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Subclass of Meromorphic Univalent Functions Defined by Hadamard Product with Multiplier Transformation

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Abstract

In this paper, we have introduced a class $\mathcal{M}(\beta, \alpha, \lambda, r)$ of meromorphic univalent functions defined by Hadamard product with multiplier transformation in the punctured unit disk U^*. We study several properties like , coefficient estimates and closure theorems. Also we obtain some results connected with (n, δ)- neighborhoods on $\mathcal{M}^{\eta}(\beta, \alpha, \lambda, r)$ inclusive property and integral operator.

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1 Introduction

Let $\sum$ denote the class of functions f of the form :

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n$$

which are analytic and meromorphic univalent in the punctured unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \setminus \{0\}$ and suppose H denote the subclass of meromorphic univalent functions in the punctured unit disk U^* of the form:

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \quad (a_n \geq 0, \quad n \in \mathbb{N} = \{1, 2, \cdots\}). \tag{1}$$

A function $f \in H$ is meromorphic starlike function of order $\rho, (0 \leq \rho < 1)$ if

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho, \quad (z \in U). \tag{2}$$

The class of all such functions is denoted by $H^*(\rho)$. A functions $f \in H$ is meromorphic convex function of order $\rho, (0 \leq \rho < 1)$ if

$$-Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho, \quad (z \in U). \tag{3}$$

The Hadamard product (or convolution) of two functions f is given by (1) and

$$g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n, \quad (b_n \geq 0, n \in \mathbb{N} = \{1, 2, \cdots\}). \tag{4}$$

is defined by

$$(f * g)(z) = z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n.$$

We shall need the multiplier transformation on H, we define the operator $\mathcal{I}_1(r, \lambda)$ by the following infinite series when

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n,$$

then

$$\mathcal{I}_1(r, \lambda)f(z) = z^{-1} + \sum_{n=1}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^r \quad a_n z^n, \quad \lambda \geq 0. \tag{5}$$

The operator $\mathcal{I}_1(r, \lambda)$ was studied recently by Cho and Srivastava [4] and Cho and Kim [3]. The operator $\mathcal{I}_p(r, \lambda)$ was studied on class of meromorphic multivalent functions with complex order by Tehranchi and Kulkarni in [7].

Definition 1 : Let $f \in H$ be given by (1). The class $\mathcal{M}(\beta, \alpha, \lambda, r)$ is defined by:

$$\mathcal{M}(\beta, \alpha, \lambda, r) = \left\{ f \in H : Re\left\{ \frac{1 + \frac{z(\mathcal{I}_1(r,\lambda)(f*g)(z))''}{(\mathcal{I}_1(r,\lambda)(f*g)(z))'}}{(2\alpha - 1) + \alpha \frac{z(\mathcal{I}_1(r,\lambda)(f*g)(z))''}{(\mathcal{I}_1(r,\lambda)(f*g)(z))'}} \right\} > \beta, \right.$$

$$\left. 0 \leq \beta < 1,\ 0 \leq \alpha < 1,\ r \in \mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}, \lambda \geq 0 \right\}. \tag{6}$$

Lemma 1(see [1]) : If w is any complex number, then $Re(w) > \mu$ if and only if $|w - (1 + \mu)| < |w + (1 - \mu)|$ where $0 \leq \mu$.

The same some of properties have been studied for other class in [5].

2 Main Results

The first theorem gives a necessary and sufficient condition for a function f to be in the class $\mathcal{M}(\beta, \alpha, \lambda, r)$.

