On Some Certain Properties of a New Subclass of Univalent Functions Defined by Differential Subordination Property

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Abstract: ***In this paper, we have studied a new subclass of univalent functions defined by differential subordination property by using the linear operator $\mathcal{L}_{\lambda,\iota,m}^{\gamma+c,\alpha_1}$. Coefficient bounds, some properties of neighborhoods, convolution properties; Integral mean inequalities for the fractional integral for this class are obtained.***

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1. Introduction

Let S be the class of all functions of from the:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (n \in N), \tag{1}$$

which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

Let D denote the subclass of S containing of functions of the from:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0, n \in N). \tag{2}$$

The Hadamard product (or convolution) of two power series

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ \ and \ \ g(z) = z - \sum_{n=2}^{\infty} b_n z^n \tag{3}$$

in D is defined by:

$$(f * g)(z) = f(z) * g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n. \tag{4}$$

For positive real values of $\alpha_1, \dots, \alpha_\iota$ and $\beta_1, \dots, \beta_m (\beta_j \neq 0, -1, \dots, j = 1,2, \dots, m)$,
the generalized hypergeometric function ${}_\iota F_m(z)$ is defined by:

$$ {}_\iota F_m(z) = {}_\iota F_m(\alpha_1, \dots, \alpha_\iota; \beta_1, \dots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_\iota)_n}{(\beta_1)_n \dots (\beta_m)_n} \cdot \frac{z^n}{n!} \tag{5}$$

$(\iota \leq m+1\,;\ \iota, m \in N_0 = N \cup \{0\}; z \in U)$,
where $(\alpha)_n$ is the pochhammer symbol defined by

$$(\alpha)_n = \begin{cases} 1, & n = 0 \\ \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1), & \alpha \in N. \end{cases} \tag{6}$$

The notation ${}_\iota F_m$ is quite useful for representing many well-know functions such as the exponential, the Bessel and laguerre polynomial. Let

$$H[\alpha_1, \dots, \alpha_\iota; \beta_1, \dots, \beta_m]: D \to D$$

be a linear operator defined by

$$H[\alpha_1, \dots, \alpha_\iota; \beta_1, \dots, \beta_m] f(z) = z \ {}_\iota F_m(\alpha_1, \dots, \alpha_\iota; \beta_1, \dots, \beta_m; z) * f(z) = z - \sum_{n=2}^{\infty} W_n(\alpha_1; \iota; m) a_n z^n, \tag{7}$$

where

$$W_n(\alpha_1; \iota; m) = \frac{(\alpha_1)_{n-1} \dots (\alpha_\iota)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_m)_{n-1}} \cdot \frac{1}{(n-1)!}. \tag{8}$$

For notational simplicity, we use shorter notation $H_m^\iota[\alpha_1]$ for $H[\alpha_1, \dots, \alpha_\iota; \beta_1, \dots, \beta_m]$.
In the sequel. It follows from (7) that

$$H_0^1[1] f(z) = f(z), H_0^1[2] f(z) = z f'(z).$$

The linear operator $H_m^\iota[\alpha_1]$ is called Dozik-Srivastava operator (see [5]) introduced by Dozik and Srivastava which was subsequently extended by Dziok and Raina [4] by using the generalized hypergeometric function, recently Srivastava et. al. [12] defined the linear operator $\mathcal{L}_{\lambda,\iota,m}^{\gamma+c,\alpha_1}$ as follows:

$$\mathcal{L}_{\lambda,\iota,m}^{0} f(z) = f(z)$$

$$\mathcal{L}_{\lambda,\iota,m}^{1,\alpha_1} f(z) = (1-\lambda) H_m^\iota[\alpha_1] f(z) + \lambda (H_m^\iota[\alpha_1] f(z))' \mathcal{L}_{\lambda,\iota,m}^{\alpha_1} f(z), \quad (\lambda \geq 0), \tag{9}$$

