

The relation Between the Rational Valued Character And Artin Character of the Quaternion Group

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المستخلص

الهدف من هذا البحث هو التعبير عن كل شاخص من شواخص الزمرة Q_{2n} بصيغة تركيب خطي للشواخص المحتثة من الشواخص الأحادية $1_{H_i}^{Q_{2n}}$ حيث ان معاملات أعداد صحيحة وان H_i زمر جزئية دائرية من الزمرة Q_{2n} .

Abstract

In this research, we present the relation between the rational valued character and Artin character of the group Q_{2n} when n is odd, and the main result is the following: Every rational valued characters of quaterion group can be written as Z -linear combination of induced unit characters $1_{H_i}^{Q_{2n}}$, with H_i runs over all cyclic subgroups of Q_{2n} .

Introduction:

Let Q_{2n} be a quaternions group and Q be a rational field, two element x and y of Q_{2n} are called Q -conjugate if the cyclic group generated by x is conjugate to the cyclic group generated by y . this defines an equivalence relation on Q_{2n} their equivalence classes are called Γ -class of Q_{2n} . let $cf(G, Z)$ be the set of all Z -valued class function of G which are constant on Q -classes. Let $R(Q, G)$ be the intersection of $cf(G, Z)$ with $R(G)$ the group of generalized characters of G . $R(Q, G)$ is a finitely generated Z -module with bases Q -characters $\sum_{\sigma \in \text{Gal}(Q(x_i)/Q)} \chi_i^\sigma$, $\chi_i \in \text{Irr}(G)$ and rank m . we will denote by $K(G)$ the factor group $cf(G, Z)/R(Q, Z)$.

Preliminaries:

Definition 1: Let H be a cyclic subgroup of group G . Given ϕ a character of H and $g \in G$ and let $x_1, x_2, \dots, x_m$ be m conjugate classes of H contained in $cl(g)$ in G then:

$$\phi^G(g) = \frac{|C_G(g)|}{|C_H(g)|} \sum_{i=1}^m \phi(x_i), \text{ where } C(g) \text{ is the centralizer of } g. \quad \dots (*)$$

Definition 2: Let G be a finite group. All characters of G induced from a principle character of cyclic subgroup of G are called Artin character of G .

Definition 3: For each positive integer n the general quatenin group of order $4n$ can be defined as follows:

$$Q_{2n} = \{ \langle x, y \rangle : x^{2n} = y^4 = 1, yxy^{-1} = x^{-1} \mid$$

Theorem 1: Let G be a finite group then

$$|K(G)| = \left[\frac{\prod_{i=1}^m n_i |N(H_i)|}{\phi(|H_i|)} \right]^{\frac{1}{2}}. [3]$$

Theorem 2: Let G be a finite group, $\{H_i = \langle x_i \rangle, 1 \leq i \leq m\}$ be a full set of non-conjugate cyclic group of G and n_i be the number of conjugate cyclic subgroup of G contained in the Q -class Γ_i , then each rational valued character of G can be written as a $\mathbb{Z}$ -linear combination of $1_{H_i}^G$ if and only if

$$\left(\prod_{i=1}^m \frac{n_i}{\phi(|H_i|)} \right)^{\frac{1}{2}} = \prod_{i=1}^m \frac{(|N(H_i)|)^{\frac{1}{2}}}{|H_i|}. [3]$$

Lemma 1: Let m be the number of Q -classes Γ of Q_{2n} , when n is odd number then $m = m_i + 1$ where m_i is the number of Q -classes Γ_i of the cyclic subgroup $H_{2n} = \langle x \rangle$. [4]

Corollary 1: $K(Q_{2n}) = K(H_{2n}) \oplus H_4$ where n is odd number. [4]

The main result:

A well-known theorem due to Artin asserts, if G is a finite groups and θ is a rational valued character of G , then

$$\theta = \sum_{i=1}^m a_i 1_{H_i}^{Q_{2n}}, a_i \in \mathbb{Q} \quad \dots\dots (**)$$

So according to this theorem each rational valued character χ of Q_{2n} can be written as:

$$\chi = \sum_{i=1}^m a_i 1_{H_i}^{Q_{2n}}, a_i \in \mathbb{Q}$$

In this research, we have proved that each rational valued character χ of Q_{2n} , which is expressible as in (**) with $a_i \in \mathbb{Z}$ when n odd,

$$\text{i.e. } \chi = \sum_{i=1}^m a_i 1_{H_i}^{Q_{2n}}, a_i \in \mathbb{Z}$$

Proof: Let n be odd number, by theorem (1) we have:

$$|K(Q_{2n})| = \left(\prod_{i=1}^m \frac{n_i |N(H_i)|}{\phi(|H_i|)} \right)^{\frac{1}{2}} \quad \dots\dots (1)$$

And by corollary (1) we get

$$|K(Q_{2n})| = 4 |K(H_{2n})| \quad \dots\dots (2)$$

Furthermore by equation (1) and lemma (1) we get

$$|K(Q_{2n})| = \left(\frac{\prod_{i=1}^{m_1} n_i |N(H_i)|}{\phi(|H_i|)} \right)^{\frac{1}{2}} \left(\frac{n_i |N(H_4)|}{\phi(|H_4|)} \right)^{\frac{1}{2}}$$

But $n_m=2$ and $\phi(|H_4|)=\phi(4)=2$ then

$$|K(Q_{2n})| = |K(H_{2n})| (|N(H_4)|)^{\frac{1}{2}} \quad \dots\dots (3)$$

Also by equation (2) and equation (3) we get

$$4 |K(H_{2n})| = |K(H_{2n})| (|N(H_4)|)^{\frac{1}{2}}, \text{ then } |N(H_4)|=16$$

$$\text{Hence } \prod_{i=1}^m \frac{(|N(H_i)|)^{\frac{1}{2}}}{|H_i|} = \prod_{i=1}^{m_1} \frac{(|N(H_i)|)^{\frac{1}{2}}}{|H_i|} \frac{(N(H_4))^{\frac{1}{2}}}{|H_4|}$$

and since $|H_4|=4$ thus

$$\prod_{i=1}^m \frac{(|N(H_i)|)^{\frac{1}{2}}}{|H_i|} = \prod_{i=1}^{m_1} \frac{(|N(H_i)|)^{\frac{1}{2}}}{|H_i|} \quad \dots\dots (4)$$

On other hand

$$\left(\prod_{i=1}^m \frac{n_i}{\phi(|H_i|)} \right)^{\frac{1}{2}} = \left(\prod_{i=1}^{m_1} \frac{n_i}{\phi(|H_i|)} \right)^{\frac{1}{2}} \left(\frac{n_m}{\phi(|H_4|)} \right)^{\frac{1}{2}}$$

And as above $n_m=2$ and $\phi(|H_4|)=\phi(4)=2$

Therefore

$$\left(\prod_{i=1}^m \frac{n_i}{\phi(|H_i|)} \right)^{\frac{1}{2}} = \left(\prod_{i=1}^{m_1} \frac{n_i}{\phi(|H_i|)} \right)^{\frac{1}{2}} \dots\dots (5)$$

By comparison (4) and (5) we get:

$$\left(\prod_{i=1}^m \frac{n_i}{\phi(|H_i|)} \right)^{\frac{1}{2}} = \prod_{i=1}^{m_1} \frac{(|N(H_i)|)^{\frac{1}{2}}}{|H_i|}$$

Thus, by theorem (2) we get to the result

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