GENERALIZED JORDAN LEFT DERIVATION ON SOME GAMMA RING

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ABSTRACT

In this paper we define a Generalized Jordan left derivation on Γ-ring and show that the existence of a non-zero generalized Jordan left derivation D on a completely prime Γ-ring implies D is a generalized left derivation .Furthermore we show that every generalized Jordan left derivation on Γ-ring has a commutator left non-zero divisor is a generalized left derivation on Γ-ring.

***Key wards :** Γ-ring, prime Γ-ring, prime Γ-ring , left derivation, ,Jordan left derivation, generalized Jordan left derivation .*

1-INTRODUCTION

An additive mapping d:R→R is called a left derivation (resp.,Jordan left derivation)if d(xy)=xd(y)+yd(x)(resp.,d(x^2)=2xd(x)) holds for all x,y$\in$R.Clearly, every left derivation is a Jordan left derivation,Thus ,it is natural to ask that :whether every Jordan left derivation on a ring is a left derivation? In [1] authors answered the above question in case the underlying ring R is 2-torsion free and prime and in [5] Rajaa c.Shaheen,answered the above question in case the underlying ring R is a 2-torsion free and has a commutator left non-zero divisor and define the concept of generalized Jordan left derivation and Generalized left derivation as follows:

Let R be a ring,and let δ:R→R be an additive map,if there is a left derivation (resp.,Jordan left derivation)d: R→R such that δ(xy)=x δ(y)+yd(x) (resp., δ(x^2)=x δ(x)+xd(x) for all x,y$\in$R,then δ is called a generalized left derivation and d is called the relating left derivation(resp., then δ is called a generalized Jordan left derivation and d is called the relating Jordan left derivation).and study the same problem .In this paper we define a generalized Jordan left derivation and we show that the existence of a non-zero generalized Jordan left derivation D:M→M on a completely prime Γ-ring M which satisfy the condition

$x\alpha y\beta z=x\beta y\alpha z$ for all x,y,z$\in$M and $\alpha,\beta\in\Gamma$ implies D is a generalized left derivation and ***we show that every generalized Jordan left derivation on Γ-ring has a commutator left non-zero divisor is a generalized left derivation.***

Let M and Γ be additive abelian groups, M is called a Γ-ring if for any x,y,z $\in$M and α, β $\in$ Γ,the following conditions are satisfied

(1) $x\alpha y\in M$
(2)$(x+y)\alpha z=x\alpha z+y\alpha z$
$x(\alpha+\beta)z=x\alpha z+x\beta z$
$x\alpha(y+z)=x\alpha y+x\alpha z$
(3)$(x\alpha y)\beta z=x\alpha(y\beta z)$

The notion of Γ-ring was introduced by Nobusawa[4] and generalized by Barnes[2],many properties of Γ-ring were obtained by many researcheres.
M is called a 2-torsion free if 2x=0 implies x=0 for all x$\in$M.AΓ-ring M is called prime if aΓMΓb=0 implies a=0 or b=0 and M is called completely prime if a Γb=0 implies a=0 or b=0(a,b $\in$M),Since a Γb Γa Γb $\subset$a ΓM Γb ,then every completely prime Γ-ring is prime. In [3] Y.Ceven defined a Jordan left derivation as follows

<u>Definition 1.1</u> :-*Let M be a Γ-ring and let** ***d:M→M be an additive map.d is called a Left derivation if for any a,b$\in$ M and $\alpha\in\Gamma$,

$d(a\alpha b)=a\alpha d(b)+b\alpha d(a)$,

d is called a Jordan left derivation if for any a$\in$M and $\alpha\in\Gamma$,

$d(a\alpha a)=2a\alpha d(a)$.

