On a Subclass of Multivalent Harmonic Functions Defined by a Linear Operator

Waggas Galib Atshan[1], Huda Khalid Abid Zaid[2]

[1]Department of Mathematics, College of Computer Science and Mathematics, University of Al-Qadisiya, Diwaniya-Iraq

Abstract. ***In this paper, we define a subclass of p- valent harmonic functions defined by a linear operator and study some results as coefficient inequality, convolution property and convex set.***

Keywords: Multivalent harmonic function, convolution, linear operator.

AMS Subject Classification: 30C45.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simple connected domain $D \subset \mathbb{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in D, we call h the analytic part and g the co –analytic part of f.

A necessary and sufficient condition for f to be locally univalent and sense –preserving in D is that $|h'(z)| > |g'(z)|$ in D, see Clunie and Sheil –Small [3].

Denote by $M(p)$ the class of functions $f = h + \bar{g}$ that are harmonic multivalent and sense –preserving in the unit disk $U = \{z: |z| < 1\}$. For $f = h + \bar{g} \in M(p)$, we may express the analytic function h and g as:

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k,$$

$$\text{g}(z) = \sum_{k=n+p-1}^{\infty} b_k z^k,$$

$$|b_k| < 1. \qquad (1.1)$$

Let $N(p)$ denote the subclass of $M(p)$ consisting of functions $= h + \bar{g}$, where h and g are given by:

$$f(z) = z^p + \sum_{k=n+p}^{\infty} |a_k| z^k,$$

$$\text{g}(z) = \sum_{k=n+p-1}^{\infty} |b_k| z^k,$$

$$|b_k| < 1. \qquad (1.2)$$

We introduce here a class $N_\lambda(p,\alpha)$ of harmonic functions of the form (1.1) that satisfy the inequality

$$Re\left\{\frac{z^{p-1}}{[\mathcal{L}_p(h * \emptyset_1)(z)]' - \overline{[\mathcal{L}_p(g * \emptyset_1)(z)]'}}\right\} > \alpha,$$

where $0 \le \alpha < \frac{1}{p}$, $\lambda \ge 0, p \in \mathbb{N}$ and

$$\mathcal{L}_p f(z) = \mathcal{L}_p h(z) + \overline{\mathcal{L}_p g(z)}. \qquad (1.3)$$

The operator $\mathcal{L}_p$ denotes the linear operator introduced in [6].

For h and g given by (1.1), we obtain

$$\mathcal{L}_p h(z) = z^p + \sum_{k=n+p}^{\infty}\left[\lambda\left(\frac{k}{p} - 1\right) + 1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}\, a_k z^k,$$

$$\mathcal{L}_p g(z) = -\sum_{k=n+p-1}^{\infty}\left[\lambda\left(\frac{k}{p} + 1\right) - 1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}\, b_k z^k,$$

where a_1, a_2, c_1, c_2 are positive real numbers , $\lambda \ge 0, p \in \mathbb{N}$.

Now, the convolution of h, g is given by (1.2) and

$$\phi_1(z) = z^p + \sum_{k=n+p}^{\infty} |A_k| z^k, \qquad \phi_2(z) = \sum_{k=n+p-1}^{\infty} |B_k| z^k$$

is defined by

$$(h * \phi_1)(z) = z^p + \sum_{k=n+p}^{\infty} |A_k||a_k| z^k$$

$$(g * \phi_2)(z) = \sum_{k=n+p-1}^{\infty} |B_k||b_k| z^k, \quad |b_k| < 1,$$

we further denote by $N_\lambda(p,\alpha)$ the subclass of $M_\lambda(p,\alpha)$ that satisfies the relation

$$N_\lambda(p,\alpha) = N_\lambda \bigcap M_\lambda(p,\alpha).$$

Lemma (1.1)[1]: If $\alpha \ge 0$, then $Re\, w > \alpha$ if and only if $|w - (1+\alpha)| < |w + (1-\alpha)|$, where w be any complex number.

2. Main Results

Theorem 2.1: Let $f = h + \bar{g}$ (h and g are given by (1.1)). If

$$\sum_{k=n+p}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p} - 1\right) + 1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k| + \sum_{k=n+p-1}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p} + 1\right) - 1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k| \le p, \qquad (2.1)$$

where $(0 \le \alpha < \frac{1}{p}, \lambda \ge 0, p \in \mathbb{N}, z \in U)$, then f is harmonic p –valent sense –preserving in U and $f \in M_\lambda(p,\alpha)$.

