On Some New Results of a Subclass of Univalent Functions Defined by Ruscheweyh Derivative

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Abstract: ***In this paper, we introduce a new class of univalent functions defined by Ruscheweyh derivative in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$, we obtain basic properties, like, coefficient inequality, distortion and covering theorem, radii of starlikeness, convexity and close –to-convexity, extreme points, Hadamard product, closure theorems and convolution operator for functions belonging to the class $\sum^{+}(\sigma, c, \beta, \lambda)$.***

Keywords: Univalent function, Ruscheweyh derivative, Distortion theorem, Radius of starlikeness, Extreme points, Hadamard product.

2014 Mathematics Subject Classification: 30C45.

1.Introduction

Let $\sum$ denote the class of functions of the from:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, (1)$$

which are analytic and univalent in the open unit disk U.
If a function f is given by (1) and g is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \ (2)$$

is in the class $\sum$, then the convolution (or Hadamard product) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, z \in U. (3)$$

Let $\sum^{+}$ denote the subclass of $\sum$ consisting of functions of the from

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0, n \in N). (4)$$

We aim to study the subclass $\sum^{+}(\sigma, c, \beta, \lambda)$ consisting of function $f \in \sum^{+}$ and satisfying the condition:

$$\left| \frac{\sigma\left[z\left(D^{\lambda} f(z)\right)'' - \left(\left(D^{\lambda} f(z)\right)' - 1\right)\right]}{cz\left(D^{\lambda} f(z)\right)'' + \left((1-\sigma)\left(D^{\lambda} f(z)\right)' + 1\right)} \right| < \beta, z \in U, (5)$$

where $0 \leq \sigma < 1, 0 \leq c < 1, 0 < \beta < 1$ and $D^{\lambda} f(z)$ is the Ruscheweyh derivative [6], [7] of f of order λ defined as follow:

$$D^{\lambda} f(z) = z + \sum_{n=2}^{\infty} a_n A_n(\lambda) z^n,$$

where

$$A_n(\lambda) = \frac{(\lambda+1)(\lambda+2)\ldots(\lambda+n-1)}{(n-1)!}, \lambda > -1, z \in U \ (6)$$

Another classes studied by several authors, like, [2] and [4] consisting of functions of the from (4).

2.Coefficient Inequality

In the following theorem, we obtain anecessary and sufficient condition to be the function in the class $\sum^{+}(\sigma, c, \beta, \lambda)$.

Theorem 1: Let the function f be defined by (4). Then $f \in \sum^{+}(\sigma, c, \beta, \lambda)$ if and only if

$$\sum_{n=2}^{\infty} \left[n\left(\sigma(n+\beta) - \beta(c(n-1)+1)\right)\right] A_n(\lambda) a_n \leq \beta(2-\sigma), (7)$$

where $0 < \beta < 1, 0 < \sigma < 1, 0 \leq c < 1, and\ \lambda > -1$. The result (7) is sharp for the function

$$f(z) = z + \frac{\beta(2-\sigma)}{\left[n\left(\sigma(n+\beta) - \beta(c(n-1)+1)\right)\right] A_n(\lambda)} z^n, (n \geq 2)(8).$$

Proof: Suppose that the inequality (7) holds true and $|z| = 1$. Then we have

$$\left|\sigma\left[z\left(D^{\lambda} f(z)\right)'' - \left(\left(D^{\lambda} f(z)\right)' - 1\right)\right]\right| - \beta\left|cz\left(D^{\lambda} f(z)\right)'' + \left((1-\sigma)\left(D^{\lambda} f(z)\right)' + 1\right)\right|$$

$$= \left|\sum_{n=2}^{\infty} (\sigma n^2) A_n(\lambda) a_n z^{n-2}\right| - \beta \left|\sum_{n=2}^{\infty} (cn^2 - cn + n - \sigma n) A_n(\lambda)\, a_n z^{n-2} + (2-\sigma)\right|$$

$$\leq \sum_{n=2}^{\infty} \left[n\left(\sigma(n+\beta) - \beta(c(n-1)+1)\right)\right] A_n(\lambda) - \beta(2-\sigma) \leq 0,$$

