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Fractional Calculus on a Subclass of Spiral-Like Functions Defined by Komatu Operator

Waggas Galib Atshan

Department of Mathematics
College of Computer Science and Mathematics
Al-Qadisiya University, Diwaniya, Iraq
waggashnd@yahoo.com

Abstract

In this paper, we study an application of the fractional calculus techniques for the new subclass of spiral-like functions $S_p(\beta, \lambda, \mu, \alpha, \gamma)$. Distortion theorems for the fractional derivative and fractional integration are obtained. Also we get result about coefficient inequality.

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1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic and univalent in the disk $U = \{z : |z| < 1\}$. For β real, $|\beta| < \frac{\pi}{2}$, a function f in the form (1) is said in $S_p(\beta)$, the class of β-spiral-like functions, if and only if

$$Re\left\{e^{i\beta}\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in U. \tag{2}$$

For $\beta = 0, S_p(0) \equiv S$ is the well-known class of functions starlike with respect to the origin. For $\beta \neq 0$, it is known that $S_p(\beta)$ is not contained in S. In fact, Robertson [2] showed that the radius of starlikeness of $S_p(\beta)$ is $(\cos\beta + |\sin\beta|)^{-1}$.

The class $S_p(\beta)$ was introduced and shown to be a subfamily of S by Spaček [3]. Later, Zamorski [5] obtained sharp coefficient bounds for the class.

Definition (1) [4] : The fractional integral of order δ $(0 < \delta)$ is defined by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(t)}{(z-t)^{1-\delta}} dt \tag{3}$$

where $f(z)$ is an analytic function in a simply connected region of z-plane containing the origin and the multiplicity of $(z-t)^{\delta-1}$ is removed by required $\log(z-t)$ to be real when $(z-t) > 0$.

Definition (2) [4] : The fractional derivative of order δ $(0 \leq \delta < 1)$ is defined by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\delta}}\ dt \tag{4}$$

where $f(z)$ is as in Definition 1 and the multiplicity of $(z-t)^{-\delta}$ is removed like Definition 1.

Definition 3 [4] : [Under the conditions of Definition 2] the fractional derivative of order $n + \delta$ $(n = 0, 1, 2, \cdots)$ is defined by

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z).$$

From Definition 1 and 2 by applying a simple calculation we get

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(2+\delta)} z^{\delta+1} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\delta)}\ a_n z^{n+\delta}, \tag{5}$$

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(2-\delta)} z^{1-\delta} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)}\ a_n z^{n-\delta}. \tag{6}$$

Definition 4 : The integral operator of $f \in S_p(\beta)$ for $\lambda > -1, \mu > 0$ is denoted by P_λ^μ and defined as following:

$$\begin{aligned} P_\lambda^\mu(f(z)) &= \frac{(\lambda+1)^\mu}{\Gamma(\mu)} \int_0^1 t^{\lambda-1} (\log \frac{1}{t})^{\mu-1} f(zt) dt \\ &= z + \sum_{n=2}^{\infty} \left(\frac{\lambda+1}{\lambda+n} \right)^\mu a_n z^n \quad (\lambda > -1,\ \mu > 0,\ f \in S_p(\beta)). \end{aligned} \tag{7}$$

The operator defined by (7) is known as the Komatu operator [1].

Definition 5 : We introduce a new subclass of $S_p(\beta)$ as functions in the form (1) that satisfy the inequality

$$\left| \frac{(z(P_\lambda^\mu(f(z)))'/P_\lambda^\mu(f(z))) - 1}{2\alpha\{(z(P_\lambda^\mu(f(z)))'/P_\lambda^\mu(f(z))) - 1 + (1-\gamma)e^{-i\beta}\cos\beta\} - ((z(P_\lambda^\mu(f(z)))'/P_\lambda^\mu(f(z))) - 1)} \right| < 1$$

for $z \in U$, where $0 \le \gamma < 1, 0 < \alpha \le 1, |\beta| < \frac{\pi}{2}, \lambda > -1, \mu > 0$. We denote for our class by $S_p(\beta, \lambda, \mu, \alpha, \gamma)$.

2. Main Results

In the following theorem, we obtain the coefficient inequality for the class $S_p(\beta, \lambda, \mu, \alpha, \gamma)$.

Theorem 1 : Let $f(z) \in S_p(\beta)$. Then $f(z)$ is in the class $S_p(\beta, \lambda, \mu, \alpha, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} [(n-1)(1-\alpha) + \alpha(1-\gamma)|e^{-i\beta}\cos\beta|] \left(\frac{\lambda+1}{\lambda+n} \right)^\mu a_n \le \alpha(1-\gamma)|e^{-i\beta}\cos\beta| \tag{8}$$

where $0 \le \gamma < 1, 0 < \alpha \le 1, |\beta| < \frac{\pi}{2}, \lambda > -1$ and $\mu > 0$.