$$\mathcal{L}_{\lambda,\iota,m}^{2,\alpha_1} f(z) = \mathcal{L}_{\lambda,\iota,m}^{\alpha_1} (\mathcal{L}_{\lambda,\iota,m}^{1,\alpha_1} f(z)) \tag{10}$$

and in general,

$$\mathcal{L}_{\lambda,\iota,m}^{\gamma,\alpha_1} f(z) = \mathcal{L}_{\lambda,\iota,m}^{\alpha_1} (\mathcal{L}_{\lambda,\iota,m}^{\gamma-1,\alpha_1} f(z)), (\iota \leq m+1\,;\ \iota, m \in N_0 = N \cup \{0\}; z \in U). \tag{11}$$

If the function $f(z)$ is given by (1), then we see form (7), (8), (9) and (11) that

$$\mathcal{L}_{\lambda,\iota,m}^{\gamma,\alpha_1} f(z) = z - \sum_{n=2}^{\infty} W_n^{\gamma}(\alpha_1;\lambda;\iota;m) a_n z^n, \qquad (12)$$

where

$$W_n^{\gamma}(\alpha_1;\lambda;\iota;m) = \left(\frac{(\alpha_1)_{n-1}\dots(\alpha_\iota)_{n-1}[1+\lambda(n-1)]}{(\beta_1)_{n-1}\dots(\beta_m)_{n-1}\ (n-1)!}\right)^{\tau}, n \in N\setminus\{1\}, \gamma \in N_0). \ (13)$$

Unless otherwise stated. We note that when $\gamma = 1$ and $\lambda = 0$ the linear operator $\mathcal{L}_{\lambda,\iota,m}^{\gamma,\alpha_1}$ would reduce to the familiar Dziok-Srivastava linear operator given by (see [5]), includes (as its special cases) various other linear operators introduced and studied by Carlson and Shaffer [3], Owa [9] and Ruscheweyh [10].

For tow analytic functions $f, g \in D$, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists as Schwarsz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = (g(z)), z \in U$. Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence (see [8]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Definition 1: For any function $f \in U$ and $\phi \geq 0$, the ϕ-neighborhood of f is defined as:

$$N_\phi(f) = \left\{ g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in D : \sum_{n=2}^{\infty} n|a_n - b_n| \leq \phi \right\}. \qquad (14)$$

In particular, for the function $e(z) = z$, we see that

$$N_\phi(e) = \left\{ g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in D : \sum_{n=2}^{\infty} n|b_n| \leq \phi \right\}. \qquad (15)$$

The concept of neighborhoods was first introduced by Goodman [6] and then generalized by Ruscheweyh [11].

Definition 2: For fixed parameters A and B, with $-1 \leq B \leq 0$ and $0 < A \leq 1$. We say that $f \in D$ is in the class $K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$ if it satisfies the following subordination condition:

$$\mathcal{L}_{\lambda,\iota,m}^{\gamma,\alpha_1} f(z) \prec \frac{1+Az}{1+Bz}. \qquad (16)$$

In view of the definition of subordination (16) is equivalent to the following condition:

$$\left| \frac{\mathcal{L}_{\lambda,\iota,m}^{\gamma,\alpha_1} f(z)}{B\mathcal{L}_{\lambda,\iota,m}^{\gamma,\alpha_1} f(z) - Az} \right| < 1, \ (z \in U). \qquad (17)$$

For convenience, we write

$K(\gamma, c, \alpha_1, \lambda, \iota, m, 1-2\eta, -1) = K(\gamma, c, \alpha_1, \lambda, \iota, m, \eta)$, where $K(\gamma, c, \alpha_1, \lambda, \iota, m, \eta)$ denotes the class of function in D satisfying the inequality:

$$Re\{\mathcal{L}_{\lambda,\iota,m}^{\gamma,\alpha_1} f(z)\} > \eta, (0 \leq \eta < 1, z \in U).$$

2. Neighborhoods for the class $K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$:

Theorem 1: A function $f \in D$ belong to the class $K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$ if and only if

$$\sum_{n=2}^{\infty} W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-AB)a_n \leq [1 + A(A-B-1)], \qquad (17)$$

$for\ \gamma, c, \iota, m \in N_0 = N \cup \{0\}, \iota \leq m+1, \lambda \geq 0, -1 \leq B \leq 0$ and $0 < A \leq 1$.