by hypothesis, hence, by maximum modulus principle $f \in \sum^{+}(\sigma, c, \beta, \lambda)$.
Conversely, assume that $f \in \sum^{+}(\sigma, c, \beta, \lambda)$, so that

$$\left| \frac{\sigma\left[z\left(D^{\lambda} f(z)\right)'' - \left(\left(D^{\lambda} f(z)\right)' - 1\right)\right]}{cz\left(D^{\lambda} f(z)\right)'' + \left((1-\sigma)\left(D^{\lambda} f(z)\right)' + 1\right)} \right| < \beta, z \in U,$$

hence

$$\left|\sigma\left[z\left(D^\lambda f(z)\right)'' - \left(\left(D^\lambda f(z)\right)' - 1\right)\right]\right| < \beta\left|cz\left(D^\lambda f(z)\right)'' + \left((1-\sigma)\left(D^\lambda f(z)\right)' + 1\right)\right|.$$

Therefore, we get

$$\left|\sum_{n=2}^{\infty}(\sigma n^2)A_n(\lambda)a_n z^{n-2}\right| < \beta\left|\sum_{n=2}^{\infty}(cn^2 - cn + n - \sigma n)A_n(\lambda)\, a_n z^{n-2} + (2-\sigma)\right|,$$

thus

$$\sum_{n=2}^{\infty}\left[n\left(\sigma(n+\beta) - \beta\left(c(n-1)+1\right)\right)\right]A_n(\lambda)a_n \le \beta(2-\sigma).$$

Corollary 1: Let the function $f \in \Sigma^+(\sigma, c, \beta, \lambda)$.Then

$$a_n \le \frac{\beta(2-\sigma)}{\left[n\left(\sigma(n+\beta) - \beta\left(c(n-1)+1\right)\right)\right]A_n(\lambda)} z^n, n \ge 2.$$

3.Distortion and Covering Theorems

We introduce the growth and distortion theorems for the function f in the class $\Sigma^+(\sigma, c, \beta, \lambda)$.

Theorem 2: Let the function $f \in \Sigma^+(\sigma, c, \beta, \lambda)$. Then

$$|z| - \frac{\beta(2-\sigma)}{\left[2\left(\sigma(2+\beta) - \beta(c+1)\right)\right](\lambda+1)}|z|^2 \le |f(z)| \le |z| + \frac{\beta(2-\sigma)}{\left[2\left(\sigma(2+\beta) - \beta(c+1)\right)\right](\lambda+1)}|z|^2, |z| < 1.$$

The result is sharp and attained

$$f(z) = z + \frac{\beta(2-\sigma)}{\left[2\left(\sigma(2+\beta) - \beta(c+1)\right)\right](\lambda+1)}z^2.$$

Proof:

$$|f(z)| = \left|z + \sum_{n=2}^{\infty} a_n z^n\right| \le |z| + \sum_{n=2}^{\infty} a_n|z|^n \le |z| + |z|^2\sum_{n=2}^{\infty} a_n.$$

By Theorem (1), we get

$$\sum_{n=2}^{\infty} a_n \le \frac{\beta(2-\sigma)}{\left[2\left(\sigma(2+\beta) - \beta(c+1)\right)\right](\lambda+1)}. (9)$$

Thus

$$|f(z)| \le |z| + \frac{\beta(2-\sigma)}{\left[2\left(\sigma(2+\beta) - \beta(c+1)\right)\right](\lambda+1)}|z|^2.$$

Also

$$|f(z)| \ge |z| - \sum_{n=2}^{\infty} a_n|z|^n \ge |z| - |z|^2\sum_{n=2}^{\infty} a_n \ge |z| - \frac{\beta(2-\sigma)}{\left[2\left(\sigma(2+\beta) - \beta(c+1)\right)\right](\lambda+1)}|z|^2,$$

and this completed the proof.