Theorem 1 : Let $f \in H$. Then $f \in \mathcal{M}(\beta, \alpha, \lambda, r)$ if and only if

$$\sum_{n=1}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^r n(\beta(1-\alpha) + n(1-\beta\alpha))a_n b_n \leq 1 - \beta. \tag{7}$$

The result is sharp for the function

$$f(z) = z^{-1} + \frac{1-\beta}{\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\beta(1-\alpha) + n(1-\beta\alpha))b_n} z^n.$$

Proof : Let $|z| = 1$, in view of (6), we need to prove that $Re(w) > \beta$, where

$$\begin{aligned} w &= \frac{(\mathcal{I}_1(r,\lambda)(f*g)(z))' + z(\mathcal{I}_1(r,\lambda)(f*g)(z))''}{(2\alpha-1)(\mathcal{I}_1(r,\lambda)(f*g)(z))' + \alpha z(\mathcal{I}_1(r,\lambda)(f*g)(z))''} \\ &= \frac{z^{-2} + \sum_{n=1}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^r n^2 a_n b_n z^{n-1}}{z^{-2} + \sum_{n=1}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^r n(\alpha(1+n)-1) a_n b_n z^{n-1}} \\ &= \frac{A(z)}{B(z)}. \end{aligned} \tag{8}$$

By Lemma 1, it sufficies to show that

$$|A(z) - (1+\beta)B(z)| - |A(z) + (1-\beta)B(z)| \leq 0. \tag{9}$$

Therefore, we obtain

$$\begin{aligned}
&|A(z)-(1+\beta)B(z)|-|A(z)+(1-\beta)B(z)| \\
&\le \beta|z|^{-2}+\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^{r} n(n-(1+\beta)(\alpha(1+n)-1))a_nb_n|z|^{n-1} \\
&-(2-\beta)|z|^{-2}++\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^{r} n(n+(1-\beta)(\alpha(1+n)-1))a_nb_n|z|^{n-1} \\
&=(2\beta-2)+\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^{r} n(2n-(1+\beta)(\alpha(1+n)-1)+(1-\beta)(\alpha(1+n)-1))a_nb_n \\
&=(2\beta-2)+\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^{r} n(2n+2\beta(1-\alpha)-2\beta\alpha n)a_nb_n \\
&=2(\beta-1)+\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^{r} n(2n(1-\beta\alpha)+2\beta(1-\alpha))a_nb_n \\
&=(\beta-1)+\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^{r} n(n(1-\beta\alpha)+\beta(1-\alpha))a_nb_n \\
&=\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^{r} n(\beta(1-\alpha)+n(1-\beta\alpha))a_nb_n-(1-\beta)\le 0.
\end{aligned}$$

By hypothesis. Conversely, assume that

$$Re\left\{\frac{1+\frac{z(\mathcal{I}_1(r,\lambda)(f*g)(z))''}{(\mathcal{I}_1(r,\lambda)(f*g)(z))'}}{(2\alpha-1)+\alpha\frac{z(\mathcal{I}_1(r,\lambda)(f*g)(z))''}{(\mathcal{I}_1(r,\lambda)(f*g)(z))'}}\right\}$$

$$=Re\left\{\frac{z^{-2}+\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^{r} n^2a_nb_nz^{n-1}}{z^{-2}+\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^{r} n(\alpha(1+n)-1)a_nb_nz^{n-1}}\right\}>\beta. \tag{10}$$

we can choose the value of z on the real axis, so that $(\mathcal{I}_1(r,\lambda)(f*g)(z))'$ is real. Let $z\to 1^-$, through real values, so we can write (10) as

$$\sum_{n=1}^{\infty}\left(\frac{n+\lambda}{1+\lambda}\right)^{r} n(\beta(1-\alpha)+n(1-\beta\alpha))a_nb_n\le 1-\beta.$$

Finally, sharpness follows if we take

$$f(z)=z^{-1}+\frac{1-\beta}{\left(\frac{n+\lambda}{1+\lambda}\right)^{r} n(\beta(1-\alpha)+n(1-\beta\alpha))b_n}z^n,\quad n=1,2,\cdots. \tag{11}$$

The proof is complete.

Corollary 1 : Let $f \in \mathcal{M}(\beta, \alpha, \lambda, r)$. Then

$$a_n \leq \frac{1-\beta}{\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\beta(1-\alpha)+n(1-\beta\alpha))b_n}, \quad n = 1, 2, \cdots . \tag{12}$$

The equality in (12) is attained for the function f given by (11).