Proof: Let $f \in K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$.Then

$$\mathcal{L}_{\lambda,\iota,m}^{\gamma,\alpha_1} f(z) \prec \frac{1+Az}{1+Bz}, z \in U. \qquad (18)$$

Therefore there exists an analytic function w such that

$$w(z) = \frac{\mathcal{L}_{\lambda,\iota,m}^{\gamma,\alpha_1}(B-1) - Az}{\mathcal{L}_{\lambda,\iota,m}^{\gamma,\alpha_1}(B-AB) + A^2 z}. \qquad (19)$$

Hence

$$|w(z)| = \left| \frac{\mathcal{L}_{\lambda,\iota,m}^{\gamma,\alpha_1}(B-1) - Az}{\mathcal{L}_{\lambda,\iota,m}^{\gamma,\alpha_1}(B-AB) + A^2 z} \right|$$

$$= \left| \frac{(B-1)z - \sum_{n=2}^{\infty} W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(B-1)a_n z^n - Az}{(B-AB)z - \sum_{n=2}^{\infty} W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(B-AB)a_n z^n + A^2 z} \right| < 1.$$

Thus

$$Re\left\{ \frac{(B-1)z - \sum_{n=2}^{\infty} W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(B-1)a_n z^n - Az}{(B-AB)z - \sum_{n=2}^{\infty} W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(B-AB)a_n z^n + A^2 z} \right\} < 1. \quad (20)$$

Taking $|z| = r$, for sufficiently small r with $0 < r < 1$, the denominator of (20) is positive and so it is positive for all r with $0 < r < 1$, since $w(z)$ is analytic for $|z| < 1$. Then , the inequality (20) yields

$$\sum_{n=2}^{\infty} W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-B)a_n r^n + (B-A-1)r < \sum_{n=2}^{\infty} W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(AB-B) + (B + A^2 - AB)r.$$

Equivalently,

$$\sum_{n=2}^{\infty} W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-AB)a_n r^n \leq [1 + A(A-B-1)]r,$$

and (17)follows upon letting $r \to 1$.

Conversely, for $|z| = r, 0 < r < 1$, we have $r^n < r$. That is,

$$\sum_{n=2}^{\infty} W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)\,(1-AB)a_n r^n \le \sum_{n=2}^{\infty} W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)\,(1-AB)a_n r \le [1 + A(A-B-1)]r.$$

From (17), we have

$$\left|(B-A-1)z + \sum_{n=2}^{\infty} W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1-B)\,a_n z^n\right| \le (B-A-1)r + \sum_{n=2}^{\infty} W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1-B)\,a_n r^n$$
$$< \sum_{n=2}^{\infty} W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(AB-B)\,a_n r^n + (A^2+B-AB)r$$
$$< \left|\sum_{n=2}^{\infty} W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(AB-B)\,a_n z^n + (A^2+B-AB)z\right|.$$

This prove that

$$\mathcal{L}_{\lambda,\iota,m}^{\gamma,\alpha_1} f(z) \prec \frac{1+Az}{1+Bz}, z \in U$$

and hence $f \in K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$.

Theorem 2: If

$$\phi = \frac{[1 + A(A-B-1)]}{\left(\frac{(\alpha_1)_1 \dots (\alpha_\iota)_1}{(\beta_1)_1 \dots (\beta_m)_1}(1+\lambda)\right)^{\gamma+c}(1-AB)}, \quad (21)$$

then $K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B) \subset N_\phi(e)$.

