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On a Class of Meromorphic Univalent Functions Defined by Linear Derivative Operator

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Abstract

In this paper, we introduce the class of meromorphic univalent functions defined by linear derivative operator. We obtain a coefficient inequality, closure theorem, Hadamard product (or convolution) and integral operator for the functions in the class $k^{\lambda,1}(\beta,\alpha)$.

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1 Introduction

Let $\sum$ denote the class of functions f of the form :

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n \tag{1}$$

which are analytic and meromorphic univalent in the punctured unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \setminus \{0\}$. Let $\mathcal{A}$ be a subclass of $\sum$ of functions of the form:

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \quad (a_n \geq 0, \quad n \in \mathbb{N}). \tag{2}$$

A function $f \in \mathcal{A}$ is meromorphic starlike function of order $\rho, (0 \leq \rho < 1)$ if

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho, \quad (z \in U). \tag{3}$$

The class of all such functions is denoted by $\mathcal{A}^*(\rho)$. A function $f \in \mathcal{A}$ is meromorphic convex function of order $\rho, (0 \leq \rho < 1)$ if

$$-Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho, \quad (z \in U). \tag{4}$$

The Hadamard product (or convolution) of two functions, f is given by (2) and

$$g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n, \quad (b_n \geq 0, n \in \mathbb{N}) \tag{5}$$

is defined by

$$(f * g)(z) = z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n.$$

We shall need to state the extended linear derivative operator of Ruscheweyh type for the function belonging to the class $\mathcal{A}$ which is defined by the following convolution

$$D_*^{\lambda,1} f(z) = \frac{z^{-1}}{(1-z)^{\lambda+1}} * f(z), \quad (\lambda > -1; \quad f \in \mathcal{A}). \tag{6}$$

In terms of binomial coefficients , (6) can be written as

$$D_*^{\lambda,1} f(z) = z^{-1} + \sum_{n=1}^{\infty} \binom{\lambda+n}{n} a_n z^n, \quad (\lambda > -1; \quad f \in \mathcal{A}). \tag{7}$$

The linear operator $D^{\lambda,1}$ analogous to $D_*^{\lambda,1}$ (defined by 6) , was consider recently by Raina and Srivastava [5] on the space of analytic and p-valent function in U $(U = U^* \cup \{0\})$. Also the linear operator $D_*^{\lambda,p}$ was studied on meromorphic multivalent functions for other class in [2].

Definition 1 : Let $f \in \mathcal{A}$ be given by (2). The class $k^{\lambda,1}(\beta, \alpha)$ is defined by:

$$k^{\lambda,1}(\beta,\alpha) = \left\{ f \in \mathcal{A} : \left| \frac{\frac{z(D_*^{\lambda,1} f(z))''}{(D_*^{\lambda,1} f(z))'} + 2}{\frac{z(D_*^{\lambda,1} f(z))''}{(D_*^{\lambda,1} f(z))'} + 2\alpha} \right| < \beta, \lambda > -1, 0 \leq \alpha < 1, 0 < \beta \leq 1 \right\}. \tag{8}$$

Some of the following properties have been found on other classes in [4], [1] and [3].

2 Coefficient Inequality

Theorem 1 : Let $f \in \mathcal{A}$. Then $f \in k^{\lambda,1}(\beta,\alpha)$ if and only if

$$\sum_{n=1}^{\infty} \binom{\lambda+n}{n} n[n(1+\beta)+(1+\beta(2\alpha-1))]a_n \leq 2\beta(1-\alpha),$$

$$\lambda > -1, \quad 0 \leq \alpha < 1, 0 < \beta \leq 1. \tag{9}$$

The result is sharp for the function

$$f(z) = z^{-1} + \frac{2\beta(1-\alpha)}{\binom{\lambda+n}{n} n[n(1+\beta)+(1+\beta(2\alpha-1))]} z^n, \quad n \geq 1.$$

Proof : Assume that the inequality (9) holds true and let $|z| = 1$, then from (8), we have

$$\begin{aligned}
&|z(D_*^{\lambda,1} f(z))'' + 2(D_*^{\lambda,1} f(z))'| - \beta|z(D_*^{\lambda,1} f(z))'' + 2\alpha(D_*^{\lambda,1} f(z))'| \\
&= \left|\sum_{n=1}^{\infty} n(n+1) \binom{\lambda+n}{n} a_n z^{n-1}\right| \\
&-\beta \left|2(1-\alpha)z^{-2} + \sum_{n=1}^{\infty} n(n-1+2\alpha) \binom{\lambda+n}{n} a_n z^{n-1}\right| \\
&\leq \sum_{n=1}^{\infty} \binom{\lambda+n}{n} n[n(1+\beta)+(1+\beta(2\alpha-1))]a_n - 2\beta(1-\alpha) \leq 0.
\end{aligned} \tag{10}$$

