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On a New Subclass of Meromorphically Multivalent Functions Defined by Linear Operator

Ali Hussein Battor*, **Waggas Galib Atshan****
and Ahmed Abbas Abdalreda*

* Department of Mathematics, College of Education for Girls
University of Kufa, Najaf-Iraq

** Department of Mathematics
College of Computer Science and Mathematics
University of Al-Qadisiya, Diwaniya -Iraq
* battor_ali@yahoo.com
** waggashnd@yahoo.com
* ahmedaljnahe@yahoo.com

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Abstract

Making use a linear operator, which is defined here by the convolution (or Hadamard product) involving the generalized hypergeometric function, we discuss the new subclass $\Sigma_p^*(q, s, \alpha_1, \sigma)$ of meromorphically multivalent functions with negative coefficients in the punctured unit disk U^*. In the present paper, we attempt to give the various important properties of this new subclass $\Sigma_p^*(q, s, \alpha_1, \sigma)$, like, coefficient inequality, convex set, growth and distortion bounds. We also derive some interesting results of our class, like, integral representation, neighborhoods of the class $\Sigma_p^*(q, s, \alpha_1, \sigma)$, radii of starlikeness and convexity and integral operator.

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1 Introduction

Let Σ_p^* denote the class of functions of the form:

$$f(z) = \frac{1}{z^p} - \sum_{n=p}^{\infty} a_n z^n; \quad (a_n \geq 0, p \in N = \{1, 2, \cdots\}), \tag{1.1}$$

which are analytic and meromorphic multivalent in the punctured unit disk

$$U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U - \{0\}.$$

Recall [3,9] that a function $f \in \Sigma_p^*$ is said to be meromorphically multivalent starlike of order ρ if it is satisfying the following condition :-

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho; \quad (0 \leq \rho < p; z \in U^*). \tag{1.2}$$

Similarly, recall[14] a function $f \in \Sigma_p^*$ is said to be meromorphically multivalent convex of order ρ if it is satisfying the following condition :-

$$-Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho; \quad (0 \leq \rho < p; z \in U^*). \tag{1.3}$$

For functions $f \in \Sigma_p^*$ given by (1.1) and $g \in \Sigma_p^*$ given by

$$g(z) = \frac{1}{z^p} - \sum_{n=p}^{\infty} b_n z^n, \quad (b_n \geq 0, p \in N),$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g) = \frac{1}{z^p} - \sum_{n=p}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.4}$$

For positive real values of $\alpha_1, \cdots, \alpha_q$ and $\beta_1, \cdots, \beta_s$ ($\beta_j \neq 0, -1, \cdots; j = 1, 2, \cdots, s$), we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$ by

$${}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!}, \tag{1.5}$$

($q \leq s+1; q, s \in N_0 = N \cup \{0\}; z \in U$), where $(\theta)_v$ is the Pochhammer symbol defined

$$(\theta)_v = \begin{cases} 1 & v = 0 \\ \theta(\theta+1)(\theta+2)\cdots(\theta+v-1), & v \in N. \end{cases} \tag{1.6}$$

Corresponding to the function $h_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$, defined by

$$h_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) = z^{-p} \ {}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z); \qquad (1.7)$$

we consider a linear operator

$$H_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s): \quad \Sigma_p^* \to \Sigma_p^*,$$

which is defined by means of the following Hadamard product (or convolution):

$$H_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s) f(z) = h_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) * f(z). \qquad (1.8)$$

We observe that, for a function $f(z)$ of the form (1.1); we have

$$H_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s) f(z) = \frac{1}{z^p} - \sum_{n=p}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{a_n}{n!} z^n. \qquad (1.9)$$

If, for convenience, we write

$$H_{p,q,s}(\alpha_1) = H_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s), \qquad (1.10)$$

then one can easily verify from the definition (1.8) that

$$z(H_{p,q,s}(\alpha_1) f(z))' = \alpha_1 H_{p,q,s}(\alpha_1 + 1) f(z) - (\alpha_1 + p) H_{p,q,s}(\alpha_1) f(z). \qquad (1.11)$$

The linear operator $H_{p,q,s}(\alpha_1)$ was investigated recently by Lin and Srivastava [11]. Some interesting subclasses of analytic functions associated with the generalized hypergeometric function, were considered recently by (for example) Dziok and Srivastava {[5] and [6]}, Gangadharan et. al. [7] and Liu [10].