In this paper ,we generalized the above definition by giving the following definition

<u>Definition 1.2</u>:- Let M be a Γ-ring and let D:M→M ***be an additive map.Then D is called a <u>Generalized left derivation</u> if there exist a left derivation*** d:M→M such that

$D(a\alpha b)=a\alpha D(b)+b\alpha d(a)$,for all a,b$\in$ M ***and $\alpha\in\Gamma$,***

<u>Definition 1.3</u>:- Let M be a Γ-ring and let D:M→M be an additive map.Then D is ***called a <u>Generalized Jordan left derivation</u> if there exist a Jordan left derivation*** d:M→M such that

$D(a\alpha a)=a\alpha D(a)+a\alpha d(a)$,for all a$\in$ M ***and $\alpha\in\Gamma$,***

2. RESULT

***Lemma 2.1**:- Let M be a Γ-ring, $D:M\rightarrow M$ be a Generalized Jordan left derivation and $d:M\rightarrow M$ be the relating Jordan left derivation then the following statements hold:*

(i) $D(a\alpha b+b\alpha a)=a\alpha D(b)+b\alpha D(a)+a\alpha d(b)+b\alpha d(a)$ *for all* $a,b\in M$ *and* $\alpha\in\Gamma$,

Especially if M is 2-torsion free and $a\alpha b\beta c=a\beta b\alpha c$, *for all* $a,b,c\in M$ *and* $\alpha,\beta\in\Gamma$, *then*

(ii) $D(a\alpha b\beta a)=a\alpha b\beta D(a)+2a\alpha b\beta d(a)+a\alpha a\beta d(b)-b\alpha a\beta d(a)$

(iii) $D(a\alpha b\beta c+c\alpha b\beta a)=a\alpha b\beta D(c)+c\alpha b\beta D(a)+2a\alpha b\beta d(c)+2c\alpha b\beta d(a)+a\alpha c\beta d(b)+c\alpha a\beta d(b)-b\alpha a\beta d(c)-b\alpha c\beta d(a)$.

***Proof**:-(i)Since D is a Generalized Jordan left derivation then*

$D(a\alpha a)=a\alpha D(a)+a\alpha d(a)$ *,for all* $a\in M$ *and* $\alpha\in\Gamma$,……………(1)

By linearizing (1),we get for all $a,b\in M$ *and* $\alpha\in\Gamma$,

$D(a\alpha b+b\alpha a)=a\alpha D(b)+b\alpha D(a)+a\alpha d(b)+b\alpha d(a)$,…………….(2)

(ii)In (2) replace b by $a\beta b+b\beta a$, $\beta\in\Gamma$

$W= D(a\alpha (a\beta b+b\beta a) +(a\beta b+b\beta a) \alpha a)$

$=a\alpha D(a\beta b+b\beta a)+(a\beta b+b\beta a)\alpha D(a)+$

$a\alpha d(a\beta b+b\beta a)+(a\beta b+b\beta a)\alpha d(a)$

$=a\alpha a\beta D(b)+a\alpha b\beta D(a)+a\alpha a\beta d(b)+a\alpha b\beta d(a)+a\beta b\alpha D(a)+b\beta a\alpha D(a)+2a\alpha a\beta d(b)+2a\alpha b\beta d(a)+a\beta b\alpha d(a)+b\beta a\alpha d(a)$

on the other hand,

$W= D(a\alpha (a\beta b+b\beta a) +(a\beta b+b\beta a) \alpha a)$

$=D(a\alpha a\beta b+2a\alpha b\beta a+b\beta a\alpha a)$

$= D(a\alpha a\beta b+ b\beta a\alpha a)+2D(a\alpha b\beta a)$

$= a\alpha a\beta D(b)+b\beta D(a\alpha a)+ a\alpha a\beta d(b)+b\beta d(a\alpha a) +2D(a\alpha b\beta a)$

$=a\alpha a\beta D(b)+b\beta a\alpha D(a)+b\beta a\alpha d(a)+ a\alpha a\beta d(b)+2b\beta a\alpha d(a)+2D(a\alpha b\beta a)$

then by comparing these two expression of W and by using the fact of 2-torsion free ring ,we get

$D(a\alpha b\beta a)=a\alpha b\beta D(a)+2a\alpha b\beta d(a)+a\alpha a\beta d(b)-b\alpha a\beta d(a)$………..(3 *(iii)by linearizing (3) we find that*