3 Closure Theorem

Theorem 2 : Let the functions f_k defined by

$$f_k(z) = z^{-1} + \sum_{n=1}^{\infty} a_{n,k} z^n, \quad (a_{n,k} \geq 0, n \in \mathbb{N}, k = 1, 2, \cdots, \ell)$$

be in the class $\mathcal{M}(\beta, \alpha, \lambda, r)$ for every $k = 1, 2, \cdots, \ell$. Then the function h defined by

$$h(z) = z^{-1} + \sum_{n=1}^{\infty} e_n z^n, \quad (e_n \geq 0, n \in \mathbb{N})$$

also belongs to the class $\mathcal{M}(\beta, \alpha, \lambda, r)$, where

$$e_n = \frac{1}{\ell} \sum_{k=1}^{\ell} a_{n,k}, \quad (n = 1, 2, \cdots).$$

Proof : Since $f_k \in \mathcal{M}(\beta, \alpha, \lambda, r)$ it follows from Theorem 1 that

$$\sum_{n=1}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^r n(\beta(1-\alpha)+n(1-\beta\alpha))a_{n,k}b_n \leq 1-\beta, \tag{13}$$

for every $k = 1, 2, \cdots, \ell$. Hence

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^r n(\beta(1-\alpha)+n(1-\beta\alpha))e_n b_n \\
&= \sum_{n=1}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^r n(\beta(1-\alpha)+n(1-\beta\alpha))b_n \left(\frac{1}{\ell}\sum_{k=1}^{\ell} a_{n,k}\right) \\
&= \frac{1}{\ell}\sum_{k=1}^{\ell} \left(\sum_{n=1}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^r n(\beta(1-\alpha)+n(1-\beta\alpha))a_{n,k}b_n\right) \leq 1-\beta.
\end{aligned}$$

By Theorem 1, it follows that $h \in \mathcal{M}(\beta, \alpha, \lambda, r)$.

4 Neighborhood Property and Partial Sums

We define the (n,δ)-neighborhood of a function $f \in H$ by

$$N_{n,\delta}(f) = \left\{ g \in H : g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n \ \text{ and } \ \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta, 0 \leq \delta < 1 \right\}. \tag{14}$$

For the identity function $e(z) = z$, we have

$$N_{n,\delta}(e) = \left\{ g \in H : g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n \ \text{ and } \ \sum_{n=1}^{\infty} n|b_n| \leq \delta \right\}.$$

Definition 2 : A function $f \in H$ is said to be in the class $\mathcal{M}^{\eta}(\beta, \alpha, \lambda, r)$ if there exists a function $g \in \mathcal{M}(\beta, \alpha, \lambda, r)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \eta, \quad (z \in U, 0 \leq \eta < 1).$$

Theorem 3 : If $g \in \mathcal{M}(\beta, \alpha, \lambda, r)$ and

$$\eta = 1 - \frac{\delta(1 + \beta(1 - 2\alpha))a_1}{(1 + \beta(1 - 2\alpha))a_1 - (1 - \beta)}. \tag{15}$$

Then $N_{n,\delta}(g) \subset \mathcal{M}^{\eta}(\beta, \alpha, \lambda, r)$.

Proof : Let $f \in N_{n,\delta}(g)$. We want to find from (14) that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta,$$

which readily implies the following coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, \quad (n \in \mathbb{N}). \tag{16}$$

Next, since $g \in \mathcal{M}(\beta, \alpha, \lambda, r)$, we have from Theorem 1

$$\sum_{n=1}^{\infty} b_n \leq \frac{(1 - \beta)}{(1 + \beta(1 - 2\alpha))a_1}. \tag{17}$$

So that

$$\left| \frac{f(z)}{g(z)} - 1 \right| \leq \frac{\sum_{n=1}^{\infty} |a_n - b_n|}{1 - \sum_{n=1}^{\infty} b_n} \leq \frac{\delta(1 + \beta(1 - 2\alpha))a_1}{(1 + \beta(1 - 2\alpha))a_1 - (1 - \beta)} = 1 - \eta.$$

Thus by Definition 2, $f \in \mathcal{M}^{\eta}(\beta, \alpha, \lambda, r)$ for η given by (15). This completes the proof.

