

# Some Interesting Properties of a Subclass of Meromorphic Univalent Functions Defined by Hadamard Product

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**Abstract:** In the present paper, we define a new subclass  $SM_u(\tau, \gamma, \lambda, \alpha)$  of meromorphic univalent with positive coefficients defined by Hadamard product in the punctured unit disk  $U^*$ . We obtain some interesting properties, like, coefficient estimates, extreme points, distortion theorem, partial sums, integral representation.

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## 1. Introduction

Let  $M_u$  denote the class of functions of the form:

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \quad (1)$$

which are analytic and meromorphic univalent in the punctured unit disk

$U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ .

Let  $SM_u$  be a subclass of  $M_u$  consisting of functions of the form:

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \quad (a_n \geq 0). \quad (2)$$

For the function  $f \in SM_u$  given by (2) and  $g \in SM_u$  defined by

$$g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n, \quad (b_n \geq 0), \quad (3)$$

the convolution (or Hadamard product) of  $f$  and  $g$  is defined by

$$(f * g)(z) = z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n. \quad (4)$$

We shall need to state the extended linear derivative operator of Ruscheweyh type for the function belong to the class  $SM_u$

$D_*^{\lambda,1}: SM_u \rightarrow SM_u$  is defined by the following convolution:

$$D_*^{\lambda,1} f(z) = \frac{z^{-1}}{(1-z)^{\lambda+1}} * f(z), \quad (\lambda > -1; f \in SM_1). \quad (5)$$

In terms of binomial coefficients, (5) can be written as

$$D_*^{\lambda,1} f(z) = z^{-1} + \sum_{n=1}^{\infty} \binom{\lambda+n}{n} a_n z^n \quad (\lambda > -1; f \in SM_u). \quad (6)$$

The linear operator  $D_*^{\lambda,1}$  analogous to  $D_*^{\lambda,1}$  was consider recently by Raina and Srivastava [7] on the space of analytic and p-valent functions in  $U$  ( $U = U^* \cup \{0\}$ ).

A function  $f \in M_u$  is said to be in the class  $M_u S$  of meromorphic univalent starlike function of order  $\alpha$  if :

$$-Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (z \in U^*, 0 \leq \alpha < 1). \quad (7)$$

A function  $f \in M_u$  is said to be in the class  $M_u C$  of meromorphic univalent convex function of order  $\alpha$  if :

$$-Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad (z \in U^*, 0 \leq \alpha < 1). \quad (8)$$

**Definition 1:** A function  $f \in SM_u$  is said to be in the class of  $SM_u(\tau, \gamma, \lambda, \alpha)$  if it satisfies the following condition:

$$\left| \frac{\frac{\frac{z\tau}{2} (D_*^{\lambda,1}(f * g)(z))' + \frac{\tau}{2}}{(D_*^{\lambda,1}(f * g)(z))} + \frac{\tau}{2}}{\alpha(1+3\gamma) + \frac{2z\alpha\gamma (D_*^{\lambda,1}(f * g)(z))'}{(D_*^{\lambda,1}(f * g)(z))}} \right| < 1, \quad (9)$$

for  $0 < \gamma < 1, 0 < \tau < 1, 0 < \alpha < 1$ .

Atshan and Kulkarni [3], Atshan and Buti [1], Atshan and Joudah [2], Dziok et al. [4], Khairnar and More [5] and Najafzadeh and Ebadian [6] studied meromorphic univalent and Multivalent functions for different classes.

## 2. Coefficient Inequality

The following theorem gives a necessary and sufficient condition for a function  $f$  to be in the class  $SM_u(\tau, \gamma, \lambda, \alpha)$ .

**Theorem 1:** Let  $f \in SM_u$ . Then  $f \in SM_u(\tau, \gamma, \lambda, \alpha)$  if and only if

$$\sum_{n=1}^{\infty} \binom{\lambda+n}{n} \left( \frac{\tau}{2}(n+1) + \gamma + \alpha(1+\gamma(3+2n)) \right) a_n b_n \leq \alpha(1+\gamma), \quad (10)$$

where  $0 < \gamma < 1, 0 < \tau < 1, 0 < \alpha < 1$ .