Definition 1.1 : Let $\Sigma_p^*(q, s, \alpha_1, \sigma)$ denote the new class of functions $f \in \Sigma_p^*$, which satisfy the condition,

$$\left| \frac{\frac{z(H_{p,q,s}(\alpha_1)f(z))''}{(H_{p,q,s}(\alpha_1)f(z))'} + (1+p)}{\frac{z(H_{p,q,s}(\alpha_1)f(z))''}{(H_{p,q,s}(\alpha_1)f(z))'} - (1+p) + 2\sigma} \right| < 1 \qquad (1.12)$$

where $0 \leq \sigma < p, p \in N$ and $z \in U^*$.

Such type of study was carried out by several different authors for another classes, like, Aouf [1,2], Mogra et. al. [13], Miller [12] and Cho et.al. [4].

2 Main Results

First, we derive the coefficient inequality for the class $\Sigma_p^*(q, s, \alpha_1, \sigma)$ contained in:-

Theorem 2.1 : Let $f \in \Sigma_p^*$. Then f is in the class $\Sigma_p^*(q,s,\alpha_1,\sigma)$ if and only if

$$\sum_{n=p}^{\infty}[n+\sigma-1]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)}\frac{a_n}{(n-1)!} \le p((p+1)-\sigma), \tag{2.1}$$

where $0 \le \sigma < p, p \in N$. The result is sharp for the function

$$f(z) = \frac{1}{z^p} + \frac{p((p+1)-\sigma)(n-1)!}{[n+\sigma-1]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}}\, z^n, \quad (n \ge p). \tag{2.2}$$

Proof : Suppose that the inequality (2.1) holds true and $|z| = 1$. Then, we have

$$\begin{aligned}
&|z(H_{p,q,s}(\alpha_1)f(z))'' + (1+p)(H_{p,q,s}(\alpha_1)f(z))'| \\
&\quad - |z(H_{p,q,s}(\alpha_1)f(z))'' + (2\sigma-(1+p))(H_{p,q,s}(\alpha_1)f(z))'| \\
&= \left|-\sum_{n=p}^{\infty} n(n+p)\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\cdot\frac{a_n}{n!}z^{n-1}\right| \\
&- \left|2p(p+1)-\sigma)z^{-(p+1)} - \sum_{n=p}^{\infty} n(n+2(\sigma-1)-p)\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\cdot\frac{a_n}{n!}z^{n-1}\right| \\
&\le \sum_{n=p}^{\infty} n(n+p)\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\cdot\frac{a_n}{n!}|z|^{n-1} - 2p((p+1)-\sigma) \\
&\quad + \sum_{n=p}^{\infty} n[n+2(\sigma-1)-p]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\cdot\frac{a_n}{n!}|z|^{n-1} \\
&= \sum_{n=p}^{\infty} 2n(n+\sigma-1)\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\cdot\frac{a_n}{n!} - 2p((p+1)-\sigma) \le 0,
\end{aligned}$$

hence

$$\sum_{n=p}^{\infty}(n+\sigma-1)\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\frac{a_n}{(n-1)!} - p(p+1)-\sigma) \le 0,$$

by hypothesis. Thus by maximum modulus principle, $f \in \Sigma_p^*(q,s,\alpha_1,\sigma)$. To show the converse, suppose that $f \in \Sigma_p^*(q,s,\alpha_1,\sigma)$. Then from (1.12), we have

$$\begin{aligned}
&\left|\frac{\frac{z(H_{p,q,s}(\alpha_1)f(z))''}{(H_{p,q,s}(\alpha_1)f(z))'} + (1+p)}{\frac{z(H_{p,q,s}(\alpha_1)f(z))''}{(H_{p,q,s}(\alpha_1)f(z))'} - (1+p) + 2\sigma}\right| \\
&= \left|\frac{-\sum_{n=p}^{\infty} n[n+p]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\cdot\frac{a_n}{n!}z^{n-1}}{2p((p+1)-\sigma)z^{-(p+1)} - \sum_{n=p}^{\infty} n[n+2(\sigma-1)-p]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\cdot\frac{a_n}{n!}z^{n-1}}\right| < 1.
\end{aligned}$$

Since $Re(z) \leq |z|$ for all z $(z \in U)$, we have

$$Re\left\{\frac{\sum\limits_{n=p}^{\infty} n(n+p)\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\cdot\frac{a_n}{n!}z^{n-1}}{2p((p+1)-\sigma)z^{-(p+1)}-\sum\limits_{n=p}^{\infty} n[n+2(\sigma-1)-p]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\cdot\frac{a_n}{n!}z^{n-1}}\right\}<1. \tag{2.3}$$