Proof: It follows from (17), that if $f \in K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$, then

$$W_2^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1-AB)\sum_{n=2}^{\infty} n a_n \le [1 + A(A-B-1)],$$

hence

$$\left(\frac{(\alpha_1)_1 \dots (\alpha_\iota)_1}{(\beta_1)_1 \dots (\beta_m)_1}(1+\lambda)\right)^{\gamma+c}(1-AB)\sum_{n=2}^{\infty} n a_n \le [1 + A(A-B-1)], \quad (22)$$

which implies,

$$\sum_{n=2}^{\infty} n a_n \le \frac{[1 + A(A-B-1)]}{\left(\frac{(\alpha_1)_1 \dots (\alpha_\iota)_1}{(\beta_1)_1 \dots (\beta_m)_1}(1+\lambda)\right)^{\gamma+c}(1-AB)} = \phi. \quad (23)$$

Using (15), we get the result.

Definition 3: The function g defined by

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n$$

is said to be member of the class $K_\beta(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$ if there exists a function $f \in K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$ such that

$$\left|\frac{g(z)}{f(z)} - 1\right| \le 1 - \beta, \quad (z \in U, 0 \le \beta < 1). \quad (24)$$

Theorem 3: If $f \in K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$ and

$$\beta = 1 - \frac{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1-AB)}{2\left[W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1-AB) - [1 + A(A-B-1)]\right]}, \quad (25)$$

then $N_\phi(f) \subset K_\beta(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$.

Proof: Let $g \in N_\phi(f)$. Then we have from (14) that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \le \phi,$$

which implies the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \le \frac{\phi}{2}.$$

Also since $f \in K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$, we have from (17)

$$\sum_{n=2}^{\infty} a_n \le \frac{[1 + A(A-B-1)]}{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1-AB)},$$

where

$$W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m) = \left(\frac{(\alpha_1)_1 \dots (\alpha_\iota)_1}{(\beta_1)_1 \dots (\beta_m)_1}(1+\lambda)\right)^{\gamma+c},$$

so that

$$\left|\frac{g(z)}{f(z)} - 1\right| = \left|\frac{\sum_{n=2}^{\infty}(a_n - b_n)z^n}{z - \sum_{n=2}^{\infty} a_n z^n}\right| < \frac{\sum_{n=2}^{\infty}|a_n - b_n|}{1 - \sum_{n=2}^{\infty} a_n}$$
$$\le \frac{\phi}{2} \cdot \frac{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1-AB)}{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)[1 + A(A-B-1)]} = 1 - \beta.$$

Thus by Definition (3), $g \in K_\beta(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$ for β given by (25). This completes the proof.

3. Convolution Properties

Theorem 4: Let the function $f_j (j = 1,2)$ defined by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad (a_{n,j} \ge 0, j = 1,2), \quad (26)$$

be in the class $K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$.

Then $f_1 * f_2 \in K(\gamma, c, \alpha_1, \lambda, \iota, m, A, \sigma)$, where

$$\sigma \le \frac{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(A^2 - A + 1) - [1 + A(A-B-1)]^2}{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)A(1-AB)^2 - [A + [1 + A(A-B-1)]^2]}.$$

Proof: We must find the largest σ such that

$$\sum_{n=2}^{\infty} \frac{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1-A\sigma)}{[1 + A(A-\sigma-1)]} a_{n,1} a_{n,2} \le 1.$$

Since $f_j \in K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)(j = 1,2)$, then

$$\sum_{n=2}^{\infty} \frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-AB)}{[1+A(A-B-1)]} a_{n,j} \leq 1, \quad (j = 1,2). \qquad (27)$$

By Cahuch-Schwarz inequality, we get

$$\sum_{n=2}^{\infty} \frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-AB)}{[1+A(A-B-1)]} \sqrt{a_{n,1}a_{n,2}} \leq 1. \qquad (28).$$