The result is sharp for the function

$$f(z) = z^{-1} + \frac{\alpha(1 + \gamma)}{\binom{\lambda+n}{n} \left(\frac{\tau}{2}(n+1) + \alpha(1 + \gamma(3 + 2n))\right) b_n} z^n, n \in N.$$

**Proof:** Suppose that the inequality (10) holds true and  $|z| = 1$ . Then from (9), we have

$$\left| \frac{z\tau}{2} (D_*^{\lambda,1}(f * g)(z))' + \frac{\tau}{2} (D_*^{\lambda,1}(f * g)(z)) \right| - \left| \alpha(1 + \gamma) (D_*^{\lambda,1}(f * g)(z)) + \gamma (D_*^{\lambda,1}(f * g)(z)) + z\alpha\gamma (D_*^{\lambda,1}(f * g)(z))' \right|$$

$$= \left| \sum_{n=1}^{\infty} \frac{\tau}{2} (n+1) \binom{\lambda+n}{n} a_n b_n z^n \right| - \left| \alpha(1 + 3\gamma) (z^{-1} + \sum_{n=1}^{\infty} \binom{\lambda+n}{n} a_n b_n z^n) + 2z\alpha\gamma (-z^{-2} + \sum_{n=1}^{\infty} n \binom{\lambda+n}{n} a_n b_n z^{n-1}) \right|$$

$$= \left| \sum_{n=1}^{\infty} \frac{\tau}{2} (n+1) \binom{\lambda+n}{n} a_n b_n z^n \right| - \left| \alpha(1 + 3\gamma) z^{-1} + \alpha(1 + 3\gamma) \sum_{n=1}^{\infty} \binom{\lambda+n}{n} a_n b_n z^n - 2\alpha\gamma z^{-1} + \sum_{n=1}^{\infty} 2\alpha\gamma n \binom{\lambda+n}{n} a_n b_n z^n \right|$$

$$= \left| \sum_{n=1}^{\infty} \frac{\tau}{2} (n+1) \binom{\lambda+n}{n} a_n b_n z^n \right| - \left| \alpha(1 + \gamma) z^{-1} + \sum_{n=1}^{\infty} \alpha(1 + \gamma(3 + 2n)) \binom{\lambda+n}{n} a_n b_n z^n \right| \quad (11)$$

$$\leq \sum_{n=1}^{\infty} \frac{\tau}{2} (n+1) \binom{\lambda+n}{n} a_n b_n - \alpha(1 + \gamma) + \sum_{n=1}^{\infty} \alpha(1 + \gamma(3 + 2n)) \binom{\lambda+n}{n} a_n b_n$$

$$= \sum_{n=1}^{\infty} \binom{\lambda+n}{n} \left( \frac{\tau}{2} (n+1) + \alpha(1 + \gamma(3 + 2n)) \right) a_n b_n - \alpha(1 + \gamma) \leq 0,$$

by hypothesis.

Hence, by maximum modulus principle,  $f \in SM_u(\tau, \gamma, \lambda, \alpha)$

Conversely, assume that defined by (2) is un the class  $SM_u(\tau, \gamma, \lambda, \alpha)$ .

Then from (9), we have

$$\left| \frac{\frac{z\tau}{2} (D_*^{\lambda,1}(f * g)(z))' + \frac{\tau}{2} (D_*^{\lambda,1}(f * g)(z))}{\alpha(1 + 3\gamma) + \frac{2z\alpha\gamma (D_*^{\lambda,1}(f * g)(z))'}{(D_*^{\lambda,1}(f * g)(z))}} \right| = \left| \frac{\sum_{n=1}^{\infty} \frac{\tau}{2} (n+1) \binom{\lambda+n}{n} a_n b_n z^n}{(\alpha + \alpha\gamma) z^{-1} + \sum_{n=1}^{\infty} \alpha(1 + \gamma(3 + 2n)) \binom{\lambda+n}{n} a_n b_n z^n} \right| < 1.$$

Since  $\text{Re}(z) \leq |z|$  for all  $z (z \in U^*)$ , we get

$$\text{Re} \left\{ \frac{\sum_{n=1}^{\infty} \frac{\tau}{2} (n+1) \binom{\lambda+n}{n} a_n b_n z^n}{(\alpha + \alpha\gamma) z^{-1} + \sum_{n=1}^{\infty} \alpha(1 + \gamma(3 + 2n)) \binom{\lambda+n}{n} a_n b_n z^n} \right\} < 1. \quad (12)$$

We choose the value of  $z$  on the real axis so that  $\frac{z(D_*^{\lambda,1}(f * g)(z))'}{(D_*^{\lambda,1}(f * g)(z))}$  is real.