Hence by the principle of maximum modulus , $f \in k^{\lambda,1}(\beta,\alpha)$. Conversely, suppose that f defined by (2) is in the class $k^{\lambda,1}(\beta,\alpha)$, then from (7) , we have

$$\left|\frac{\frac{z(D_*^{\lambda,1} f(z))''}{(D_*^{\lambda,1} f(z))'} + 2}{\frac{z(D_*^{\lambda,1} f(z))''}{(D_*^{\lambda,1} f(z))'} + 2\alpha}\right| = \left|\frac{\sum_{n=1}^{\infty} n(n+1) \binom{\lambda+n}{n} a_n z^{n-1}}{2(1-\alpha)z^{-2} + \sum_{n=1}^{\infty} n(n-1+2\alpha) \binom{\lambda+n}{n} a_n z^{n-1}}\right| < \beta.$$

Since $|Re(z)| \leq |z|$ for all z, we have

$$Re\left\{\frac{\sum_{n=1}^{\infty} n(n+1) \binom{\lambda+n}{n} a_n z^{n-1}}{2(1-\alpha)z^{-2} + \sum_{n=1}^{\infty} n(n-1+2\alpha) \binom{\lambda+n}{n} a_n z^{n-1}}\right\} < \beta,$$

we choose the value of z on the real axis so that $\frac{z(D_*^{\lambda,1}f(z))''}{(D_*^{\lambda,1}f(z))'}$ is real. Upon clearing the denominator of (10) and letting $z \to 1^-$ through real values, we get the inequality (9).

Sharpness of the result follows by setting

$$f(z) = z^{-1} + \frac{2\beta(1-\alpha)}{\begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\beta)+(1+\beta(2\alpha-1))]} z^n, \quad n \geq 1. \tag{11}$$

Corollary 1 : Let $f \in k^{\lambda,1}(\beta,\alpha)$. Then

$$a_n \leq \frac{2\beta(1-\alpha)}{\begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\beta)+(1+\beta(2\alpha-1))]}$$

where $\lambda > -1, 0 \leq \alpha < 1, 0 < \beta \leq 1$.

Theorem 2 : The class $k^{\lambda,1}(\beta,\alpha)$ is convex set.

Proof : Let f and g be the arbitrary elements of $k^{\lambda,1}(\beta,\alpha)$. Then for every γ $(0 < \gamma < 1)$, we show that $(1-\gamma)f(z) + \gamma g(z) \in k^{\lambda,1}(\beta,\alpha)$. Thus, we have

$$(1-\gamma)f(z) + \gamma g(z) = z^{-1} + \sum_{n=1}^{\infty} [(1-\gamma)a_n + \gamma b_n]$$

and

$$\sum_{n=1}^{\infty} \frac{\begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)} [(1-\gamma)a_n + \gamma b_n]$$
$$= (1-\gamma)\sum_{n=1}^{\infty} \frac{\begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)} a_n$$
$$+\gamma \sum_{n=1}^{\infty} \frac{\begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)} b_n \leq 1.$$

This completes the proof.

3 Closure Theorem

Theorem 3 : Let the function f_i defined by

$$f_i(z) = z^{-1} + \sum_{n=1}^{\infty} a_{n,i} z^n, \quad (a_{n,i} \geq 0, \quad n \in \mathbb{N}, n \geq 1)$$

be in the class $k^{\lambda,1}(\beta_i, \alpha_i), (0 \leq \alpha_i < 1, 0 < \beta_i \leq 1, n \in \mathbb{N}, n \geq 1)$ for each $i = 1, 2, \cdots, m$. Then the function h defined by

$$h(z) = z^{-1} + \frac{1}{m} \sum_{n=1}^{\infty} \left(\sum_{i=1}^{m} a_{n,i} \right) z^n$$

is in the class $k^{\lambda,1}(\beta, \alpha)$, where

$$\beta = \min_{1 \leq i \leq m} \{\beta_i\} \quad \text{and} \quad \alpha = \min_{1 \leq i \leq m} \{\alpha_i\}. \tag{12}$$