Theorem 3: Let $f \in \Sigma^+(\sigma, c, \beta, \lambda)$. Then

$$1 - \frac{\beta(2-\sigma)}{\left[2\left(\sigma(2+\beta) - \beta(c+1)\right)\right](\lambda+1)}|z| \le |f'(z)| \le 1 + \frac{\beta(2-\sigma)}{\left[2\left(\sigma(2+\beta) - \beta(c+1)\right)\right](\lambda+1)}|z|$$

with equality for

$$f(z) = z + \frac{\beta(2-\sigma)}{\left[2\left(\sigma(2+\beta) - \beta(c+1)\right)\right](\lambda+1)}z^2.$$

Proof: Notice that

$$\left[2\left(\sigma(2+\beta) - \beta(c+1)\right)\right](\lambda+1)\sum_{n=2}^{\infty} na_n \le \sum_{n=2}^{\infty}\left[n\left(\sigma(n-2+\beta) - \beta\left(c(n-1)+1\right)\right)\right]A_n(\lambda)a_n \le \beta(2-\sigma), (10)$$

from Theorem 1. Thus

$$|f'(z)| = \left|1 + \sum_{n=2}^{\infty} na_n z^{n-1}\right| \ge 1 - \sum_{n=2}^{\infty} na_n|z|^{n-1} \ge 1 - |z|\sum_{n=2}^{\infty} na_n \ge 1 - |z|\frac{\beta(2-\sigma)}{\left[2\left(\sigma(2+\beta) - \beta(c+1)\right)\right](\lambda+1)}. (11)$$

Combining (10)and(11),we get the result.

4.Radii of starlikeness, convexity and close-to-convexity:

In the following theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class$\Sigma^+(\sigma, c, \beta, \lambda)$.

Theorem 4: Let $f \in \Sigma^+(\sigma, c, \beta, \lambda)$. Then f isstarlike in the disk $|z| < R_1$, of order $\alpha, 0 \le \alpha < 1$, where

$$R_1 = \inf_n\left[\frac{(1-\alpha)\left[n\left(\sigma(n+\beta) - \beta\left(c(n-1)+1\right)\right)\right]A_n(\lambda)}{(n-\alpha)\beta(2-\sigma)}\right]^{\frac{1}{n-1}}, n \ge 2. (12)$$

Proof: f is starlike of order $\alpha, 0 \le \alpha < 1$, if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha.$$

We must show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \alpha,$$

for$|z| < R_1$.

Indeed we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n|z|^{n-1}} \le 1 - \alpha, (0 \le \alpha < 1)\ (13)$$

hence by Theorem 1, (13) will be true if

$$\frac{n-\alpha}{1-\alpha}|z|^{n-1} \le \frac{\left[n\left(\sigma(n+\beta) - \beta\left(c(n-1)+1\right)\right)\right]A_n(\lambda)}{\beta(2-\sigma)}$$

or if

$$|z| \le \left[\frac{(1-\alpha)\left[n\left(\sigma(n+\beta) - \beta\left(c(n-1)+1\right)\right)\right]A_n(\lambda)}{(n-\alpha)\beta(2-\sigma)}\right]^{\frac{1}{n-1}}, n \ge 2\ (14)$$

the theorem follows easily from (14).

Theorem 5: Let $f \in \sum^{+}(\sigma, c, \beta, \lambda)$. Then f is convex in $|z| < R_2$ of order $\alpha, 0 \le \alpha < 1$, where

$$R_2 = \inf_n \left[\frac{(1-\alpha)[n(\sigma(n+\beta) - \beta(c(n-1)+1))]A_n(\lambda)}{n(n-\alpha)\beta(2-\sigma)}\right]^{\frac{1}{n-2}}, n \ge 2 \quad (15)$$

Proof: f is convex of order $\alpha, 0 \le \alpha < 1$, if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha.$$

Thus it is enough to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \le 1 - \alpha,$$

for $|z| < R_2$.