The result (8) is sharp for the function $f(z)$ given by the following form

$$f(z) = z + \frac{\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(1 + \alpha((1-\gamma)|e^{-i\beta}\cos\beta| - 1)) \left(\frac{\lambda+1}{\lambda+2} \right)^\mu} z^2 \tag{9}$$

Proof : For $|z| = 1$, we have

$$|z(P^{\mu}_{\lambda}(f(z)))' - P^{\mu}_{\lambda}(f(z))| - |2\alpha\{z(P^{\mu}_{\lambda}(f(z)))' - (1-(1-\gamma)e^{-i\beta}\cos\beta)P^{\mu}_{\lambda}(f(z))\}$$
$$-[z(P^{\mu}_{\lambda}(f(z)))' - P^{\mu}_{\lambda}(f(z))]|$$
$$= \left|\sum_{n=2}^{\infty}(n-1)\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n z^n\right| - \Big|2\alpha(1-\gamma)e^{-i\beta}\cos\beta z$$
$$-\sum_{n=2}^{\infty}[(n-1)(1-2\alpha) - 2\alpha(1-\gamma)e^{-i\beta}\cos\beta]\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n z^n\Big|$$
$$\leq \sum_{n=2}^{\infty}(n-1)\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n - 2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|$$
$$+\sum_{n=2}^{\infty}[(n-1)(1-2\alpha) + 2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|]\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n$$
$$= \sum_{n=2}^{\infty}[2(n-1)(1-\alpha) + 2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|]\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n - 2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|$$
$$\leq 0, \quad \text{by hypothesis.}$$

Thus by Maximum Modulus theorem $f \in S_p(\beta, \lambda, \mu, \alpha, \gamma)$. Conversely, assume that

$$\left|\frac{\frac{z(P^{\mu}_{\lambda}(f(z)))'}{P^{\mu}_{\lambda}(f(z))} - 1}{2\alpha\left[\frac{z(P^{\mu}_{\lambda}(f(z)))'}{P^{\mu}_{\lambda}(f(z))} - 1 + (1-\gamma)e^{-i\beta}\cos\beta\right] - \left[\frac{z(P^{\mu}_{\lambda}(f(z)))'}{P^{\mu}_{\lambda}(f(z))} - 1\right]}\right|$$
$$= \left|\frac{z(P^{\mu}_{\lambda}(f(z)))' - P^{\mu}_{\lambda}(f(z))}{2\alpha(z(P^{\mu}_{\lambda}(f(z)))' - (1-(1-\gamma)e^{-i\beta}\cos\beta)P^{\mu}_{\lambda}(f(z))) - (z(P^{\mu}_{\lambda}(f(z)))' - P^{\mu}_{\lambda}(f(z)))}\right|$$
$$= \Big|[\sum_{n=2}^{\infty}(n-1)\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n z^{n-1}]/[2\alpha(1-\gamma)e^{-i\beta}\cos\beta$$
$$+\sum_{n=2}^{\infty}2\alpha(n-1+(1-\gamma)e^{-i\beta}\cos\beta)\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n z^{n-1} - \sum_{n=2}^{\infty}(n-1)\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n z^{n-1}\Big|$$
$$= \left|\frac{\sum_{n=2}^{\infty}(n-1)\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n z^{n-1}}{2\alpha(1-\gamma)e^{-i\beta}\cos\beta - \sum_{n=2}^{\infty}[(n-1)(1-2\alpha) - 2\alpha(1-\gamma)e^{-i\beta}\cos\beta]\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n z^{n-1}}\right| < 1.$$

Since $|Re(z)| < |z|$ for all z, we have

$$Re\left\{\frac{\sum_{n=2}^{\infty}(n-1)\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n z^{n-1}}{2\alpha(1-\gamma)|e^{-i\beta}\cos\beta| - \sum_{n=2}^{\infty}[(n-1)(1-2\alpha) - 2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|]\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n z^{n-1}}\right\} < 1 \qquad (10)$$

We can choose value of z on the real axis so that $P^{\mu}_{\lambda}(f(z))$ is real. Let $z \to 1^-$, through real values, so we can write (10) as

$$\sum_{n=2}^{\infty}[(n-1)(1-\alpha) + \alpha(1-\gamma)|e^{-i\beta}\cos\beta|]\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu} a_n \leq \alpha(1-\gamma)|e^{-i\beta}\cos\beta|.$$

Corollary 1 ; Let $f(z) \in S_p(\beta, \lambda, \mu, \alpha, \gamma)$, then

$$a_n \leq \frac{\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{[(n-1)(1-\alpha)+\alpha(1-\gamma)|e^{-i\beta}\cos\beta|]\left(\frac{\lambda+1}{\lambda+n}\right)^{\mu}}, \quad n \geq 2$$

where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U.$

Theorem 2 : Let $f(z) \in S_p(\beta, \lambda, \mu, \alpha, \gamma)$, then

$$|D_z^{-\delta} f(z)| \leq \frac{1}{\Gamma(2+\delta)}|z|^{1+\delta}\left[1 + \frac{2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(2+\delta)(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}|z|\right] \tag{11}$$

and

$$|D_z^{-\delta} f(z)| \geq \frac{1}{\Gamma(2+\delta)}|z|^{1+\delta}\left[1 - \frac{2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(2+\delta)(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}|z|\right] \tag{12}$$

The inequalities in (11) and (12) are attained for the function $f(z)$ given by (9).