Proof : Since $f_i \in k^{\lambda,1}(\beta_i, \alpha_i)$ for each $i = 1, 2, \cdots, m$, we note that

$$\sum_{n=1}^{\infty} \begin{pmatrix} \lambda + n \\ n \end{pmatrix} n[n(1 + \beta_i) + (1 + \beta_i(2\alpha_i - 1))] a_{n,i} \leq 2\beta_i(1 - \alpha_i).$$

Therefore,

$$\begin{aligned} &\sum_{n=1}^{\infty} \begin{pmatrix} \lambda + n \\ n \end{pmatrix} n[n(1 + \beta_i) + (1 + \beta_i(2\alpha_i - 1))] \left(\frac{1}{m} \sum_{i=1}^{m} a_{n,i} \right) \\ &= \frac{1}{m} \sum_{i=1}^{m} \left(\sum_{n=1}^{\infty} \begin{pmatrix} \lambda + n \\ n \end{pmatrix} n[n(1 + \beta_i) + (1 + \beta_i(2\alpha_i - 1))] a_{n,i} \right) \\ &\leq \frac{1}{m} \sum_{i=1}^{m} 2\beta_i(1 - \alpha_i) \leq 2\beta(1 - \alpha). \end{aligned}$$

Thus we get

$$\sum_{n=1}^{\infty} \begin{pmatrix} \lambda + n \\ n \end{pmatrix} n[n(1 + \beta_i) + (1 + \beta_i(2\alpha_i - 1))] \left(\frac{1}{m} \sum_{i=1}^{m} a_{n,i} \right) \leq 2\beta(1 - \alpha).$$

Hence by Theorem 1, we have $h \in k^{\lambda,1}(\beta, \alpha)$, where α and β is given by (12). This completes the proof of the theorem.

4 Hadamard Product

Theorem 4 : Let $f, g \in k^{\lambda,1}(\beta,\alpha)$. Then$(f*g) \in k^{\lambda,1}(\delta,\alpha)$ for

$$\begin{aligned} f(z) &= z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \\ g(z) &= z^{-1} + \sum_{n=1}^{\infty} b_n z^n \end{aligned}$$

and

$$(f*g)(z) = z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n,$$

where

$$\delta = \frac{2\beta^2(\alpha-1)(n+1)}{2\beta^2(1-\alpha)(n+2\alpha-1) - \begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\beta)+(1+\beta(2\alpha-1))]^2}.$$

Proof : Since f and g are in the class $k^{\lambda,1}(\beta,\alpha)$, then

$$\sum_{n=1}^{\infty} \frac{\begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)} \quad a_n \leq 1 \tag{13}$$

and

$$\sum_{n=1}^{\infty} \frac{\begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)} \quad b_n \leq 1. \tag{14}$$

We have to find the largest δ such that

$$\sum_{n=1}^{\infty} \frac{\begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\delta)+(1+\delta(2\alpha-1))]}{2\delta(1-\alpha)} \quad a_n b_n \leq 1. \tag{15}$$

By Cauchy-Schwarz inequality, we get

$$\sum_{n=1}^{\infty} \frac{\begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)} \quad \sqrt{a_n b_n} \leq 1. \tag{16}$$

We want only to show that

$$\frac{\binom{\lambda+n}{n} n[n(1+\delta)+(1+\delta(2\alpha-1))]}{2\delta(1-\alpha)} \; a_n b_n$$
$$\le \frac{\binom{\lambda+n}{n} n[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)} \; \sqrt{a_n b_n}.$$

This equivalently to

$$\sqrt{a_n b_n} \le \frac{\delta[n(1+\beta)+(1+\beta(2\alpha-1))]}{\beta[n(1+\delta)+(1+\delta(2\alpha-1))]}. \tag{17}$$

From (16), we get

$$\sqrt{a_n b_n} \le \frac{2\beta(1-\alpha)}{\binom{\lambda+n}{n} n[n(1+\beta)+(1+\beta(2\alpha-1))]}.$$

Thus it is enough to show that

$$\frac{2\beta(1-\alpha)}{\binom{\lambda+n}{n} n[n(1+\beta)+(1+\beta(2\alpha-1))]} \le \frac{\delta[n(1+\beta)+(1+\beta(2\alpha-1))]}{\beta[n(1+\delta)+(1+\delta(2\alpha-1))]}$$

which simplifies to

$$\delta \le \frac{2\beta^2(\alpha-1)(n+1)}{2\beta^2(1-\alpha)(n+2\alpha-1) - \binom{\lambda+n}{n} n[n(1+\beta)+(1+\beta(2\alpha-1))]^2}.$$