Now, we introduce the partial sums and the same property has been found for other class in [6].

Theorem 4 : Let $f \in H$ be given by (1) and define the partial sums $s_1(z)$ and $s_k(z)$ by $s_1(z) = z^{-1}$ and

$$s_k(z) = z^{-1} + \sum_{n=1}^{k-1} a_n z^n,$$

suppose also that

$$\sum_{n=1}^{\infty} d_n a_n \leq 1, \quad \left(d_n = \frac{\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\beta(1-\alpha) + n(1-\beta\alpha))b_n}{(1-\beta)} \right) \tag{18}$$

Then we have

$$Re\left\{\frac{f(z)}{s_k(z)}\right\} > 1 - \frac{1}{d_k} \tag{19}$$

and

$$Re\left\{\frac{s_k(z)}{f(z)}\right\} > 1 - \frac{d_k}{1+d_k}. \tag{20}$$

Each of the bounds in (19) and (20) is the best possible for $n \in \mathbb{N}$.

Proof : For the coefficients d_n given by (18), it is not difficult to verify that $d_{n+1} > d_n > 1, n = 1, 2, \cdots$.

Therefore, by using the hypothesis (18), we have

$$\sum_{n=1}^{k-1} a_n + d_k \sum_{n=k}^{\infty} a_n \leq \sum_{n=1}^{\infty} d_n a_n \leq 1. \tag{21}$$

By setting

$$g_1(z) = d_k \left(\frac{f(z)}{s_k(z)} - \left(1 - \frac{1}{d_k}\right) \right) = 1 + \frac{d_k \sum\limits_{n=k}^{\infty} a_n z^{n+1}}{1 + \sum\limits_{n=1}^{k-1} a_n z^{n+1}} \tag{22}$$

and applying (21) we find that

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{d_k \sum\limits_{n=k}^{\infty} a_n}{2 - 2\sum\limits_{n=1}^{k-1} a_n - d_k \sum\limits_{n=k}^{\infty} a_n} \leq 1, \tag{23}$$

which readily yields the assertion (19). If we take

$$f(z) = z^{-1} - \frac{z^k}{d_k}, \tag{24}$$

then

$$\frac{f(z)}{s_k(z)} = 1 - \frac{z^k}{d_k} \to 1 - \frac{1}{d_k} \quad (z \to 1^-),$$

which shows that the bound in (19) is the best possible for each $n \in \mathbb{N}$.

Similarly, if we put

$$g_2(z) = (1+d_k)\left(\frac{s_k(z)}{f(z)} - \frac{d_k}{1+d_k}\right) = 1 - \frac{(1+d_k)\sum_{n=k}^{\infty} a_n z^{n+1}}{1 + \sum_{n=1}^{k-1} a_n z^{n+1}}$$

and make use of (21) we obtain

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+d_k)\sum_{n=k}^{\infty} a_n}{2 - 2\sum_{n=1}^{k-1} a_n + (1-d_k)\sum_{n=k}^{\infty} a_n} \leq 1 \tag{25}$$

which leads us to the assertion (20). The bounds given in (19) and (20) is sharp with the function given by (24). The proof of the theorem is complete.

5 Integral Operator

Next, we consider integral transforms of functions in the class $\mathcal{M}(\beta, \alpha, \lambda, r)$, some of these integral transforms was studied by Atshan on the other class in [2].

Theorem 5 : Let the function f given by (1) be in the class $\mathcal{M}(\beta, \alpha, \lambda, r)$. Then the integral operator

$$F(z) = c\int_0^1 u^c f(uz)du, \qquad (0 < u \leq 1, 0 < c < \infty) \tag{26}$$

is in the class $\mathcal{M}(\gamma, \alpha, \lambda, r)$, where

$$\gamma = \frac{(c+n+1)(\beta(1-\alpha)+n(1-\beta\alpha)) - c(1-\beta)n}{c(1-\beta)(1-2\alpha)+(c+n+1)(\beta(1-\alpha)+n(1-\beta\alpha))}.$$