$D(a\alpha b\beta c+c\alpha b\beta a)=a\alpha b\beta D(c)+c\alpha b\beta D(a)+2a\alpha b\beta d(c)+$

$2c\alpha b\beta d(a)+a\alpha c\beta d(b)+c\alpha a\beta d(b)-b\alpha a\beta d(c)-b\alpha c\beta d(a)$…….(4)

Now we shall give the following lemma which is necessary to prove [lemma 2.3]

Lemma 2.2*:-let M be a 2-torsion free Γ-ring and* $d:M\rightarrow M$ *is a Jordan left derivation on M and* $a\alpha b\beta c=a\beta b\alpha c$ *for all* $a,b,c\in M$ *and* $\alpha,\beta\in\Gamma$ *then*

$(a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)\beta d(b)=0.$

Proof*:- From [3 ,Lemma 2.2,i],we have*

$(a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)\beta d(a)=0$…………….(5)

by replacing a by a+b in (5),we find that

$((a+b)\alpha(a+b)\alpha b-2(a+b)\alpha b\alpha(a+b)+b\alpha(a+b)\alpha(a+b))\beta d(a+b)=0$

$(a\alpha a\alpha b+b\alpha a\alpha b+a\alpha b\alpha b+b\alpha b\alpha b-2a\alpha b\alpha a-2a\alpha b\alpha b-2b\alpha b\alpha a-2b\alpha b$
$\alpha b+b\alpha a\alpha a+b\alpha b\alpha b+b\alpha a\alpha b+b\alpha b\alpha a)\beta(d(a)+d(b))=0$

$(a\alpha a\alpha b+2b\alpha a\alpha b-2a\alpha b\alpha a-a\alpha b\alpha b-b\alpha b\alpha a+b\alpha a\alpha a)\beta(d(a)+d(b)=0$

$((a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)-(b\alpha b\alpha a-2b\alpha a\alpha b+a\alpha b\alpha b))\beta d(a)+$

$((a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)-(b\alpha b\alpha a-2b\alpha a\alpha b+a\alpha b\alpha b))\beta d(b)=0$

and since $(a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)\beta d(a)=0$*,we get*

$-(b\alpha b\alpha a-2b\alpha a\alpha b+a\alpha b\alpha b)\beta d(a)+(a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)\beta d(b)=0$…………..(6)

since

$0=d([a,b]\beta[a,b])$

$=d(a\beta(b\alpha a\alpha b)+(b\alpha a\alpha b)\beta a)-d(a\alpha(b\alpha b)\alpha a)-d(b\alpha(a\alpha a)\alpha b)$

$=2(a\beta d(b\alpha a\alpha b)+(b\alpha a\alpha b)\beta d(a))-a\alpha a\beta d(b\alpha b)-3a\alpha b\alpha b\beta d(a)+b$
$\alpha b\alpha a\beta d(a)-b\alpha b\beta d(a\alpha a)-3b\alpha a\alpha a\beta d(b)+a\alpha a\alpha b\beta d(b)$

$=-3(a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)\beta d(b)-(a\alpha b\alpha b-2b\alpha a\alpha b+b\alpha b\alpha a)\beta d(a)$

and hence

$(a\alpha b\alpha b-2b\alpha a\alpha b+b\alpha b\alpha a\beta d(a)+3(a\alpha a\alpha b-a\alpha b\alpha a+b\alpha a\alpha a)\beta d(b)=0$

………………………(7)

from(6) and (7) ,we get

$4(a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)\beta d(b)=0$

and since M is 2-torsion free Γ*-ring,then*

$(a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)\beta d(b)=0$◙

Lemma2.3:- *Let M be a 2-torsion free Γ-ring which satisfy the condition $a\alpha b\beta c=a\beta b\alpha c$, for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$, then $D:M\to M$ be a Generalized Jordan left derivation and $d:M\to M$ be the relating Jordan left derivation then*

