b)\alpha\phi_i(a))=0$$

if $\sum_{i=1}^{n}\phi_i(a\alpha b)-\sum_{i=1}^{n}\phi_i(a)\alpha\phi_i(b)=0\Rightarrow$

$$\phi_n(a\alpha b)-\sum_{i=1}^{n}\phi_i(a)\alpha\phi_i(b)=0$$

$\Rightarrow\theta$ *is HH*

and if $\sum_{i=1}^{n} \phi_i(a\ \alpha\ b)$**-** $\sum_{i=1}^{n} \phi_i(b)\ \alpha\ \phi_i(a)$**=0**

$\Rightarrow \phi_n(a\ \alpha\ b) - \sum_{i=1}^{n} \phi_i(b)\ \alpha\ \phi_i(a) = 0 \Rightarrow \theta$ *is A HH.*

3-Generalized Jordan Higher Homomorphism on Completely Prime Gamma Rings

<u>Definition 3.1</u>:-Let F= $(f_i)_{i\in N}$ ***be a family of additive mappings of a*** Γ***-ring M into a*** Γ***-ring N,F is said to be***

-a generalized higher homomorphism on Gamma ring (GHH,for short) if there exist a HH $\theta = (\Phi_i)_{i\in N}$ ***of M into N such that for every n*** $\in$***N,we have***

$$f_n(a\alpha b) = \sum_{i=1}^{n} f_i(a)\ \alpha\ \phi_i(b) \qquad \forall\ a,b\in M,\ \alpha \in \Gamma.$$

<u>Definition 3.2</u>:-Let F= $(f_i)_{i\in N}$ ***be a family of additive mappings of a*** Γ***-ring M into a*** Γ***-ring N,F is said to be***

-a generalized anti- higher homomorphism on Gamma ring (GAHH,for short) if there exist an AHH $\theta = (\Phi_i)_{i\in N}$ ***of M into N such that for every n*** $\in$***N,we have***

$$f_n(a\alpha b) = \sum_{i=1}^{n} f_i(b)\ \alpha\ \phi_i(a) \qquad \forall\ a,b\in M,\ \alpha \in \Gamma.$$

<u>Definition 3.3</u>:-Let F= $(f_i)_{i\in N}$ ***be a family of additive mappings of a*** Γ***-ring M into a*** Γ***-ring N,F is said to be***

-a generalized Jordan higher homomorphism on Gamma ring (GJHH,for short) if there exist a JHH $\theta = (\Phi_i)_{i\in N}$ ***of M into N such that for every n*** $\in$***N,we have***

$$f_n(a\alpha b + b\alpha a) = \sum_{i=1}^{n} f_i(a)\ \alpha\ \phi_i(b) + \sum_{i=1}^{n} f_i(b)\ \alpha\ \phi_i(a) \qquad \forall\ a,b\in M,\ \alpha \in \Gamma.$$

We should mentioned the reader that $\phi_i(a)\ \alpha\ \phi_j(b)=0\quad \forall\, a,b\in M,\ \alpha\in\Gamma$ and $a\alpha b\ \beta c=a\ \beta b\alpha c\quad \forall\, a,b,c\in M$(resp.,N), $\alpha,\beta\in\Gamma$ and so $f_i(a)\ \alpha\ \phi_j(b)=0$ $\forall\, a,b\in M,\ \alpha\in\Gamma$ will be used freely in this section .

***Lemma 3.4**:- Let F=$(f_i)_{i\in N}$ be a GJHH of a Γ-ring M into a Γ-ring N and $\theta=(\Phi_i)_{i\in N}$ be the relating JHH ,then*

(i) If N is 2-torsion free Γ-ring,then

$$f_n(a\alpha b\beta a)=\sum_{i=1}^{n} f_i(a)\ \alpha\ \phi_i(b)\ \beta\ \phi_i(a)$$

(ii) $f_n(a\ \alpha b\ \beta c+c\ \alpha b\ \beta a)=\sum_{i=1}^{n} f_i(a)\ \alpha\ \phi_i(b)\ \beta\ \phi_i(c)+\sum_{i=1}^{n} f_i(c)\ \alpha\ \phi_i(b)\ \beta\ \phi_i(a)$

***Proof**:-by the same technique of the proof[Lemma2.4] we can prove this lemma◙*

***Theorem3.5** :- Let F=$(f_i)_{i\in N}$ be a generalized Jordan Higher Homomorphism from a Gamma ring M into a 2-torsion free completely prime Gamma ring N then F is either a generalized higher homomorphism or an generalized anti-higher homomorphism.*