We went only show that

$$\frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-A\sigma)}{[1+A(A-\sigma-1)]} a_{n,1}a_{n,2} \leq \frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-AB)}{[1+A(A-B-1)]} \sqrt{a_{n,1}a_{n,2}}.$$

This equivalently to

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{[1+A(A-\sigma-1)](1-AB)}{[1+A(A-B-1)](1-A\sigma)}$$

from (28), we have

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{[1+A(A-B-1)]}{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-AB)}.$$

Thus it is sufficient to show that

$$\frac{[1+A(A-B-1)]}{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-AB)} \leq \frac{[1+A(A-\sigma-1)](1-AB)}{[1+A(A-B-1)](1-A\sigma)},$$

which implies to

$$\sigma \leq \frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(A^2-A+1) - [1+A(A-B-1)]^2}{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)A(1-AB)^2 - [A+[1+A(A-B-1)]^2]}.$$

Theorem 5: Let the function $f_j(j = 1,2)$ defined by (26) be in the class $K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$. Then the function h defined by

$$h(z) = z - \sum_{n=2}^{\infty} (a_{n,1}{}^2 + a_{n,2}{}^2) z^n, \qquad (29)$$

belong to the class $K(\gamma, c, \alpha_1, \lambda, \iota, m, A, \varepsilon)$, where

$$\varepsilon \leq \frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)^2 A(1-AB)^2 - 2A[1+A(A-B-1)]^2}{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)^2(1-AB)^2(1+A+A^2) - W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)2[[1+A(A-B-1)]]^2},$$

$$\varepsilon \leq \frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)^2 A(1-AB)^2 - 2A[1+A(A-B}{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)^2(1-AB)^2(1+A+A^2) - W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)2}$$

Proof: We must find the largest ε such that

$$\sum_{n=2}^{\infty} \frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-A\varepsilon)}{[1+A(A-\varepsilon-1)]} (a_{n,1}{}^2 + a_{n,2}{}^2) \leq 1.$$

Since $f_j \in K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B, \varepsilon)(j = 1,2)$, we get

$$\sum_{n=2}^{\infty} \left(\frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-AB)}{[1+A(A-B-1)]}\right)^2 a_{n,1}{}^2 \leq \left(\sum_{n=2}^{\infty} \frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-AB)}{[1+A(A-B-1)]} a_{n,1}\right)^2 \leq 1, \qquad (30)$$

and

$$\sum_{n=2}^{\infty} \left(\frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-AB)}{[1+A(A-B-1)]}\right)^2 a_{n,2}{}^2 \leq \left(\sum_{n=2}^{\infty} \frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-AB)}{[1+A(A-B-1)]} a_{n,2}\right)^2 \leq 1. \qquad (31)$$

Combining the inequalities (30) and (31), gives

$$\sum_{n=2}^{\infty} \frac{1}{2}\left(\frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-AB)}{[1+A(A-B-1)]}\right)^2 (a_{n,1}{}^2 + a_{n,2}{}^2) \leq 1. \qquad (32)$$

But $h \in K(\gamma, c, \alpha_1, \lambda, \iota, m, A, \varepsilon)$, if and only if

$$\sum_{n=2}^{\infty} \frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-A\varepsilon)}{[1+A(A-\varepsilon-1)]} (a_{n,1}{}^2 + a_{n,2}{}^2) \leq 1, \qquad (33)$$

the inequality (33) will be satisfied if

$$\frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-A\varepsilon)}{[1+A(A-B-1)]} \leq \frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)^2(1-AB)^2}{2[1+A(A-B-1)]^2}, \quad (n = 2,3,\ldots)$$

so that

and this completes the proof.