The result is sharp for the function

$$f(z) = z^{-1} + \frac{1-\beta}{(\beta(1-\alpha)+(1-\beta\alpha))b_1} z.$$

Proof : Let

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n$$

in the class $\mathcal{M}(\beta,\alpha,\lambda,r)$. Then

$$\begin{aligned} F(z) &= c\int_0^1 u^c f(uz)du \\ &= c\int_0^1 \left(u^{c-1}z^{-1} + \sum_{n=1}^{\infty} a_n u^{n+c} z^n\right) du \\ &= z^{-1} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n z^n. \end{aligned} \quad (27)$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\gamma(1-\alpha)+n(1-\gamma\alpha))}{(c+n+1)(1-\gamma)} a_n b_n \leq 1. \quad (28)$$

Since $f \in \mathcal{M}(\beta,\alpha,\lambda,r)$, we have

$$\sum_{n=1}^{\infty} \frac{\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\beta(1-\alpha)+n(1-\beta\alpha))}{(1-\beta)} a_n b_n \leq 1.$$

Note that (26) it satisfied if

$$\frac{c\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\gamma(1-\alpha)+n(1-\gamma\alpha))}{(c+n+1)(1-\gamma)} b_n \leq \frac{\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\beta(1-\alpha)+n(1-\beta\alpha))}{(1-\beta)} b_n.$$

Rewriting the inequality, we have

$$c(1-\beta)(\gamma(1-\alpha)+n(1-\gamma\alpha)) \leq (c+n+1)(1-\gamma)(\beta(1-\alpha)+n(1-\beta\alpha))$$

solving for γ, we have

$$\gamma \leq \frac{(c+n+1)(\beta(1-\alpha)+n(1-\beta\alpha)) - c(1-\beta)n}{c(1-\beta)(1-2\alpha)+(c+n+1)(\beta(1-\alpha)+n(1-\beta\alpha))} = F(n). \quad (29)$$

A simple computation will show that $F(n)$ is increasing $F(n) \geq F(1)$. Using this, the results follows.

Theorem 6 : Let the function f given by (1) be in the class $\mathcal{M}(\beta, \alpha, \lambda, r)$. Then the function F defined by (27) is convex in the disk $|z| < R_1$, where

$$R_1 = \inf_n \left\{ \frac{\left(\frac{n+\lambda}{1+\lambda}\right)^r (c+n+1)(\beta(1-\alpha)+n(1-\beta\alpha))b_n}{c(1-\beta)(n+2)} \right\}^{\frac{1}{n+1}}. \quad (30)$$

Proof : We show that

$$\left| \frac{zF''(z)}{F'(z)} + 2 \right| \leq 1 \quad \text{in} \quad |z| < R_1 \quad (31)$$

R_1 is given by (30). In view of (27) we have

$$\begin{aligned} \left| \frac{zF''(z)+2F'(z)}{F'(z)} \right| &= \left| \frac{\sum\limits_{n=1}^{\infty} \frac{c}{n+c+1} n(n+1)a_n z^{n+1}}{-1+\sum\limits_{n=1}^{\infty} \frac{c}{n+c+1} n a_n z^{n+1}} \right| \\ &\leq \frac{\sum\limits_{n=1}^{\infty} \frac{c}{n+c+1} n(n+1)a_n |z|^{n+1}}{1-\sum\limits_{n=1}^{\infty} \frac{c}{n+c+1} n a_n |z|^{n+1}} \leq 1. \end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \frac{c(n+2)}{n+c+1} n a_n |z|^{n+1} \leq 1.$$

This is enough to consider

$$|z|^{n+1} \leq \frac{\left(\frac{n+\lambda}{1+\lambda}\right)^r (c+n+1)(\beta(1-\alpha)+n(1-\beta\alpha))b_n}{c(1-\beta)(n+2)}.$$

Therefore,

$$|z| \leq \left\{ \frac{\left(\frac{n+\lambda}{1+\lambda}\right)^r (c+n+1)(\beta(1-\alpha)+n(1-\beta\alpha))b_n}{c(1-\beta)(n+2)} \right\}^{\frac{1}{n+1}},$$

for $n \geq 1, n \in \mathbb{N}$. The result follows by setting $|z| = R_1$.

