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On Subordination Properties of Univalent functions

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Abstract: ***In this paper, we study some results of Differential Subordination for classes of univalent functions in the unit disk.***

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1. Introduction

Let Σ_α^+ be denote the class of all functions $f(z)$, in the unit disk U, of the form

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^{n-n/\alpha}, \qquad \alpha = \{2,3,4 \ldots\}, \qquad (1.1)$$

which are analytic in the open unite disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfying $f(0) = 1$.

Also, let Σ_α^- be denote the class of all functions $f(z)$, in the unit disk U, of the form

$$f(z) = 1 - \sum_{n=1}^{\infty} a_n z^{n-n/\alpha}, \qquad \alpha = \{2,3,4 \ldots\}, \qquad (1.2)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfying
$f(0) = 1$
For two functions f and g analytic in U, we say that f is subordinate to g in U,

Written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$, $(z \in U)$.
In particular, if the function g is univalent in U then $f \prec g$ if and only if
$f(0) = g(0)$, and $f(U) \subset g(U)$.
A function $f \in \Sigma_\alpha^+$ is said to be starlike of order β if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \qquad (z \in U, \ 0 \le \beta \le 1)$$

Denote this class by $S^*(\beta)$.
A function $f \in \Sigma_\alpha^+$ is said to be convex of order β, if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta, \qquad (z \in U, \ 0 \le \beta \le 1)$$

Denote this class by $C(\beta)$.

Lemma (1)[1]:. Let q be univalent in the unit disk U, and θ be analytic in domain D containing $q(U)$. If
$zp'(z)\theta(p(z)) \prec zq'(z)\theta(q(z))$,
then
$p(z) \prec g(z)$
and q is the best dominant

Lemma (2) [3]: Let q be convex univalent in the unite disk U, and let θ be analytic in a domain D containing $q(U)$. Assume that

$$Re\left\{\theta'(q(z)) + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0$$

If p is analytic in U with $p(0) = q(0)$ and $p(U) \subset D$, then
$zp'(z) + \theta(p(z)) \prec zq'(z) + \theta(q(z))$.
then
$p(z) \prec q(z)$
and q is the best dominant.

Lemma (3)[3]: Let q be convex univalent in the unit disk U, and $\psi, t \in \mathbb{C}$ with

$$Re\left\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{t}\right\} > 0.$$

If p is analytic in U and
$\psi p(z) + tzp'(z) \prec \psi q(z) + tzq'(z)$
then
$p(z) \prec q(z)$
and q is the best dominant.

2. Main Results

Theorem (1): Let the function q be univalent in the unit disk U, $q(z) \neq 0$ and
$zq'(z)\theta(q(z)) \neq 0$
is starlike in U. If $f \in \Sigma_\alpha^+$ satisfies the subordination

$$2 - \frac{zf''(z)}{f'(z)} + \frac{z^2 f'(z)}{zf'(z) - f(z)} \prec \frac{-zq'(z)}{\delta q(z)}. \qquad (2.1)$$

Then

$$\left[\frac{z^2 f'(z)}{zf'(z) - f(z)}\right]^{\delta} \prec q(z),$$

and $q(z)$ is the best dominant.
Proof : Define the function p by

$$p(z) = \left[\frac{z^2 f'(z)}{zf'(z) - f(z)}\right]^{\delta}, \qquad z \in U, \delta \in \mathbb{C}/\{0\} \qquad (2.2)$$

then

$$zp'(z) = \delta\left[\frac{z^2 f'(z)}{zf'(z) - f(z)}\right]^{\delta}\left[-2 + \frac{zf''(z)}{f'(z)} - \frac{z^2 f''(z)}{zf'(z) - f(z)}\right] \prec \frac{-zq'(z)}{\delta q(z)}, \qquad (2.3)$$

by setting $\theta(w)=\frac{-1}{\delta w}$, it can easily observed that $\theta(w)$ is analytic in $\mathbb{C}/\{0\}$. Then we obtain that $\theta(p(z))=\frac{-1}{\delta p(z)}$ and $\theta(q(z))=\frac{-1}{\delta q(z)}$

From (2.3), we have

$$zp'(z)\theta(p(z))=2-\frac{zf''(z)}{f'(z)}+\frac{z^{2}f''(z)}{zf'(z)-f(z)},\qquad(2.4)$$

form (2.1) and (2.4), we get

$$zp'(z)\theta(p(z))\prec\frac{-zq'(z)}{\delta q(z)}=zq'(z)\theta(q(z)).$$

Therefore by Lemma (1), we get $p(z)\prec q(z)$ by using (2.2) we obtain the result.

By taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in the Theorem (1), we obtain the following corollary :

Corollary (1): If $f\in\Sigma_{\alpha}^{+}$ satisfies the subordination

$$2-\frac{zf''(z)}{f'(z)}+\frac{z^{2}f''(z)}{zf'(z)-f(z)}\prec\frac{(B-A)z}{\delta(1+Az)(1+Bz)},$$

Then,

$$\left[\frac{z^{2}f'(z)}{zf'(z)-f(z)}\right]^{\delta}\prec\frac{1+Az}{1+Bz},$$

and $q(z)=\frac{1+Az}{1+Bz}$ is the best dominant.

By taking $q(z)=\frac{1+z}{1+z}$ in the Theorem (1), we obtain the following corollary :

Corollary (2): If $\in\Sigma_{\alpha}^{+}$ satisfies the subordination

$$2-\frac{zf''(z)}{f'(z)}+\frac{z^{2}f''(z)}{zf'(z)-f(z)}\prec\frac{-2z}{\delta(1-z)(1+z)}$$

then

$$\left[\frac{z^{2}f'(z)}{zf'(z)-f(z)}\right]^{\delta}\prec\frac{1+z}{1-z}$$

and $q(z)=\frac{1+z}{1-z}$ is the best dominant.

Theorem (2): Let q be convex univalent in the unit disk U and $q(0)=0$. If $f\in\Sigma_{\alpha}^{+}$ satisfies

$$\frac{\delta zf''(z)}{f'(z)}\left[\frac{z^{2}f'(z)}{zf'(z)-f(z)}\right]^{\delta}\prec zq'(z)+2\delta q(z)+\delta q^{\delta+1}(z),\ (2.5)$$

then

$$\left[\frac{z^{2}f'(z)}{zf'(z)-f(z)}\right]^{\delta}\prec q(z),$$

and $q(z)$ is the best dominant.

Proof : define the function p by

$$p(z)=\left[\frac{z^{2}f'(z)}{zf'(z)-f(z)}\right]^{\delta},\ (z\in U,\delta\in\mathbb{C}\setminus\{0\}),\qquad(2.6)$$

then

$$zp'(z)=\delta\left[\frac{z^{2}f'(z)}{zf'(z)-f(z)}\right]^{\delta}\left[-2+\frac{zf''(z)}{f'(z)}-\frac{z^{2}f''(z)}{zf'(z)-f(z)}\right]\ (2.7)$$

By setting $\theta(w)=\delta w^{\delta+1/\delta}+2\delta w$, it can easily observed that $\theta(w)$ is analytic in $\mathbb{C}$. Then we obtain that

$$\theta(p(z))=\delta[p(z)]^{\delta+1/\delta}+2\delta p(z)\ and\ \theta(q(z))=\delta[q(z)]^{\delta+1/\delta}+2\delta q(z)\qquad(2.8)$$

From (2-7) and (2-8), we get

$$zp'(z)+\theta(p(z))=\frac{\delta zf''(z)}{f'(z)}\left[\frac{z^{2}f'(z)}{zf'(z)-f(z)}\right]^{\delta}\qquad(2.9)$$

From (2.5) and (2.9), we get

$$zp'(z)+\theta(p(z))\prec zq'(z)+2\delta q(z)+\delta q^{\delta+1}(z)=zq'(z)+\theta(q(z)).$$

then by Lemma (2), we get $p(z)\prec q(z)$.

By using (2.6), we obtain the result.

By taking $q(z)=ze^{\gamma Az}$ in the Theorem (2), we obtain the following corollary :

Corollary (3): If $f\in\Sigma_{\alpha}^{-}$ satisfies the subordination

$$\frac{\delta zf''(z)}{f'(z)}\left[\frac{z^{2}f'(z)}{zf'(z)-f(z)}\right]^{\delta}\prec\left(z^{2}\gamma A+z+2\delta+\delta e^{\delta+1/\delta}\right)e^{\gamma Az},$$

then

$$\left[\frac{z^{2}f'(z)}{zf'(z)-f(z)}\right]^{\delta}\prec ze^{\gamma Az}$$

and $q(z)=ze^{\gamma Az}$ is the best dominant.

By taking $q(z)=(1-z)^{-2ab}$ in the Theorem (2), we obtain the following corollary :

Corollary (4): If $f\in\Sigma_{\alpha}^{-}$ satisfies the subordination

$$\frac{\delta zf''(z)}{f'(z)}\left[\frac{z^{2}f'(z)}{zf'(z)-f(z)}\right]^{\delta}\prec 2zab(1-z)^{-(2ab+1)}+2\delta(1-z)^{-2ab}+\delta(1-z)^{-2ab(\frac{\delta+1}{\delta}}$$

then

$$\left[\frac{z^{2}f'(z)}{zf'(z)-f(z)}\right]^{\delta}\prec(1-z)^{-2ab},$$

and $q(z)=(1-z)^{-2ab}$ is the best dominant.