4. Integral Mean Inequalities for the Fractional Integral

Definition 4[8]: The fractional integral of order s $(s > 0)$ is defined for a function by

$$D_z^{-s} f(z) = \frac{1}{\Gamma(s)} \int_0^z \frac{f(t)}{(z-t)^{1-s}} dt, \qquad (34)$$

where the function f is analytic in a simply-connected region of the complex $z-plane$containing, and multiplicity of $(z-t)^{s-1}$ is removed by requiring $log(z-t)$ to be real, when $(z-t) > 0$.

In 1925, Littlweood [7] proved the following subordination theorem:

Theorem 6 (Littlweood [7]) : If f and g are analytic in U with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}(0 < r < 1)$

$$\int_0^{2\pi} |f(z)|^{\mu} \, d\theta \leq \int_0^{2\pi} |g(z)|^{\mu} \, d\theta.$$

Theorem 7: Let $f \in K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$ and suppose that f_n is defined by

$$f_n = z - \frac{[1 + A(A - B - 1)]}{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1 - AB)} z^n, \quad (n \geq 2). \quad (35)$$

Also let

$$\sum_{i=2}^{\infty} (i - \eta)_{\eta+1} a_i \leq \frac{[1 + A(A - B - 1)]\Gamma(n + 1)\Gamma(s + \eta + 3)}{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1 - AB)\Gamma(n + s + \eta + 1)\Gamma(2 - \eta)}, \quad (36)$$

for $0 \leq \eta \leq i, s > 0,$ *where* $(i - \eta)_{\eta+1}$ denote the Pochhammer symbol defined by

$(i - \eta)_{\eta+1} = (i - \eta)(i - \eta + 1) \dots i$.

If there exists an analytic function q defined by

$$(q(z))^{n-1} = \frac{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1 - AB)\Gamma(n + s + \eta + 1)}{[1 + A(A - B - 1)]\Gamma(n + 1)} \sum_{i=2}^{\infty} (i - \eta)_{\eta+1} H(i) a_i z^{i-1}, \quad (37)$$

where $i \geq \eta$ and

$$H(i) = \frac{\Gamma(i - \eta)}{\Gamma(i + s + \eta + 1)}, \quad (s > 0, i \geq 2), \quad (38)$$

then, for $z = re^{i\theta}$ *and* $0 < r < 1$

$$\int_0^{2\pi} \left|D_z^{-s-\eta} f(z)\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|D_z^{-s-\eta} f_n(z)\right|^{\mu} d\theta, (s > 0, \mu > 0). \quad (39)$$

Proof: Let

$$f(z) = z - \sum_{i=2}^{\infty} a_i z^i .$$

For $\eta \geq 0$ and Definition 4, we get

$$D_z^{-s-\eta} f(z) = \frac{\Gamma(2) z^{s+\eta+1}}{\Gamma(s + \eta + 2)} \left(1 - \sum_{i=2}^{\infty} \frac{\Gamma(i + 1)\Gamma(s + \eta + 2)}{\Gamma(2)\Gamma(i + s + \eta + 1)} a_i z^{i-1}\right) = \frac{\Gamma(2) z^{s+\eta+1}}{\Gamma(s + \eta + 2)} \left(1 - \sum_{i=2}^{\infty} \frac{\Gamma(s + \eta + 2)}{\Gamma(2)} (i - \eta)_{\eta+1} H(i) a_i z^{i-1}\right),$$

where

$$H(i) = \frac{\Gamma(i - 1)}{\Gamma(i + s + \eta + 1)}, \quad (s \geq 0, i \geq 2).$$

Since H is decreasing function of i, we have

$$0 < H(i) \leq H(2) = \frac{\Gamma(2 - \eta)}{\Gamma(s + \eta + 3)}.$$

Similarly, from (35) and Definition 4, we get

$$D_z^{-s-\eta} f(z) = \frac{\Gamma(2) z^{s+\eta+1}}{\Gamma(s + \eta + 2)} \left(1 - \frac{[1 + A(A - B - 1)]\Gamma(n + 1)\Gamma(s + \eta + 2)}{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1 - AB)\Gamma(n + s + \eta + 1)} z^{n-1}\right).$$