Theorem 7 : Let the function f given by (1) be in the class $\mathcal{M}(\beta, \alpha, \lambda, r)$

$$F(z) = \frac{1}{c}[(c+1)f(z) + zf'(z)] = z^{-1} + \sum_{n=1}^{\infty} \frac{c+n+1}{c} a_n z^n, \quad c > 0. \quad (32)$$

Then $F(z)$ is in the class $\mathcal{M}(\beta,\alpha,\lambda,r)$ for $|z| \le r\ (\beta,\alpha,c,\xi)$ where

$$r(\beta,\alpha,c,\xi) = \inf_n \left\{ \frac{c(\beta(1-\alpha)+n(1-\beta\alpha))(1-\xi)}{(c+n+1)(\xi(1-\alpha)+n(1-\xi\alpha))} \right\}^{\frac{1}{n+1}}, \quad n = 1, 2, \cdots .$$

The result is sharp for the function

$$f(z) = z^{-1} + \frac{1-\beta}{\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\beta(1-\alpha)+n(1-\beta\alpha))b_n} z^n.$$

Proof : Let

$$w = \frac{(\mathcal{I}_1(r,\lambda)(f*g)(z))' + z(\mathcal{I}_1(r,\lambda)(f*g)(z))''}{(2\alpha-1)(\mathcal{I}_1(r,\lambda)(f*g)(z))' + \alpha z(\mathcal{I}_1(r,\lambda)(f*g)(z))''}.$$

Then it is sufficient to show that

$$|w-1| < |w+1-2\xi|.$$

A computation shows that this is satisfied if

$$\sum_{n=1}^{\infty} \frac{\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\xi(1-\alpha)+n(1-\xi\alpha))(c+n+1)}{c(1-\xi)} a_n b_n |z|^{n+1} \le 1. \tag{33}$$

Since $f \in \mathcal{M}(\beta,\alpha,\lambda,r)$, then by Theorem 1, we have

$$\sum_{n=1}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^r n(\beta(1-\alpha)+n(1-\beta\alpha)) a_n b_n \le 1-\beta.$$

The equation (33) is satisfied if

$$\frac{\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\xi(1-\alpha)+n(1-\xi\alpha))(c+n+1)}{c(1-\xi)} a_n b_n |z|^{n+1} \le \frac{\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\beta(1-\alpha)+n(1-\beta\alpha))}{(1-\beta)} a_n b_n$$

solving for $|z|$ we obtain

$$|z|^{n+1} \le \frac{c(\beta(1-\alpha)+n(1-\beta\alpha))(1-\xi)}{(c+n+1)(\xi(1-\alpha)+n(1-\xi\alpha))}.$$

Therefore,

$$|z| \le \left\{ \frac{c(\beta(1-\alpha)+n(1-\beta\alpha))(1-\xi)}{(c+n+1)(\xi(1-\alpha)+n(1-\xi\alpha))} \right\}^{\frac{1}{n+1}}.$$

Solving for $|z|$, we obtain the result.

Theorem 8 : If $f \in \mathcal{M}(\beta,\alpha,\lambda,r)$, then the integral operator

$$F(z) = c\int_0^1 u^c f(uz)du, \qquad (0 < u \leq 1, 0 < c < \infty)$$

is in the class $\mathcal{M}\left(\frac{1+\beta c}{2+c},\alpha,\lambda,r\right)$.