Indeed we have

$$\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} na_n|z|^{n-1}} \le 1 - \alpha, (0 \le \alpha < 1) \quad (16)$$

Hence by Theorem 1, (16) will be true if

$$\frac{n(n-\alpha)|z|^{n-1}}{(1-\alpha)} \le \frac{[n(\sigma(n+\beta) - \beta(c(n-1)+1))]A_n(\lambda)}{\beta(2-\sigma)}$$

or if

$$|z| \le \left[\frac{(1-\alpha)[n(\sigma(n+\beta) - \beta(c(n-1)+1))]A_n(\lambda)}{n(n-\alpha)\beta(2-\sigma)}\right]^{\frac{1}{n-1}}, n \ge 2. \quad (17)$$

Theorem 6: Let $\in \sum^{+}(\sigma, c, \beta, \lambda)$. Then f is close-to-convex function in $|z| < R_3$ of order $\alpha, 0 \le \alpha < 1$, where

$$R_3 = \inf_n \left[\frac{(1-\alpha)[n(\sigma(n+\beta) - \beta(c(n-1)+1))]A_n(\lambda)}{n\beta(2-\sigma)}\right]^{\frac{1}{n-1}}. \quad (18)$$

Proof: f is close-to-convex function of order $\alpha, 0 \le \alpha < 1$, if

$$Re\{f'(z)\} > \alpha.$$

Thus it is enough to show that

$$|f'(z) - 1| = \left|\sum_{n=2}^{\infty} na_n z^{n-1}\right| \le \sum_{n=2}^{\infty} na_n|z|^{n-1}.$$

Thus

$$|f'(z) - 1| \le 1 - \alpha \; if \; \sum_{n=2}^{\infty} \frac{na_n|z|^{n-1}}{1-\alpha} \le 1, (19)$$

hence by Theorem1, (19) will be true if

$$\frac{n|z|^{n-1}}{1-\alpha} \le \frac{[n(\sigma(n+\beta) - \beta(c(n-1)+1))]A_n(\lambda)}{\beta(2-\sigma)}$$

or if

$$|z| \le \left[\frac{(1-\alpha)[n(\sigma(n+\beta) - \beta(c(n-1)+1))]A_n(\lambda)}{n\beta(2-\sigma)}\right]^{\frac{1}{n-1}}, n \ge 2 \quad (20)$$

the result follows easily from (20).

5.Extreme Points:

In the following theorem, we obtain extreme points for the class $\sum^{+}(\sigma, c, \beta, \lambda)$.

Theorem 7: Let $f_1(z) = z$ and

$$f_n(z) = z + \frac{\beta(2-\sigma)}{[n(\sigma(n+\beta) - \beta(c(n-1)+1))]A_n(\lambda)} z^n, for \; n = 2,3, \dots.$$

Then $f \in \sum^{+}(\sigma, c, \beta, \lambda)$ if and only if it can be expressed in the from

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z),$$

where

$$\left(\mu_n \ge 0 \; and \; \sum_{n=1}^{\infty} \mu_n = 1 \; or \; 1 = \mu_1 + \sum_{n=2}^{\infty} \mu_n\right).$$

Proof: Let

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) = z + \sum_{n=2}^{\infty} \frac{\beta(2-\sigma)}{[n(\sigma(n+\beta) - \beta(c(n-1)+1))]A_n(\lambda)} \mu_n,$$

then

$$\sum_{n=2}^{\infty} \left(\frac{[n(\sigma(n+\beta) - \beta(c(n-1)+1))]A_n(\lambda)}{\beta(2-\sigma)}\right) \mu_n \frac{\beta(2-\sigma)}{[n(\sigma(n+\beta) - \beta(c(n-1)+1))]A_n(\lambda)}$$

$$= \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 \le 1.$$

Using Theorem 1, we easily get $(z) \in \sum^{+}(\sigma, c, \beta, \lambda)$.

Conversely, let $f(z) \in \sum^{+}(\sigma, c, \beta, \lambda)$ is of the from (4). Then

$$a_n \le \frac{\beta(2-\sigma)}{[n(\sigma(n+\beta) - \beta(c(n-1)+1))]A_n(\lambda)}, (n \ge 2).$$

Setting

$$\mu_n = \frac{[n(\sigma(n+\beta) - \beta(c(n-1)+1))]A_n(\lambda)}{\beta(2-\sigma)} a_n, for \; n \ge 2$$

and

$$\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n.$$

Then

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

$$= z + \sum_{n=2}^{\infty} \frac{\beta(2-\sigma)}{[n(\sigma(n+\beta) - \beta(c(n-1)+1))]A_n(\lambda)} \mu_n z^n$$

$$= \mu_1 z + \sum_{n=2}^{\infty} \mu_n f_n(z).$$

Thus

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) = \mu_1 f_1(z) + \sum_{n=2}^{\infty} \mu_n f_n(z).$$

6.Hadamard Product

In the following theorem, we obtain the convolution result for function belong to the class $\sum^{+}(\sigma, c, \beta, \lambda)$.