Theorem (3): Let the function q be convex univalent in the unit disk U, $q'(z)\neq 0$ and assume that

$$Re\left\{1+\frac{zq''(z)}{q'(z)}+2\delta\right\}>0\ ,\qquad(2.10)$$

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If $f \in \Sigma_\alpha^+$ satisfies the subordination

$$t\delta\left[\frac{z^2 f''(z)}{zf'(z)-f(z)}\right]^\delta\left[\frac{zf''(z)}{f'(z)}-\frac{z^2 f''(z)}{zf'(z)\text{-}f(z)}\right] \prec 2t\delta q(z)+tzq'(z), \quad (2.11)$$

then

$$\left[\frac{z^2 f''(z)}{zf'(z)-f(z)}\right]^\delta \prec q(z).\ (z \in U, \delta \in \mathbb{C}/\{0\}),$$

and $q(z)$ is the best dominant .

Proof: Define the function p by

$$p(z) = \left[\frac{z^2 f''(z)}{zf'(z)-f(z)}\right]^\delta,\ (z \in U, \delta \in \mathbb{C}/\{0\}), \quad (2.12)$$

then

$$tzp'(z) = t\delta\left[\frac{z^2 f''(z)}{zf'(z)-f(z)}\right]^\delta\left[-2+\frac{zf'''(z)}{f''(z)}-\frac{z^2 f''(z)}{zf'(z)-f(z)}\right]$$

It can easily observed that

$$2t\delta p(z)+tzp'(z)=t\delta\left[\frac{z^2 f''(z)}{zf'(z)-f(z)}\right]^\delta\left[\frac{zf''}{f'(z)}-\frac{z^2 f''(z)}{zf'(z)-f(z)}\right], \quad (2.13)$$

Then by (2.11) and (2.13), we get

$2t\delta p(z)+t\delta p'(z) \prec 2t\delta q(z)+tzq'(z)$

By setting $\psi = 2t\delta$ in Lemma (3), we get

$p(z) \prec q(z)$

By using (2.12), we obtain the result.

By taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in the Theorem (3), we obtain the following corollary:

__Corollary (5):__ Let $f \in \Sigma_\alpha^+$ and assume that

$$Re\left\{\frac{1-Bz}{1+Bz}+2\delta\right\} > 0.$$

If f satisfies the subordination

$$t\delta\left[\frac{z^2 f''(z)}{zf'(z)-f(z)}\right]^\delta\left[\frac{zf''}{f'(z)}-\frac{z^2 f''(z)}{zf'(z)-f(z)}\right] \prec 2t\delta\frac{1-Bz}{1+Bz}+\frac{t(A-B)z}{(1+Bz)^2},$$

then

$$\left[\frac{z^2 f''(z)}{zf'(z)-f(z)}\right]^\delta \prec \frac{1+Az}{1+Bz}, \qquad (-1 \le B < A \le 1)$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant .

By taking $q(z) = e^{\gamma Az}$ in the Theorem (3), we obtain the following corollary:

__Corollary (6):__ Let $f \in \Sigma_\alpha^+$ and assume that

$Re\{1+\gamma Az+2\delta\} > 0.$

If f satisfies the subordination

$$t\delta\left[\frac{z^2 f''(z)}{zf'(z)-f(z)}\right]^\delta\left[\frac{zf''(z)}{f'(z)}-\frac{z^2 f''(z)}{zf'(z)-f(z)}\right] \prec (2\delta+z)te^{\gamma Az},$$

then

$$\left[\frac{z^2 f''(z)}{zf'(z)-f(z)}\right]^\delta \prec e^{\gamma Az},$$

and $q(z) = e^{\gamma Az}$ is the best dominant.

References

[1] O. Juneja, T. Reddy, M. Mogra, A convolution approach for analytic functions with negative coefficients, Soochow, J. Math. , 11(1988), 69-81.
[2] K.S. Miller, B. Ross, An introduction to the fractional Calculus and fractional differential equations, John – Wily and Sons, Tnc. , (1993).
[3] S. S. Miller and P.T. Mocanu, Differential Subordinations; Theory and Applications, Series On Monographs and Textbooks in Pure and Applied Mathematics, Vol.225, Marcel Dekker, Inc. , New York and Basel, 2000.

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