***Proof**:-Since*

$$f_n(a\alpha b\beta c+c\ \alpha b\beta a)=\sum_{i=1}^{n} f_i(a)\alpha\ \phi_i(b)\beta\ \phi_i(c)+\sum_{i=1}^{n} f_i(c)\alpha\ \phi_i(b)\beta\ \phi_i(a)$$

replace a by $a\alpha b$ and c by $b\alpha a$

$$W=f_n((a\ \alpha b)\ \alpha x\ \beta(b\alpha a)+(b\alpha a)\ \alpha x\ \beta(a\alpha b))$$

$$=\sum_{i=1}^{n} f_i(a\alpha b)\ \alpha\ \phi_i(x)\ \beta\ \phi_i(b\alpha a)+\sum_{i=1}^{n} f_i(b\alpha a)\ \alpha\ \phi_i(x)\ \beta\ \phi_i(a\alpha b)$$

On the other hand

$$W=f_n(a\ \alpha b\alpha x\ \beta(b\alpha a)+(b\alpha a\ \alpha x\ \beta a\alpha b)$$

$$=\sum_{i=1}^{n} f_i(a)\alpha\ \phi_i(b)\ \alpha\ \phi_i(x)\beta\ \phi_i(b)\ \alpha\ \phi_i(a)+$$

$$\sum_{i=1}^{n} f_i\ (b)\alpha\ \phi_i(\ a)\alpha\ \phi_i(x)\ \beta\ \phi_i(a)\alpha\ \phi_i(b)$$

By comparing these two expression of W ,we get

$$\sum_{i=1}^{n}\ (\ f_i\ (\ a\ \alpha\ b)\ \alpha\ \phi_i(x)\ \beta\ \phi_i(b\alpha\ a)+\sum_{i=1}^{n}\ f_i\ (b\alpha\ a)\ \alpha\ \phi_i(x)\ \beta\ \phi_i(\ a\ \alpha\ b)-$$

$$\sum_{i=1}^{n}\ f_i(a)\ \alpha\ \phi_i(b)\ \alpha\ \phi_i(x)\beta\ \phi_i(b)\ \alpha\ \phi_i(a)-\sum_{i=1}^{n}\ f_i(b)\ \alpha\ \phi_i(a)\ \alpha\ \phi_i(x)\ \beta\ \phi_i(a)\alpha$$

$\phi_i(b)=0$

Since $\theta=(\Phi_i)_{i\in N}$ *is JHH on completely prime Gamma ring then* θ *is either a HH or AHH by[Theorem 2.5]*

Case 1:-*if* θ *is a HH ,then if we suppose that*

$$a_b=\sum_{i=1}^{n}\ f_i\ (\ a\ \alpha\ b)-\sum_{i=1}^{n}\ f_i(b)\ \alpha\ \phi_i(a))$$

and $a^b=\sum_{i=1}^{n}\ f_i\ (\ a\ \alpha\ b)-\sum_{i=1}^{n}\ f_i(a)\ \alpha\ \phi_i(b)$

$$(\sum_{i=1}^{n}\ f_i\ (\ a\ \alpha\ b)-\sum_{i=1}^{n}\ f_i(b)\ \alpha\ \phi_i(a))\ \beta\ \phi_i(b\alpha\ a\text{-}a\alpha\ b)+$$

$$(\sum_{i=1}^{n}\ f_i\ (\ a\ \alpha\ b)-\sum_{i=1}^{n}\ f_i(a)\ \alpha\ \phi_i(b))\ \beta\ \phi_i(a\alpha\ b\text{-}b\alpha\ a)=0$$

$\Rightarrow a_b\ \beta\ \phi_i(b\alpha\ a\text{-}a\alpha\ b)+a^b\ \beta\ \phi_i(a\alpha\ b\text{-}b\alpha\ a)=0$

and it is easy to see that

$a^b=-a_b$

$2a_b\ \beta\ \phi_i(b\alpha\ a\text{-}a\alpha\ b)=0$

Since N is 2-torsion free Gamma ring ,then

$a_b\ \beta\ \phi_i(b\alpha\ a\text{-}a\alpha\ b)=0$

Since N is completely prime Gamma ring then either $a_b=0$ *or* $\phi_i(b\alpha\ a\text{-}a\alpha\ b)=0$

If $\phi_i(b\alpha\ a\text{-}a\alpha\ b)=0$ *then* $\theta=(\Phi_i)_{i\in N}$ *is a higher anti-homomorphism which is contradictions with* $\theta=(\Phi_i)_{i\in N}$ *is Higher homomorphism then* $a_b=0$ *and so F is Generalized anti-higher homomorphism .*

***Case2*:- by the same way ,if $\theta = (\Phi_i)_{i\in N}$ is anti- higher homomorphism we have F is a Generalized higher homomorphism .**

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