Theorem 8: Let f and $g \in \sum^+(\sigma,c,\beta,\lambda)$. Then $f*g \in \sum^+(\sigma,c,\delta,\lambda)$ for

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, g(z)=z+\sum_{n=2}^{\infty}b_nz^n,$$

where

$$\delta \le \frac{\beta^2(2-\sigma)[n\sigma(n-2)]}{A_n(\lambda)[n(\sigma(n+\beta))-\beta(c(n-1)+1)]^2-\beta^2(2-\sigma)[n(\sigma-(c(n-1)+1))]}.$$

Proof: *Since* $f,g\in\sum^+(\sigma,c,\beta,\lambda)$, then we have

$$\sum_{n=2}^{\infty}\frac{[n(\sigma(n+\beta)-\beta(c(n-1)+1))]A_n(\lambda)}{\beta(2-\sigma)}a_n \le 1\ (21)$$

and

$$\sum_{n=2}^{\infty}\frac{[n(\sigma(n+\beta)-\beta(c(n-1)+1))]A_n(\lambda)}{\beta(2-\sigma)}b_n \le 1.\ (22)$$

We must find the smallest number δ *such that*

$$\sum_{n=2}^{\infty}\frac{[n(\sigma(n+\delta)-\delta(c(n-1)+1))]A_n(\lambda)}{\delta(2-\sigma)}a_nb_n \le 1.\ (23)$$

By Cauchy-Schwarz inequality, we have

$$\sum_{n=2}^{\infty}\frac{[n(\sigma(n+\beta)-\beta(c(n-1)+1))]A_n(\lambda)}{\beta(2-\sigma)}\sqrt{a_nb_n} \le 1.\ (24)$$

Thus, it is enough to show that

$$\frac{[n(\sigma(n+\delta)-\delta(c(n-1)+1))]A_n(\lambda)}{\delta(2-\sigma)}a_nb_n \le \frac{[n(\sigma(n+\beta)-\beta(c(n-1)+1))]A_n(\lambda)}{\beta(2-\sigma)}\sqrt{a_nb_n},$$

that is

$$\sqrt{a_nb_n}\le\frac{[(\sigma(n+\beta))-\beta(c(n-1)+1)]\delta}{[(\sigma(n+\delta))-\delta(c(n-1)+1)]\beta},\ (25)$$

from (24), we get

$$\sqrt{a_nb_n}\le\frac{\beta(2-\sigma)}{[n(\sigma(n+\beta)-\beta(c(n-1)+1))]A_n(\lambda)}.\ (26)$$

Therefore, in view of (25) and (26) it is enough to show that

$$\frac{\beta(2-\sigma)}{[n(\sigma(n+\beta)-\beta(c(n-1)+1))]A_n(\lambda)} \le \frac{[(\sigma(n+\beta))-\beta(c(n-1)+1)]\delta}{[(\sigma(n+\delta))-\delta(c(n-1)+1)]\beta}$$

and

$$\delta \le \frac{\beta^2(2-\sigma)[n\sigma(n-2)]}{A_n(\lambda)[n(\sigma(n+\beta))-\beta(c(n-1)+1)]^2-\beta^2(2-\sigma)[n(\sigma-(c(n-1)+1))]}.$$

This complete the proof.