Proof. Let

$$w(z) = \left\{\frac{z^{p-1}}{[\mathcal{L}_p(h * \emptyset_1)(z)]' - \overline{[\mathcal{L}_p(g * \emptyset_1)(z)]'}}\right\} = \frac{A(z)}{B(z)}.$$

By using the fact that in Lemma (1.1) $Re(w) \ge \alpha$ if and only if $|w - (1+\alpha)| < |w + (1-\alpha)|$, it is sufficient to show that

$$|A(z) - (1+\alpha)B(z)| - |A(z) + (1-\alpha)B(z)| \le 0. \qquad (2.2)$$

Substituting for $A(z)$ and $B(z)$ the appropriate expressions (2.2) , we get

$$|A(z)-(1+\alpha)B(z)|-|A(z)+(1-\alpha)B(z)|$$

$$= \left| z^{p-1}-(1+\alpha)\left[pz^{p-1} + \sum_{k=n+p}^{\infty} k\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k|z^{k-1} - \sum_{k=n+p-1}^{\infty} k\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k|z^{k-1}\right]\right|$$

$$-\left| z^{p-1}+(1-\alpha)\left[pz^{p-1} + \sum_{k=n+p}^{\infty} k\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k|z^{k-1} - \sum_{k=n+p-1}^{\infty} k\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k|z^{k-1}\right]\right|$$

$$\le \sum_{k=n+p}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k| + \sum_{k=n+p-1}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k| - p \le 0,$$

by inequality (2.1), which implies that $f\in N_\lambda(p,\alpha)$. The harmonic functions

$$f(z) = z^p + \sum_{k=n+p}^{\infty} \frac{x_k}{k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}} z^k + \sum_{k=n+p-1}^{\infty} \frac{\overline{y_k}}{k\alpha\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}} (\bar z)^k, \qquad (2.3)$$

where

$$\sum_{k=n+p}^{\infty}|x_k| + \sum_{k=n+p-1}^{\infty}|\overline{y_k}| = p,$$

show that the coefficients bounds given by (2.1) is sharp. The function of the form (2.3) are in $M_\lambda(p,\alpha)$ because in view of (2.3) we infer that

$$\sum_{k=n+p}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k| + \sum_{k=n+p-1}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k| = \sum_{k=n+p}^{\infty}|x_k| + \sum_{k=n+p-1}^{\infty}|\overline{y_k}| = p.$$

Now , we need to prove that the condition (2.1) is also necessary for function of (1.2) to be in the class $N_\lambda(p,\alpha)$.

Theorem 2.2. Let $f=h+\bar g$ (h and g are given by (1.2)). Then $f\in N_\lambda(p,\alpha)$ if and only if

$$\sum_{k=n+p}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k| + \sum_{k=n+p-1}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k| \le p,$$

where $(0\le\alpha<\frac{1}{p}, \lambda\ge 0, p\in\mathbb{N}, z\in U)$.

Proof. By notation $N_\lambda(p,\alpha)\subset M_\lambda(p,\alpha)$, the sufficient part of Theorem (2.2) follows at once from Theorem (2.1), we get

$$Re\left\{\frac{z^{p-1}}{[\mathcal{L}_p(h*\emptyset_1)(z)]' - \overline{[\mathcal{L}_p(g*\emptyset_1)(z)]'}}\right\}$$

$$= Re\left\{\frac{z^{p-1}}{pz^{p-1} + \sum_{k=n+p}^{\infty} k\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k|z^{k-1} + \sum_{k=n+p}^{\infty} k\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k|z^{k-1}}\right\} > \alpha,$$

if we choose z to be real and let $z\to 1^-$, we obtain the condition (2.1).

Theorem 2.3. The class $N_\lambda(p,\alpha)$ is a convex set.

Proof. Let the function $f_j(z)(j=1,2)$ be in the class $N_\lambda(p,\alpha)$. It is sufficient to show that the function H defined by :

$$H(z)=(1-\gamma)f_1(z)+\gamma f_2(z), \quad (0\le\gamma<1)$$

is in the class $N_\lambda(p,\alpha)$, where $f_j=h_j+\overline{g_j}$ and

$$h_j(z)=z^p+\sum_{k=n+p}^{\infty}|a_{k,j}|z^k,$$

$$g_j(z)=\sum_{k=n+p-1}^{\infty}|b_{k,j}|(\bar z)^k.$$

Since for $0\le\gamma<1$

$$H(z) = z^p + \sum_{k=n+p}^{\infty} \left((1-\gamma)|a_{k,1}| - \gamma|a_{k,2}|\right) z^k - \sum_{k=n+p-1}^{\infty} \left((1-\gamma)|b_{k,1}| - \gamma|b_{k,2}|\right)(\bar{z})^k .$$

In view of Theorem (2.2), we have

$$\sum_{k=n+p}^{\infty} k\alpha \left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k|\left((1-\gamma)|a_{k,1}| - \gamma|a_{k,2}|\right) + \sum_{k=n+p-1}^{\infty} k\alpha \left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k|\left((1-\gamma)|b_{k,1}| - \gamma|b_{k,2}|\right)$$

$$= (1-\gamma)\left(\sum_{k=n+p}^{\infty} k\alpha \left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_{k,1}| + \sum_{k=n+p-1}^{\infty} k\alpha \left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_{k,1}|\right)$$

$$+\gamma\left(\sum_{k=n+p}^{\infty} k\alpha \left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_{k,2}| + \sum_{k=n+p-1}^{\infty} k\alpha \left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_{k,2}|\right)$$

$$\le (1-\gamma)p + \gamma p = p ,$$

hence $H(z) \in N_\lambda(p,\alpha)$. For harmonic functions

$$f(z) = z^p + \sum_{k=n+p}^{\infty} |a_k| z^k + \sum_{k=n+p-1}^{\infty} |b_k|(\bar{z})^k \qquad (2.4)$$

and

$$F(z) = z^p + \sum_{k=n+p}^{\infty} |r_k| z^k + \sum_{k=n+p-1}^{\infty} |s_k|(\bar{z})^k , \qquad (2.5)$$

we define the convolution of f and F as

$$(f*F)(z) = z^p + \sum_{k=n+p}^{\infty} |a_k r_k| z^k + \sum_{k=n+p-1}^{\infty} |b_k s_k|(\bar{z})^k . (2.6)$$