$[a,b]\beta(D(b\alpha a)-b\alpha D(a)-a\alpha d(b))=0$

Proof:- In (4) replace c by $[a,b]=a\alpha b-b\alpha a$

$Y=D(a\alpha b\beta[a,b]+[a,b]\alpha b\beta a)$

$=a\alpha b\beta D([a,b])+[a,b]\alpha b\beta D(a)+2a\alpha b\beta d([a,b])+2[a,b]\alpha b\beta d(a)+a\alpha[a,b]$

$\beta d(b)+[a,b]\alpha a\beta d(b)-b\alpha a\beta d([a,b])-b\alpha[a,b]\beta d(a)$

$=a\alpha b\beta D(a\alpha b)-a\alpha b\beta D(b\alpha a)+[a,b]\alpha b\beta D(a)+2a\alpha b\beta d([a,b])+$

$2[a,b]\alpha b\beta d(a)+a\alpha[a,b]\beta d(b)+[a,b]\alpha a\beta d(b)-b\alpha a\beta d([a,b])-b\alpha[a,b]\beta d(a)$

since $[a,b]\beta d([a,b])=0$ from [3,lemma 2.2,(iii)]

$Y=a\alpha b\beta D(a\alpha b)-a\alpha b\beta D(b\alpha a)+[a,b]\alpha b\beta D(a)+a\alpha b\beta d([a,b])+$

$2[a,b]\alpha b\beta d(a)-b\alpha[a,b]\beta d(a)+a\alpha[a,b]\beta d(b)+[a,b]\alpha a\beta d(b)$

On the other hand,

$Y=D(a\alpha b\beta[a,b]+[a,b]\alpha b\beta a)$

$=D(a\alpha b\beta a\alpha b-a\alpha b\beta b\alpha a+a\alpha b\beta b\alpha a-b\alpha a\beta b\alpha a)$

$=D(a\alpha b\beta a\alpha b)-D(b\alpha a\beta b\alpha a)$

$=a\alpha b\beta D(a\alpha b)+a\alpha b\beta d(a\alpha b)-b\alpha a\beta D(b\alpha a)+b\alpha a\beta d(b\alpha a)$

Then by comparing these two expression of Y and since from [3,Lemma2.2],we have

$[a,b]\beta d(a\alpha b)=a\alpha[a,b]\beta d(b)+b\alpha[a,b]\beta d(a)$………………..(8)

then we get

$-[a,b]\beta(D(b\alpha a)-b\alpha D(a)-a\alpha d(b))+a\alpha b\beta d([a,b])+2[a,b]\alpha b\beta d(a)-$
$b\alpha[a,b]\beta d(a)+[a,b]\beta d(a\alpha b)-b\alpha[a,b]\beta d(a)-$
$a\alpha b\beta d(a\alpha b)+b\alpha a\beta d(b\alpha a)=0$

then

$-[a,b]\beta(D(b\alpha a)-b\alpha D(a)-a\alpha d(b))+a\alpha b\beta d([a,b])+2[a,b]\alpha b\beta d(a)-2b\alpha[a,b]$

$\beta d(a)+a\alpha b\beta d(a\alpha b)-b\alpha a\beta d(a\alpha b)-a\alpha b\beta d(a\alpha b)+b\alpha a\beta d(b\alpha a)=0$

$-[a,b]\beta(D(b\alpha a)-b\alpha D(a)-a\alpha d(b))+[a,b]\beta d([a,b])+2[a,b]\alpha b\beta d(a)$

$-2b\alpha [a,b] \beta d(a)=0$

by [3,Lemma2.2,iii],we have

$[a,b] \beta d([a,b])=0$ for all $a,b\in M$ and $\alpha,\beta \in \Gamma$.