Theorem 5 : Let the functions f_j $(j=1,2)$ defined by

$$f_j(z) = z^{-1} + \sum_{n=1}^{\infty} a_{n,j} z^n, \quad (a_{n,j} \ge 0, j = 1,2)$$

be in the class $k^{\lambda,1}(\beta,\alpha)$. Then the function h defined by

$$h(z) = z^{-1} + \sum_{n=1}^{\infty} (a_{n,1}^2 + a_{n,2}^2) z^n \tag{18}$$

belongs to the class $k^{\lambda,1}(\beta,\alpha)$, where

$$\eta = \frac{4\beta^2(\alpha-1)(n+1)}{4\beta^2(1-\alpha)(n+2\alpha-1) - \begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\beta)+(1+\beta(2\alpha-1))]^2}.$$

Proof : Note that

$$\sum_{n=1}^{\infty} \left(\frac{\begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)} \right)^2 a_{n,j}^2$$

$$\leq \left(\sum_{n=1}^{\infty} \frac{\begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)} a_{n,j} \right)^2 \leq 1 \quad (j=1,2). \tag{19}$$

For $f_j \in k^{\lambda,1}(\beta,\alpha)$ $(j=1,2)$, we have

$$\sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{\begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)} \right)^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1. \tag{20}$$

In order to obtain our result , we have to find the largest η such that

$$\frac{[n(1+\eta)+(1+\eta(2\alpha-1))]}{\eta} \leq \frac{\begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\beta)+(1+\beta(2\alpha-1))]^2}{4\beta^2(1-\alpha)}, \quad n \geq 1$$

so that

$$\eta \leq \frac{4\beta^2(\alpha-1)(n+1)}{4\beta^2(1-\alpha)(n+2\alpha-1) - \begin{pmatrix} \lambda+n \\ n \end{pmatrix} n[n(1+\beta)+(1+\beta(2\alpha-1))]^2}.$$

5 Integral Operator and Partial Sums

Next, we consider some properties have been found on the other class in [3].

Theorem 6 : The $f \in k^{\lambda,1}(\beta,\alpha)$ if and only if the function F given by

$$F(z) = \frac{\lambda}{z^{\lambda+1}} \int_0^z t^{\lambda} f(t)dt, \qquad \lambda > -1 \tag{21}$$

is in the class $k^{\lambda+1,1}(\beta,\alpha)$.

Proof : By using of (21), we have

$$\lambda f(z) = (\lambda+1)F(z) + zF'(z), \tag{22}$$

which, in the right hand of (7), implies

$$\lambda(D_*^{\lambda,1} f(z)) = (\lambda+1)(D_*^{\lambda,1}F(z)) + z(D_*^{\lambda,1}F(z))' = \lambda(D_*^{\lambda+1,1}F(z)).$$

Therefore, we have

$$D_*^{\lambda,1} f(z) = D_*^{\lambda+1,1}F(z),$$

and the desired result follows at once.

Theorem 7 ; Let $f \in k^{\lambda,1}(\beta,\alpha)$. Then the function F defined by

$$F(z) = \frac{\lambda}{z^{\lambda+1}} \int_0^z t^{\lambda} f(t)dt = z^{-1} + \sum_{n=1}^{\infty} \frac{\lambda}{\lambda+n+1} a_n z^n, \quad \lambda > -1 \tag{23}$$

is meromorphically starlike in the disk $|z| < R_1$, where

$$R_1 = \inf_n \left\{ \frac{\binom{\lambda+n}{n} n[n(1+\beta) + (1+\beta(2\alpha-1))](\lambda+n+1)}{2\beta\lambda(n+1)(1-\alpha)} \right\}^{\frac{1}{n+1}}. \tag{24}$$

The result is sharp for the function

$$f(z) = z^{-1} + \frac{2\beta(1-\alpha)}{\binom{\lambda+n}{n} n[n(1+\beta) + (1+\beta(2\alpha-1))]} z^n.$$

Proof : We show that

$$\left| \frac{zF'(z)}{F(z)} + 1 \right| \leq 1 \quad \text{in} \quad |z| < R_1. \tag{25}$$