In the following theorem , we examine the convolution property of the class $N_\lambda(p,\alpha)$.

Theorem 2.4. If f and F are in $N_\lambda(p,\alpha)$, then $(f*F) \in N_\lambda(p,\alpha)$.

Proof. Let f and F of the forms (2.4) and (2.5) belongs to $N_\lambda(p,\alpha)$. Then the convolution of f and F is given by (2.6). Note that $|r_k| \le 1$ and $|s_k| \le 1$, since $F \in N_\lambda(p,\alpha)$. Then we can write

$$\sum_{k=n+p}^{\infty} k\alpha \left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k r_k| + \sum_{k=n+p-1}^{\infty} k\alpha \left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k s_k|$$

$$\le \sum_{k=n+p}^{\infty} k\alpha \left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k| + \sum_{k=n+p-1}^{\infty} k\alpha \left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k|.$$

The right hand side of the above inequality is bounded by p because $f \in N_\lambda(p,\alpha)$. Therefore $(f*F) \in N_\lambda(p,\alpha)$.

Now, we will examine the closure property of the class $N_\lambda(p,\alpha)$ under the generalized Bernardi –Libera –Livingston integral operator (see [2],[4] and [5]) $D_{c,p}(f)$ which is defined by

$$D_{c,p}(f)(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt , \quad (c > -p). \qquad (2.7)$$

Theorem 2.5. Let $f \in N_\lambda(p,\alpha)$. Then $D_{c,p}(f)$ belong to the class $N_\lambda(p,\alpha)$.

Proof. From the representation of $D_{c,p}(f)$, it follows that

$$D_{c,p}(f) = \frac{c+p}{z^c}\int_0^z t^{c-1}\{h(t)+\overline{g(t)}\}dt$$

$$= \frac{c+p}{z^c}\left\{\int_0^z t^{c-1}\left(t^p + \sum_{k=n+p}^{\infty}|a_k|t^k\right)dt + \overline{\int_0^z t^{c-1}\left(\sum_{k=n+p-1}^{\infty}|b_k|t^k\right)dt}\right\}$$

$$= z^p + \sum_{k=n+p}^{\infty} v_k z^k + \sum_{k=n+p-1}^{\infty} w_k(\bar{z})^k ,$$

where $v_k = \frac{c+p}{c+k}|a_k|$and $w_k = \frac{c+p}{c+k}|b_k|$. Therefore

$$\sum_{k=n+p}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|\,A_k|\left(\frac{c+p}{c+k}\right)|a_k|$$
$$+\sum_{k=n+p-1}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}+1\right)\right.$$
$$\left.-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|\,B_k|\left(\frac{c+p}{c+k}\right)|a_k|$$
$$\leq \sum_{k=n+p}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|\,A_k|\,|a_k|$$
$$+\sum_{k=n+p-1}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}+1\right)\right.$$
$$\left.-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|\,B_k||b_k| \leq p\,.$$

Since $\in N_\lambda(p,\alpha)$, by Theorem (2.2) , we have $D_{c,p}(f) \in N_\lambda(p,\alpha)$.

References

[1] E. S. Aqlan , Some problems connected with geometric function theory, Ph.D. Thesis (2004), Pune university , Pune.
[2] S. D. Bernardi , Convex and starlike univalent functions, Trans. Amer. Math. Soc, 135(1969), 429-446.
[3] J. Clunie and T. Shell –Small , Harmonic univalent function , Ann. Acad. Aci. Fenn. Ser. AI Math ,9(1984),3-25.
[4] R. J. Libera , Some classes of regular univalent functions , Proc. Amer. Math. Soc.,16(1965), 755-758.
[5] A. E. Livingston , On the radius of univalence of certain analytic function , Proc. Amer. Math. Soc., 17(1966), 325-327.
[6] E. Yasar and S. Yalcn , Properties of a subclass of multivalent harmonic functions defined by a linear operator , Gen. Math., 13(1)(2012), 10-20.

Author Profile

Waggas Galib Atshan, Assist. Prof. Dr. in Mathematics (Complex Analysis), teacher at University of Al-Qadisiya,College of Computer Science & Mathematics, Department of Mathematics, he has 90 papers published in various journals in mathematics till now, he teached seventeen subjects in mathematics till now (undergraduate , graduate),he is supervisor on 20 students (Ph.D. , M.Sc.) till now, he attended 23 international and national conferences.