Then

$$-[a,b] \beta (D(b \alpha a)-b \alpha D(a)-a \alpha d(b))=2(-[a,b] \alpha b+b \alpha [a,b]) \beta d(a)$$
$$=2(b \alpha [a,b] -[a,b] \alpha b) \beta d(a)$$

since

$$(b \alpha [a,b] -[a,b] \alpha b) \beta d(a)=[a,b] \beta d(a\alpha b)-a\alpha [a,b] \beta d(b)-[a,b] \alpha b\beta d(a)$$
$$=[a,b] \beta (d(a\alpha b)-b \alpha d(a))- a\alpha [a,b] \beta d(b)$$
$$= [a,b] \beta a \alpha d(b)-a \alpha [a,b] \beta d(b)$$
$$=([a,b] \alpha a-a \alpha [a,b]) \beta d(b)$$

then

$$-[a,b] \beta (D(b \alpha a)-b \alpha D(a)-a \alpha d(b))=-2(a \alpha [a,b]- [a,b] \alpha a) \beta d(b)$$
$$=-2(a\alpha a\alpha b-2a\alpha b\alpha a+b\alpha a\alpha a)\beta d(b)\text{…..(9)}$$

so by [Lemma 2.2],we get

$$[a,b] \beta (D(b \alpha a)-b \alpha D(a)-a \alpha d(b))=0\text{………………..(10)}$$

Theorem2.4:- *let M be a 2-torsion free Γ-ring has a commuator left non-Zero divisor and $a\alpha b\beta c=a\beta b\alpha c$ for all $a,b,c\in M$ and $\alpha,\beta \in \Gamma$ and $D:M\rightarrow M$ is a Generalized Jordan left derivation on M* and *$d:M\rightarrow M$ is the relating Jordan left derivation Then D is a Generalized left derivation on M.*

Proof:- *if we suppose that*

$G(b,a)= D(b \alpha a)-b \alpha D(a)-a \alpha d(b)$

Then (10) becomes

$[a,b] \beta G(b,a)=0$ for all $a,b\in M$ and $\alpha,\beta \in \Gamma$ …………(11)

Since M has a commuator left non-Zero divisor then $\exists x,y \in M$ and $\alpha,\beta \in \Gamma$

such that $[x,y] \beta c=0$ implies that $c=0$

It is easy to see that from (11),

$[x,y] \beta G(y,x)=0$ and so $G(y,x)=0$

In (11) replace a by a+x,then we get

$[x,b]\ \beta\ G(b,a)+[a,b]\ \beta\ G(b,x)=0$……………………(12)

replace b by b+y in (12),then we get

$[x,y]\ \beta\ G(b,a)=0 \Rightarrow G(b,a)=0$

i.e $D(b\ \alpha\ a)=b\ \alpha\ D(a)+a\ \alpha\ d(b)$

$\Rightarrow$D is a Generalized left derivation on M.

Theorem 2.5:- *let M be a 2-torsion free completely prime Γ-ring and $a\alpha b\beta c=a\beta b\alpha c$ for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$ and $D:M\rightarrow M$ is a Generalized Jordan left derivation on M .Then D is a Generalized left derivation on M.*

***Proof**:- Since $[a,b]\ \beta\ G(b,a)=0$ for all $a,b\in M$ and $\alpha,\beta\in\Gamma$*

And since M is completely prime Γ-ring,then either $[a,b]=0$ or $G(b,a)=0$

If $[a,b]=0$ for all $a,b\in M$ and $\alpha\in\Gamma\Rightarrow M$ is commutative and so D is a Generalized left derivation on M .if $G(b,a)=0$ then D is a Generalized left derivation on M.

REFERENCES

[1]Ashraf,M.,Rehman,N.:*On Lie ideals and Jordan left derivations of prime* rings,Arch.Math.(Brno)36,201-206(2000)

[2]W.E.Barnes"*On the Γ-rings of Nobusawa*"Pacific J.Math.,18(1966),411-422.

[3] Y.Ceven,"*Jordan left derivations on completely prime Gamma rings*",Fenbilimleri Dergisi(2002)cilt 23say12.

[4]N. Nobusawa."*on a generalization of the ring theory* ,Osaka J.Math.,1(1964)

[5] Rajaa.C.Shaheen,"*Generalized Jordan homomorphisms and Jordan left derivation on some rings*,M.S.C thesis,2005.