Theorem 9: Let $,g\in\sum^+(\sigma,c,\beta,\lambda)$. Then

$$h(z)=z+\sum_{n=2}^{\infty}(a_n^2+b_n^2)z^n$$

belong to the class $\sum^+(\sigma,c,\delta,\lambda)$, where

$$\delta\ge\frac{2\beta^2(2-\sigma)n\sigma}{[n(\sigma(n+\beta))-\beta(c(n-1)+1)]^2A_n(\lambda)}.$$

Proof: Since $f,g\in\sum^+(\sigma,c,\beta,\lambda)$ so by Theorem 1, yields

$$\sum_{n=2}^{\infty}\left[\frac{[n(\sigma(n+\beta)-\beta(c(n-1)+1))]A_n(\lambda)}{\beta(2-\sigma)}a_n\right]^2\le 1$$

and

$$\sum_{n=2}^{\infty}\left[\frac{[n(\sigma(n+\beta)-\beta(c(n-1)+1))]A_n(\lambda)}{\beta(2-\sigma)}b_n\right]^2\le 1,$$

we obtain from the last two inequalities

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{[n(\sigma(n+\beta)-\beta(c(n-1)+1))]A_n(\lambda)}{\beta(2-\sigma)}\right]^2(a_n^2+b_n^2)\le 1,\ (27)$$

but $h(z)\in\sum^+(\sigma,c,\delta,\lambda)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[n(\sigma(n+\delta)-\delta(c(n-1)+1))]A_n(\lambda)}{\delta(2-\sigma)}(a_n^2+b_n^2)\le 1,\ (28)$$

where $0<\ \delta<1$, however (27) implies (28) if

$$\frac{[n(\sigma(n+\delta)-\delta(c(n-1)+1))]A_n(\lambda)}{\delta(2-\sigma)}\le\left[\frac{[n(\sigma(n+\beta)-\beta(c(n-1)+1))]A_n(\lambda)}{\beta(2-\sigma)}b_n\right]^2.$$

Simplifying, we get

$$\delta\ge\frac{2\beta^2(2-\sigma)n\sigma}{[n(\sigma(n+\beta))-\beta(c(n-1)+1)]^2A_n(\lambda)}.$$

7.Closure theorems

We shall prove the following closure theorems for the class $\sum^+(\sigma,c,\beta,\lambda)$, let the function $f_i(z)(i=1,2,\dots,m)$ defined by

$$f_i(z)=z+\sum_{n=2}^{\infty}a_{n,i}z^n,(a_{n,i}\ge 0,n\in N,n\ge 2)(29).$$

Theorem 10: Let the functions $f_i(z)$ defined by (29) be in the class $\sum^+(\sigma,c,\beta,\lambda)$ for every $=1,2,\dots,m$. Then the function $h(z)$ defined by

$$h(z)=z+\sum_{n=2}^{\infty}c_nz^n,(c_n\ge 0,n\in N,n\ge 2)$$

also belongs to the class $\sum^+(\sigma,c,\beta,\lambda)$, where

$$c_n=\frac{1}{m}\sum_{i=1}^{m}a_{n,i}.$$

Proof: Since $f_i(z)\in\sum^+(\sigma,c,\beta,\lambda)$, therefore from Theorem 1, we obtain

$$\sum_{n=2}^{\infty}[n(\sigma(n+\beta)-\beta(c(n-1)+1))]A_n(\lambda)a_{n,i}\le\beta(2-\sigma),(29)$$

then

$$\sum_{n=2}^{\infty}\left[n\big(\sigma(n+\beta)-\beta(c(n-1)+1)\big)\right]A_n(\lambda)c_n$$
$$=\sum_{n=2}^{\infty}\left[n\big(\sigma(n+\beta)-\beta(c(n-1)+1)\big)\right]A_n(\lambda)\left[\frac{1}{m}\sum_{i=1}^{m}a_{n,i}\right]$$
$$\le \beta(2-\sigma).$$

Hence $h(z)\in\sum^{+}(\sigma,c,\beta,\lambda)$.

Theorem 11: Let the functions $f_i(z)$ defined by (29) be in the class $\sum^{+}(\sigma,c,\beta,\lambda)$, for every $i=1,2,\dots,m$. Then the function $h(z)$ defined by

$$h(z)=\sum_{i=1}^{m}d_i f_i(z)\ and\ \sum_{i=1}^{m}d_i=1,d_i\ge 0$$

in the class $\sum^{+}(\sigma,c,\beta,\lambda)$.