R_1 is given by (24). In view of (23) we have

$$\left|\frac{zF'(z)+F(z)}{F(z)}\right| = \left|\frac{\sum_{n=1}^{\infty}\frac{\lambda}{\lambda+n+1}na_n z^{n+1}}{1+\sum_{n=1}^{\infty}\frac{\lambda}{\lambda+n+1}a_n z^{n+1}}\right| \leq \frac{\sum_{n=1}^{\infty}\frac{\lambda}{\lambda+n+1}na_n|z|^{n+1}}{1-\sum_{n=1}^{\infty}\frac{\lambda}{\lambda+n+1}a_n|z|^{n+1}} \leq 1.$$

Hence

$$\sum_{n=1}^{\infty}\frac{\lambda(n+1)}{\lambda+n+1}a_n|z|^{n+1} \leq 1.$$

This is enough to consider

$$|z|^{n+1} \leq \frac{\binom{\lambda+n}{n} n[n(1+\beta)+(1+\beta(2\alpha-1))](\lambda+n+1)}{2\beta\lambda(n+1)(1-\alpha)}.$$

Therefore,

$$|z| \leq \left\{\frac{\binom{\lambda+n}{n} n[n(1+\beta)+(1+\beta(2\alpha-1))](\lambda+n+1)}{2\beta\lambda(n+1)(1-\alpha)}\right\}^{\frac{1}{n+1}},$$

for $n \in \mathbb{N}, n \geq 1$. The result follows by setting $|z| = R_1$.

Theorem 8 : Let $f \in \mathcal{A}$ be given by (2) and define the partial sums $s_1(z)$ and $s_k(z)$ by $s_1(z) = z^{-1}$ and

$$s_k(z) = z^{-1} + \sum_{n=1}^{k-1} a_n z^n,$$

suppose also that

$$\sum_{n=1}^{\infty} d_n a_n \leq 1, \quad \left(d_n = \frac{\binom{\lambda+n}{n} n[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)}\right). \tag{26}$$

Then we have

$$Re\left\{\frac{f(z)}{s_k(z)}\right\} > 1-\frac{1}{d_k} \quad \text{and} \quad Re\left\{\frac{s_k(z)}{f(z)}\right\} > 1-\frac{d_k}{1+d_k}. \tag{27}$$

Each of the bounds in (27) is the best possible for $n \in \mathbb{N}$.

Proof : For the coefficients d_n given by (26), it is not difficult to verify that $d_{n+1} > d_n > 1, n = 1, 2, \cdots$.

Therefore, by using the hypothesis (26), we have

$$\sum_{n=1}^{k-1} a_n + d_k \sum_{n=k}^{\infty} a_n \leq \sum_{n=1}^{\infty} d_n a_n \leq 1. \tag{28}$$

By setting

$$g_1(z) = d_k \left(\frac{f(z)}{s_k(z)} - \left(1 - \frac{1}{d_k}\right)\right) = 1 + \frac{d_k \sum\limits_{n=k}^{\infty} a_n z^{n+1}}{1 + \sum\limits_{n=1}^{k-1} a_n z^{n+1}} \tag{29}$$

and applying (28) we find that

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{d_k \sum\limits_{n=k}^{\infty} a_n}{2 - 2\sum\limits_{n=1}^{k-1} a_n - d_k \sum\limits_{n=k}^{\infty} a_n} \leq 1 \tag{30}$$

which readily yields the left asseration (27). If we take

$$f(z) = z^{-1} - \frac{z^k}{d_k}, \tag{31}$$

then

$$\frac{f(z)}{s_k(z)} = 1 - \frac{z^k}{d_k} \to 1 - \frac{1}{d_k} (z \to 1^-),$$

which shows that the bound in (27) is the best possible for each $n \in \mathbb{N}$. Similarly, if we put

$$g_2(z) = (1 + d_k) \left(\frac{s_k(z)}{f(z)} - \frac{d_k}{1 + d_k} \right) = 1 - \frac{(1 + d_k) \sum\limits_{n=k}^{\infty} a_n z^{n+1}}{1 + \sum\limits_{n=1}^{k-1} a_n z^{n+1}}$$

and make use of (28) we obtain

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + d_k) \sum\limits_{n=k}^{\infty} a_n}{2 - 2\sum\limits_{n=1}^{k-1} a_n + (1 - d_k) \sum\limits_{n=k}^{\infty} a_n} \leq 1 \tag{32}$$

which leads us to the assertion (27). The bounds given in the right of (27) is sharp with the function given by (31). The proof of the theorem is complete.

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