The result is sharp for

$$f_n(z) = z^{-1} + \frac{1-\left(\frac{1+\beta c}{2+c}\right)}{\left(\frac{n+\lambda}{1+\lambda}\right)^r n\left(\left(\frac{1+\beta c}{2+c}\right)(1-\alpha)+n\left(1-\left(\frac{1+\beta c}{2+c}\right)\alpha\right)\right)b_n} z^n.$$

Proof : By Definition of F, we get

$$F(z) = c\int_0^1 u^c f(uz)du = z^{-1} + \sum_{n=1}^{\infty}\frac{c}{c+n+1}a_n z^n.$$

By Theorem 1, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{c\left(\frac{n+\lambda}{1+\lambda}\right)^r n\left(\left(\frac{1+\beta c}{2+c}\right)(1-\alpha)+n\left(1-\left(\frac{1+\beta c}{2+c}\right)\alpha\right)\right)b_n}{(c+n+1)\left(1-\left(\frac{1+\beta c}{2+c}\right)\right)}a_n \leq 1, \qquad (34)$$

since, if $f \in \mathcal{M}(\beta,\alpha,\lambda,r)$, then (34) satisfies if

$$\frac{c}{(c+n+1)\left(1-\left(\frac{1+\beta c}{2+c}\right)\right)} \leq \frac{1}{(1-\beta)}$$

or equivalently, when

$$\varnothing(n,c,\beta) = \frac{c(1-\beta)}{(c+n+1)\left(1-\left(\frac{1+\beta c}{2+c}\right)\right)} \leq 1.$$

Since $\varnothing(n,c,\beta)$ is a decreasing function of $n, (n \geq 1)$, then the proof is completed. The result is sharp for

$$f_n(z) = z^{-1} + \frac{1-\left(\frac{1+\beta c}{2+c}\right)}{\left(\frac{n+\lambda}{1+\lambda}\right)^r n\left(\left(\frac{1+\beta c}{2+c}\right)(1-\alpha)+n\left(1-\left(\frac{1+\beta c}{2+c}\right)\alpha\right)\right)b_n} z^n.$$

Next, we get the inclusive properties of the class $\mathcal{M}(\beta,\alpha,\lambda,r)$.

Theorem 9 : Let $0 \leq \beta < 1, 0 \leq \alpha < 1, r \in \mathbb{Z}, \lambda \geq 0, \sigma \geq 0$. Then

$$\mathcal{M}(\beta,\alpha,\lambda,r) \subseteq \mathcal{M}(\sigma,0,\lambda,r)$$

where

$$\sigma \leq 1 - \frac{(n+1)(1-\beta)}{(\beta(1-\alpha)+n(1-\beta\alpha))+(1-\beta)}, \quad n \in \mathbb{N},\ n \geq 1.$$

Proof : Let $f \in \mathcal{M}(\beta, \alpha, \lambda, r)$. Then from Theorem 1, we have

$$\sum_{n=1}^{\infty} \frac{\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\beta(1-\alpha)+n(1-\beta\alpha))}{(1-\beta)} a_n b_n \leq 1. \tag{35}$$

We want to find the value σ such that

$$\sum_{n=1}^{\infty} \frac{\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\sigma+n)}{(1-\sigma)} a_n b_n \leq 1. \tag{36}$$

The inequality (35) would obviously imply (36) if

$$\frac{\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\sigma+n)}{(1-\sigma)} \leq \frac{\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\beta(1-\alpha)+n(1-\beta\alpha))}{(1-\beta)} = A.$$

Therefore,

$$\frac{\left(\frac{n+\lambda}{1+\lambda}\right)^r n(\sigma+n)}{(1-\sigma)} \leq A. \tag{37}$$

Hence

$$\frac{(1-\sigma)}{(n+1)} \geq \frac{n\left(\frac{n+\lambda}{1+\lambda}\right)^r}{A+n\left(\frac{n+\lambda}{1+\lambda}\right)^r}, \quad (n \geq 1, n \in \mathbb{N}). \tag{38}$$

The right hand side of (38) decreases as n increases and so is maximum for $n = 1$.

So (38) is satisfied provided

$$\frac{(1-\sigma)}{(n+1)} \geq \frac{(1-\beta)}{(\beta(1-\alpha)+n(1-\beta\alpha))+(1-\beta)} = L.$$

Obviously $L < 1$ and

$$\sigma \leq 1 - \frac{(n+1)(1-\beta)}{(\beta(1-\alpha)+n(1-\beta\alpha))+(1-\beta)}.$$

This completes the proof.

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