Proof : By using Theorem 1, we have

$$\sum_{n=2}^{\infty} a_n \leq \frac{\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}. \tag{13}$$

By Definition 3, we have

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(2+\delta)} z^{1+\delta} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\delta)} a_n z^{n+\delta}$$

and

$$\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)} a_n z^n = z + \sum_{n=2}^{\infty} \theta(n) a_n z^n \tag{14}$$

where $\theta(n) = \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)}$.

We know that $\theta(n)$ is a decreasing function of n and

$$0 < \theta(n) \leq \theta(2) = \frac{2}{2+\delta}.$$

Using (13) and (14) we have

$$|\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z)| \leq |z| + \theta(2)|z|^2 \sum_{n=2}^{\infty} a_n$$
$$\leq |z| + \frac{2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(2+\delta)(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}|z|^2,$$

which gives (11); we also have

$$
\begin{aligned}
&|\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)| \geq |z| - \theta(2)|z|^2\sum_{n=2}^{\infty}a_n \\
&\geq |z| - \frac{2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(2+\delta)(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}|z|^2,
\end{aligned}
$$

which gives (12).

Theorem 3 : Let $f(z) \in S_p(\beta,\lambda,\mu,\alpha,\gamma)$, then

$$
|D_z^{\delta}f(z)| \leq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left[1+\frac{2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(2-\delta)(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}|z|\right], \tag{15}
$$

and

$$
|D_z^{\delta}f(z)| \geq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left[1-\frac{2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(2-\delta)(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}|z|\right]. \tag{16}
$$

The inequalities in (15) and (16) are attained for the function $f(z)$ given by (9).

Proof : From Definition 3, we have

$$
D_z^{\delta}f(z) = \frac{1}{\Gamma(2-\delta)}z^{1-\delta} + \sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1-\delta)}a_n z^{n-\delta}
$$

and

$$
\begin{aligned}
\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z) &= z + \sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}a_n z^n \\
&= z + \sum_{n=2}^{\infty}\Phi(n)a_n z^n,
\end{aligned}
$$

where $\Phi(n) = \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}$. For $n \geq 2, \Phi(n)$ is a decreasing function of n, then

$$
\Phi(n) \leq \phi(2) = \frac{\Gamma(3)\Gamma(2-\delta)}{\Gamma(3-\delta)} = \frac{2\Gamma(2)\Gamma(2-\delta)}{(2-\delta)\Gamma(2-\delta)} = \frac{2}{2-\delta}.
$$

Also by using Theorem 1, we have

$$
\sum_{n=2}^{\infty}a_n \leq \frac{\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}},
$$

thus

$$|\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)| \leq |z| + \Phi(2)|z|^2\sum_{n=2}^{\infty}a_n$$
$$\leq |z| + \frac{2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(2-\delta)(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}|z|^2.$$

Then

$$|D_z^{\delta}f(z)| \leq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left[1+\frac{2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(2-\delta)(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}|z|\right],$$

and by the same way we obtain

$$|D_z^{\delta}f(z)| \geq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)}\left[1-\frac{2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(2-\delta)(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}|z|\right].$$

Corollary 2 : For every $f \in S_p(\beta,\lambda,\mu,\alpha,\gamma)$ we have

$$\frac{|z|^2}{2}\left[1-\frac{2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{3(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}|z|\right]$$
$$\leq \left|\int_0^z f(t)dt\right| \leq \frac{|z|^2}{2}\left[1+\frac{2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{3(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}|z|\right] \quad (17)$$

and

$$|z|\left[1-\frac{\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}|z|\right] \leq |f(z)|$$
$$\leq |z|\left[1+\frac{\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}|z|\right]. \quad (18)$$

Proof : (i) By Definition 1 and Theorem 2 for $\delta = 1$, we have $D_z^{-1}f(z) = \int_0^z f(t)dt$, the result is true.

(ii) By Definition 2 and Theorem 3 for $\delta = 0$, we have $D_z^0 f(z) = \frac{d}{dz}\int_0^z f(t)dt = f(z)$, the result is true.

Corollary 3 : $D_z^{-\delta}f(z)$ and $D_z^{\delta}f(z)$ are included in the disk with center at the origin and radii

$$\frac{1}{\Gamma(2+\delta)}\left[1+\frac{2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(2+\delta)(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}\right],$$

$$\frac{1}{\Gamma(2-\delta)}\left[1+\frac{2\alpha(1-\gamma)|e^{-i\beta}\cos\beta|}{(2-\delta)(1+\alpha((1-\gamma)|e^{-i\beta}\cos\beta|-1))\left(\frac{\lambda+1}{\lambda+2}\right)^{\mu}}\right].$$

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