We choose the value of z on the real axis so that $\frac{z(H_{p,q,s}(\alpha_1)f(z))''}{(H_{p,q,s}(\alpha_1)f(z))'}$ is real. Upon clearing the denominator of (2.3) and letting $z \to 1^-$, through real values so we can write (2.3) as

$$\sum_{n=p}^{\infty}[n+\sigma-1]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\cdot\frac{a_n}{(n-1)!} \leq p((p+1)-\sigma).$$

Sharpness of the result follows by setting

$$f(z)=\frac{1}{z^p}+\frac{p((p+1)-\sigma)(n-1)!}{[n+\sigma-1]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}}\ z^n, \quad (n \geq 1).$$

Corollary 2.2 : Let $f \in \Sigma_p^*(q,s,\alpha_1,\sigma)$. Then

$$a_n \leq \frac{p((p+1)-\sigma)(n-1)!}{[n+\sigma-1]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}}, \quad (n \geq 1).$$

Theorem 2.3 : The class $\Sigma_p^*(q,s,\alpha_1,\sigma)$ is a convex set.
Proof : Let f_1 and f_2 be the arbitrary elements of $\Sigma_p^*(q,s,\alpha_1,\sigma)$. Then for every t $(0 \leq t \leq 1)$, we show that $(1-t)f_1+tf_2 \in \Sigma_p^*(q,s,\alpha_1,\sigma)$.

Thus, we have

$$(1-t)f_1+tf_2=\frac{1}{z^p}+\sum_{n=p}^{\infty}[(1-t)a_n+tb_n]z^n.$$

Hence,

$$\sum_{n=p}^{\infty}(n+\sigma-1)\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\cdot\frac{[(1-t)a_n+tb_n]}{(n-1)!}$$
$$=(1-t)\sum_{n=p}^{\infty}(n+\sigma-1)\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\cdot\frac{a_n}{(n-1)!}$$
$$+t\sum_{n=p}^{\infty}(n+\sigma-1)\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\cdot\frac{b_n}{(n-1)!}$$
$$\leq (1-t)p((p+1)-\sigma)+tp((p+1)-\sigma)=p((p+1)-\sigma).$$

This completes the proof.

3 Growth and Distortion Bounds

Next, we obtain the growth and distortion bounds for the linear operator $H_{p,q,s}(\alpha_1)$.

Theorem 3.1 : If $f \in \Sigma_p^*(q,s,\alpha_1,\sigma)$, then

$$\frac{1}{r^p}-\frac{p((p+1)-\sigma)}{[p+\sigma-1]}r^p \leq |H_{p,q,s}(\alpha_1)f(z)| \leq \frac{1}{r^p}+\frac{p((p+1)-\sigma)}{[p+\sigma-1]}r^p, \ \ (|z|=r<1). \tag{3.1}$$

The result is sharp for the function

$$f(z)=\frac{1}{z^p}+\frac{p((p+1)-\sigma)(p-1)!}{[p+\sigma-1]\frac{(\alpha_1)_p\cdots(\alpha_q)_p}{(\beta_1)_p\cdots(\beta_s)_p}}\ z^p, \tag{3.2}$$

Proof : Let $f \in \Sigma_p^*(q,s,\alpha_1,\sigma)$. Then by Theorem 2.1, we get

$$\begin{aligned} &[p+\sigma-1]\frac{(\alpha_1)_p\cdots(\alpha_q)_p}{(\beta_1)_p\cdots(\beta_s)_p}\cdot\frac{1}{(p-1)!}\sum_{n=p}^{\infty}a_n \\ &\leq \sum_{n=p}^{\infty}[n+\sigma-1]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\frac{a_n}{(n-1)!} \leq p(p+1)-\sigma), \end{aligned}$$

or

$$\sum_{n=p}^{\infty}a_n \leq \frac{p((p+1)-\sigma)(p-1)!}{[p+\sigma-1]\frac{(\alpha_1)_p\cdots(\alpha_q)_p}{(\beta_1)_p\cdots(\beta_s)_p}} \tag{3.3}$$

$$\begin{aligned} |H_{p,q,s}(\alpha_1)f(z)| &\leq \frac{1}{|z|^p}+\sum_{n=p}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\cdot\frac{a_n}{(n-1)!}|z|^n \\ &\leq \frac{1}{|z|^p}+\sum_{n=p}^{\infty}\frac{(\alpha_1)_p\cdots(\alpha_q)_p}{(\beta_1)_p\cdots(\beta_s)_p}\cdot\frac{|z|^p}{(p-1)!}\sum_{n=p}^{\infty}a_n \\ &= \frac{1}{r^p}+\frac{(\alpha_1)_p\cdots(\alpha_q)_p}{(\beta_1)_p\cdots(\beta_s)_p}\cdot\frac{r_p}{(p-1)!}\sum_{n=p}^{\infty}a_n \\ &\leq \frac{1}{r^p}+\frac{p((p+1)-\sigma)}{[p+\sigma-1]}r^p. \end{aligned} \tag{3.4}$$