Let  $z \rightarrow 1^-$  through real values, so we can write (12) as

$$\sum_{n=1}^{\infty} \binom{\lambda+n}{n} \left( \frac{\tau}{2} (n+1) + \alpha(1 + \gamma(3 + 2n)) \right) a_n b_n \leq \alpha(1 + \gamma).$$

Finally, sharpness follows if we take

$$f(z) = z^{-1} + \frac{\alpha(1 + \gamma)}{\binom{\lambda+n}{n} \left(\frac{\tau}{2}(n+1) + \alpha(1 + \gamma(3 + 2n))\right) b_n} z^n, n = 1, 2, \dots \quad (13)$$

The proof is complete.

**Corollary 1:** Let  $f \in SM_u(\tau, \gamma, \lambda, \alpha)$ . Then

$$a_n \leq \frac{\alpha(1 + \gamma)}{\binom{\lambda+n}{n} \left(\frac{\tau}{2}(n+1) + \alpha(1 + \gamma(3 + 2n))\right) b_n}, n = 2, 3, \dots \quad (14)$$

### 3. Distortion Bounds

Next, we obtain the growth and distortion bounds for the class  $SM_u(\tau, \gamma, \lambda, \alpha)$ .

**Theorem 2:** If  $f \in SM_u(\tau, \gamma, \lambda, \alpha)$  and  $b_n \geq b_1 (n \geq 1)$ , then

$$\frac{1}{r} - \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1} r \leq |f(z)| \leq \frac{1}{r} + \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1} r, (|z| = r < 1),$$

$$\text{and} \quad \frac{1}{r^2} - \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1} r \leq |f'(z)| \leq \frac{1}{r^2} + \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1} r, (|z| = r < 1).$$

The result is sharp for the function:

$$f(z) = z^{-1} + \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1} z. \quad (15)$$

**Proof:** Since

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n,$$

then

$$\begin{aligned} |f(z)| &= \left| \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \right| \\ &\leq \frac{1}{|z|} + \sum_{n=1}^{\infty} a_n |z|^n = \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \\ &\leq \frac{1}{r} + r \sum_{n=1}^{\infty} a_n \quad (16) \end{aligned}$$

Since for  $n \geq 1$ ,

$$\begin{aligned} (1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1 \\ \leq \binom{\lambda + n}{n} \left( \frac{\tau}{2}(n + 1) \right. \\ \left. + \alpha(1 + \gamma(3 + 2n)) \right) b_n. \end{aligned}$$

By Theorem (1), we have

$$\begin{aligned} (1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1 \sum_{n=1}^{\infty} a_n \\ \leq \sum_{n=1}^{\infty} \left( \binom{\lambda + n}{n} \left( \frac{\tau}{2}(n + 1) \right. \right. \\ \left. \left. + \alpha(1 + \gamma(3 + 2n)) \right) a_n b_n. \right. \end{aligned}$$

$\leq \alpha(1 + \gamma)$ .

That is

$$\sum_{n=1}^{\infty} a_n \leq \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1}.$$

Using the above inequality in (16), we have

$$|f(z)| \leq \frac{1}{r} + \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1} r,$$

and

$$|f(z)| \geq \frac{1}{r} - \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1} r.$$

The result is sharp for the function:

$$f(z) = \frac{1}{z} + \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1} z,$$

Similarly, we have

$$|f'(z)| \geq \frac{1}{r^2} - \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1},$$

and

$$|f'(z)| \leq \frac{1}{r^2} + \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1}.$$

#### 4. Partial Sums

**Theorem 3:** Let  $f \in SM_u$  be given by (2) and the partial sums  $S_1(z)$  and  $S_k(z)$  be defined by

$S_1(z) = z^{-1}$  and

$$S_k(z) = z^{-1} + \sum_{n=1}^{k-1} a_n z^n, \quad (k > 1).$$

Also, suppose that

$$\begin{aligned} \sum_{n=1}^{\infty} d_n a_n \\ \leq 1, \left( d_n = \frac{\binom{\lambda+n}{n} \left( \frac{\tau}{2}(n+1) + \alpha(1 + \gamma(3 + 2n)) \right)}{\alpha(1 + \gamma)} \right). \quad (17) \end{aligned}$$

Then, we have

$$Re \left\{ \frac{f(z)}{S_k(z)} \right\} > 1 - \frac{1}{d_k}, \quad (18)$$

and

$$Re \left\{ \frac{S_k(z)}{f(z)} \right\} > \frac{d_k}{1 + d_k} \quad (z \in U, k > 1). \quad (19)$$

Each of the bounds in (18) and (19) is the best possible for  $n \in \mathbb{N}$ .