Proof: By definition of $h(z)$, we have

$$h(z)=\sum_{i=1}^{m}d_i z+\sum_{n=1}^{\infty}\left[\sum_{i=1}^{m}d_i a_{n,i}\right]z^n,$$

since $f_i(z)$ are in the class $\sum^{+}(\sigma,c,\beta,\lambda)$, for every $i=1,2,\dots,m$, we obtain

$$\sum_{n=2}^{\infty}\left[n\big(\sigma(n+\beta)-\beta(c(n-1)+1)\big)\right]A_n(\lambda)a_{n,i}$$
$$\le\beta(2-\sigma)$$

for every $i=1,2,\dots,m$, hence we can see that

$$\sum_{n=2}^{\infty}\left[n\big(\sigma(n+\beta)-\beta(c(n-1)+1)\big)\right]A_n(\lambda)\left[\sum_{i=1}^{m}d_i a_{n,i}\right]$$
$$=\sum_{i=1}^{\infty}d_i\left[\sum_{n=2}^{\infty}\left[n\big(\sigma(n+\beta)-\beta(c(n-1)+1)\big)\right]A_n(\lambda)a_{n,i}\right]$$
$$\le\beta(2-\sigma)\sum_{i=1}^{m}d_i=\beta(2-\sigma).$$

Thus $h(z)\in\sum^{+}(\sigma,c,\beta,\lambda)$.

8.Convolution Operator

Definition 1 [2,5]: The Gaussian hypergeometric function denoted by

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{z^n}{n!},|z|<1,$$

where $c>b>0,c>a+b$ and

$$(x)_n=\begin{cases}x(x+1)(x+2)\dots(x+n-1)\ for\ n=1,2,3,\dots\\ 1\ n=0\end{cases}.$$

Definition 2[3]: For every $f\in\sum^{+}$, we defined the convolution operator $W_{a,b,c}(f)(z)$ as below:

$$W_{a,b,c}(f)(z)=\ {}_2F_1(a,b;c;z)*f(z)$$
$$=z+\sum_{n=2}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}a_n z^n,$$

where ${}_2F_1(a,b;c;z)$ is Gaussain hypergeometric function (see[2] and [5]) introduced in Definition 1.

Theorem 12: Let f is given by (4) be in the class $\sum^{+}(\sigma,c,\beta,\lambda)$. Then the convolution operator $W_{a,b,c}(f)$ is in the class $\sum^{+}(\sigma,c,\beta,\lambda)$ for $|z|\le r(\beta,\delta)$, where

$$r(\beta,\delta)$$
$$=inf_n\left[\frac{\delta\left[n\big(\sigma(n+\beta)-\beta(c(n-1)+1)\big)\right]}{\beta\left[n\big(\sigma(n+\delta)-\delta(c(n-1))+1\big)\right]\frac{(a)_n(b)_n}{(c)_n n!}}\right]^{\frac{1}{n-1}}.$$

The result is sharp for the function

$$f_n(z)=z+\frac{\beta(2-\sigma)}{\left[n\big(\sigma(n+\beta)-\beta(c(n-1)+1)\big)\right]A_n(\lambda)}z^n,n\ge 2.$$

Proof: Since $f\in\sum^{+}(\sigma,c,\beta,\lambda)$, we have

$$\sum_{n=2}^{\infty}\frac{\left[n\big(\sigma(n+\beta)-\beta(c(n-1)+1)\big)\right]A_n(\lambda)}{\beta(2-\sigma)}a_n\le 1.$$

It is sufficient to show that

$$\sum_{n=2}^{\infty}\frac{\left[n\big(\sigma(n+\delta)-\delta(c(n-1))+1\big)\right]\frac{(a)_n(b)_n}{(c)_n n!}}{\delta(2-\sigma)}a_n\le 1.\ (30)$$

Note that (30) is satisfied if

$$\frac{\left[n\big(\sigma(n+\delta)-\delta(c(n-1))+1\big)\right]\frac{(a)_n(b)_n}{(c)_n n!}}{\delta(2-\sigma)}a_n|z|^{n-1}\le\frac{\left[n\big(\sigma(n+\beta)-\beta(c(n-1)+1)\big)\right]A_n(\lambda)}{\beta(2-\sigma)}a_n,$$

solving for $|z|$ we get the result .

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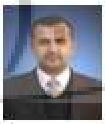
Author Profile



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