For $\mu \geq 0$ *and* $z = re^{i\theta} (0 < r < 1)$, we must show that

$$\int_0^{2\pi} \left|1 - \sum_{i=2}^{\infty} \frac{\Gamma(s + \eta + 2)}{\Gamma(2)} (i - \eta)_{\eta+1} H(i) a_i z^{i-1}\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|1 - \frac{[1 + A(A - B - 1)]\Gamma(n + 1)\Gamma(s + \eta + 2)}{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1 - AB)\Gamma(2)\Gamma(n + s + \eta + 1)} z^{n-1}\right|^{\mu} d\theta.$$

By setting

$$1 - \sum_{i=2}^{\infty} \frac{\Gamma(s + \eta + 2)}{\Gamma(2)} (i - \eta)_{\eta+1} H(i) a_i z^{i-1} = 1 - \frac{[1 + A(A - B - 1)]\Gamma(n + 1)\Gamma(s + \eta + 2)}{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1 - AB)\Gamma(2)\Gamma(n + s + \eta + 1)} (q(z))^{n-1},$$

we find that

$$(q(z))^{n-1} = \frac{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1 - AB)\Gamma(n + s + \eta + 1)}{[1 + A(A - B - 1)]\Gamma(n + 1)} \sum_{i=2}^{\infty} (i - \eta)_{\eta+1} H(i) a_i z^{i-1},$$

which readily yields $w(0) = 0$. For such a function q, we obtain

$$\left|(q(z))\right|^{n-1} \leq \frac{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1 - AB)\Gamma(n + s + \eta + 1)}{[1 + A(A - B - 1)]\Gamma(n + 1)} \sum_{i=2}^{\infty} (i - \eta)_{\eta+1} H(i) a_i |z|^{i-1}$$

$$\leq \frac{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1 - AB)\Gamma(n + s + \eta + 1)}{[1 + A(A - B - 1)]\Gamma(n + 1)} H(2)|z| \sum_{i=2}^{\infty} (i - \eta)_{\eta+1} H(i) a_i$$

$$= |z| \frac{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1 - AB)\Gamma(n + s + \eta + 1)\Gamma(2 - \eta)}{[1 + A(A - B - 1)]\Gamma(s + \eta + 3)\Gamma(n + 1)} \sum_{i=2}^{\infty} (i - \eta)_{\eta+1} H(i) a_i \leq |z| < 1.$$

This completes the proof of the theorem.

By taking $\eta = 0$ in the Theorem 7, we have the following corollary:

Corollary 1: Let $f \in K(\gamma, c, \alpha_1, \lambda, \iota, m, A, B)$ and suppose that f_n is defined by (35). Also let

$$\sum_{i=2}^{\infty} i a_i \leq \frac{[1 + A(A - B - 1)]\Gamma(n + 1)\Gamma(s + 3)}{W_n^{\gamma+c}(\alpha_1; \lambda; \iota; m)(1 - AB)\Gamma(s + \eta + 1)\Gamma(2)}, \quad n \geq 2.$$

If there exists an analytic function q defined by

$$(q(z))^{n-1} = \frac{W_n^{\gamma+c}(\alpha_1;\lambda;\iota;m)(1-AB)\Gamma(s+\eta+1)}{[1+A(A-B-1)]\Gamma(n+1)}\sum_{i=2}^{\infty} iH(i)a_i z^{i-1},$$

where

$$H(i) = \frac{\Gamma(i)}{\Gamma(i+s+1)}, \qquad (s>0, i\leq 2),$$

then, for $z = re^{i\theta}$ and $0 < r < 1$

$$\int_0^{2\pi} |D_z^{-s} f(z)|^{\mu}\, d\theta \leq \int_0^{2\pi} |D_z^{-s} f_n(z)|^{\mu}\, d\theta, (s>0, \mu > 0).$$

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