**Proof:** For the coefficients  $d_n$  given by (17), it is not difficult to verify that

$$d_{n+1} > d_n > 1, \quad n = 1, 2, \dots$$

Therefore, by using the hypothesis (17), we have

$$\sum_{n=1}^{k-1} a_n + d_k \sum_{n=k}^{\infty} a_n \leq \sum_{n=1}^{\infty} d_n a_n \leq 1. \quad (20)$$

By setting

$$\begin{aligned} g_1(z) &= d_k \left( \frac{f(z)}{S_k(z)} - \left( 1 - \frac{1}{d_k} \right) \right) \\ &= 1 + \frac{d_k \sum_{n=k}^{\infty} a_n z^{n+1}}{1 + \sum_{n=1}^{k-1} a_n z^{n+1}} \quad (21) \end{aligned}$$

and applying (20), we find that

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{d_k \sum_{n=k}^{\infty} a_n}{2 - 2 \sum_{n=1}^{k-1} a_n - d_k \sum_{n=k}^{\infty} a_n} \leq 1, \quad (22)$$

which readily yields the assertion (18), if we take

$$f(z) = z^{-1} - \frac{z^k}{d_k}, \quad (23)$$

then

$$\frac{f(z)}{S_k(z)} = 1 - \frac{z^k}{d_k} \rightarrow 1 - \frac{1}{d_k} \quad (z \rightarrow 1^-),$$

which shows that the bound (18) is the best possible for each  $n \in \mathbb{N}$ .

Similarly, if we take

$$\begin{aligned} g_2(z) &= (1 + d_k) \left( \frac{S_k(z)}{f(z)} - \frac{d_k}{1 + d_k} \right) \\ &= 1 + \frac{(1 + d_k) \sum_{n=k}^{\infty} a_n z^{n+1}}{1 + \sum_{n=1}^{k-1} a_n z^{n+1}}, \end{aligned}$$

and make use of (20), we obtain

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + d_k) \sum_{n=k}^{\infty} a_n}{2 - 2 \sum_{n=1}^{k-1} a_n + (1 + d_k) \sum_{n=k}^{\infty} a_n} \leq 1, \quad (24)$$

which leads us to the assertion (19). The bounds in (18) and (19) is sharp with the function given by (21).

**Theorem 4:** If  $f(z)$  of the form (2) satisfy the condition (10). Then

$$\begin{aligned} Re \left\{ \frac{f'(z)}{S'_k(z)} \right\} &> 1 - \frac{k + 1}{d_{k+1}} \\ Re \left\{ \frac{f(z)}{S'_k(z)} \right\} &> \frac{d_{m+1}}{k + 1 + d_{k+1}}, \end{aligned}$$

where

$$d_n \geq \begin{cases} n \text{ for } n = 2, 3, \dots, m \\ \frac{\binom{\lambda+n}{n} \left( \frac{\tau}{2}(n+1) + \alpha(1+\gamma(3+2n)) \right) b_n}{\alpha(1+\gamma)} \text{ for } n \\ = m+1, m+2, m+3, \dots \end{cases}$$

The bounds are sharp, with the extremal function  $f(z)$  of the form (15)

**Proof:** The proof is analogous to that Theorem 3, and we omit details.

### 5. Integral Representation

**Theorem 5:** Let  $f \in SM_u(\tau, \gamma, \lambda, \alpha)$ . Then

$$D_*^{\lambda,1}(f * g)(z) = \exp \int_0^z \frac{\vartheta(t)\alpha(1+3\gamma) - \frac{\tau}{2}}{\frac{\tau}{2} - \vartheta(t)2\alpha\gamma} dt,$$

where  $|\vartheta(z)| < 1, z \in U$ .

**Proof:** By putting

$$\frac{z(D_*^{\lambda,1}(f * g)(z))'}{(D_*^{\lambda,1}(f * g)(z))} = Q(z)$$

in (9), we have

$$\left| \frac{\frac{\tau}{2}Q(z) + \frac{\tau}{2}}{\alpha(1+3\gamma) + 2\alpha\gamma Q(z)} \right| < 1,$$

or equivalently

$$\frac{\frac{\tau}{2}Q(z) + \frac{\tau}{2}}{\alpha(1+3\gamma) + 2\alpha\gamma Q(z)} = \vartheta(z), (|\vartheta(z)| < 1, z \in U).$$

So

$$\frac{(D_*^{\lambda,1}(f * g)(z))'}{(D_*^{\lambda,1}(f * g)(z))} = \frac{\vartheta(z)\alpha(1+3\gamma) - \frac{\tau}{2}}{z\left(\frac{\tau}{2} - \vartheta(z)2\alpha\gamma\right)},$$

after integration, we have

$$\log(D_*^{\lambda,1}(f * g)(z)) = \int_0^z \frac{\vartheta(t)\alpha(1+3\gamma) - \frac{\tau}{2}}{\frac{\tau}{2} - \vartheta(t)2\alpha\gamma} dt.$$

Therefore

$$D_*^{\lambda,1}(f * g)(z) = \exp \int_0^z \frac{\vartheta(t)\alpha(1+3\gamma) - \frac{\tau}{2}}{\frac{\tau}{2} - \vartheta(t)2\alpha\gamma} dt,$$

and this gives the required result.

### 6. Extreme Points

**Theorem 6:** Let  $f_0(z) = z^{-1}$  and

$$f_n(z) = z^{-1}$$

$$+ \frac{\alpha(1+\gamma)}{\binom{\lambda+n}{n} \left( \frac{\tau}{2}(n+1) + \alpha(1+\gamma(3+2n)) \right) b_n} z^n, (n \geq 1). \quad (25)$$

Then  $f \in SM_u(\tau, \gamma, \lambda, \alpha)$ , if and only if it can be represented in the form

$$f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z), (\mu_n \geq 0, \sum_{n=0}^{\infty} \mu_n = 1). \quad (26)$$

**Proof:** Suppose that

$$f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z), \text{ where } \mu_n \geq 0, \sum_{n=0}^{\infty} \mu_n = 1.$$

Then

$$f(z) = \mu_0 f_0(z) + \sum_{n=1}^{\infty} \mu_n f_n(z)$$

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} \frac{\alpha(1+\gamma)\mu_n}{\binom{\lambda+n}{n} \left( \frac{\tau}{2}(n+1) + \alpha(1+\gamma(3+2n)) \right) b_n} z^n$$

$$= z^{-1} + \sum_{n=1}^{\infty} \ell_n z^n,$$

where

$$\ell_n = \frac{\alpha(1+\gamma)\mu_n}{\binom{\lambda+n}{n} \left( \frac{\tau}{2}(n+1) + \alpha(1+\gamma(3+2n)) \right) b_n}.$$

Therefore

$$\sum_{n=1}^{\infty} \mu_n \frac{\alpha(1+\gamma)}{\binom{\lambda+n}{n} \left( \frac{\tau}{2}(n+1) + \alpha(1+\gamma(3+2n)) \right) b_n} = \sum_{n=1}^{\infty} \mu_n = 1 - \mu_0 \leq 1.$$

So by Theorem (1),  $f \in SM_u(\tau, \gamma, \lambda, \alpha)$ .

Conversely, we suppose  $f \in SM_u(\tau, \gamma, \lambda, \alpha)$ . By (4), we have

$$a_n \leq \frac{\alpha(1+\gamma)}{\binom{\lambda+n}{n} \left( \frac{\tau}{2}(n+1) + \alpha(1+\gamma(3+2n)) \right) b_n}, n \geq 1.$$

We set,

$$\mu_n = \frac{\binom{\lambda+n}{n} \left( \frac{\tau}{2}(n+1) + \alpha(1+\gamma(3+2n)) \right) b_n}{\alpha(1+\gamma)} a_n, n \geq 1,$$

and

$$\mu_0 = 1 - \sum_{n=1}^{\infty} \mu_n.$$

Then, we have

$$f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z)$$

$$f(z) = \mu_0 f_0(z) + \sum_{n=1}^{\infty} \mu_n f_n(z).$$

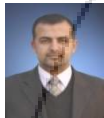
Hence the results follows.

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