Similarly,

$$\begin{aligned}
|H_{p,q,s}(\alpha_1)f(z)| &\geq \frac{1}{|z|^p} - \sum_{n=p}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \cdot \frac{a_n}{(n-1)!}|z|^n \\
&\geq \frac{1}{|z|^p} - \frac{(\alpha_1)_p \cdots (\alpha_q)_p}{(\beta_1)_p \cdots (\beta_s)_p} \cdot \frac{|z|^p}{(p-1)!} \sum_{n=p}^{\infty} a_n \\
&= \frac{1}{r^p} - \frac{(\alpha_1)_p \cdots (\alpha_q)_p}{(\beta_1)_p \cdots (\beta_s)_p} \cdot \frac{r_p}{(p-1)!} \sum_{n=p}^{\infty} a_n \\
&\geq \frac{1}{r^p} - \frac{p((p+1)-\sigma)}{[p+\sigma-1]} r^p. \qquad (3.5)
\end{aligned}$$

From (3.4) and (3.5),we get (3.1) and the proof is complete.
Theorem 3.2 : If $f \in \Sigma_p^*(q, s, \alpha_1, \sigma)$, then

$$\frac{1}{r^{p+1}} - \frac{p^2((p+1)-\sigma)}{[p+\sigma-1]} r^{p-1} \leq |(H_{p,q,s}(\alpha_1)f(z))'|$$

$$\leq \frac{1}{r^{p+1}} + \frac{p^2((p+1)-\sigma)}{[p+\sigma-1]} r^{p-1}, \quad (|z| = r < 1).(3.6)$$

The result is sharp for the function f is given by (3.2).

The proof is similar to that of Theorem (3.1).

4 Integral Representation

In the following theorem, we obtain integral representation for $H_{p,q,s}(\alpha_1)f(z)$.
Theorem 4.1 : Let $f \in \Sigma_p^*(q, s, \alpha_1, \sigma)$. Then

$$H_{p,q,s}(\alpha_1)f(z) = \int_0^z \exp\left[\int_0^z \frac{(2\sigma-(1+p))\psi(t)-(1+p)}{t(1-\psi(t))} dt\right] dt,$$

where $|\psi(t)| < 1, z \in U$.
Proof : By letting $\frac{z(H_{p,q,s}(\alpha_1)f(z))''}{(H_{p,q,s}(\alpha_1)f(z))'} = \eta(z)$ in (1.12), we have $\left|\frac{\eta(z)+(1+p)}{\eta(z)-(1+p)+2\sigma}\right| < 1$, or equivalently

$$\frac{\eta(z)+(1+p)}{\eta(z)-(1+p)+2\sigma} = \psi(z), \quad (|\psi(z)| < 1, z \in U).$$

So

$$\frac{(H_{p,q,s}(\alpha_1)f(z))''}{(H_{p,q,s}(\alpha_1)f(z))'} = \frac{(2\sigma-(1+p))\psi(z)-(1+p)}{z(1-\psi(z))},$$

after integration, we have

$$\log((H_{p,q,s}(\alpha_1)f(z))') = \int_0^z \frac{(2\sigma-(1+p))\psi(t)-(1+p)}{t(1-\psi(t))}dt.$$

Therefore,

$$(H_{p,q,s}(\alpha_1)f(z))' = \exp\left[\int_0^z \frac{(2\sigma-(1+p))\psi(t)-(1+p)}{t(1-\psi(t))}dt\right].$$

After integration, we have

$$H_{p,q,s}(\alpha_1)f(z) = \int_0^z \exp\left[\int_0^z \frac{(2\sigma-(1+p))\psi(t)-(1+p)}{t(1-\psi(t))}dt\right]dt,$$

and this gives the required result.

5 Neighborhoods for the class $\Sigma_p^*(q,s,\alpha_1,\sigma)$

Following the earlier works on neighborhoods of analytic functions by Goodman [8] and Ruscheweyh [15], we begin by introducing here the δ-neighborhood of a function $f \in \Sigma_p^*$ of the form (1.1) by means of the definition below:-

$$N_\delta(f) = \left\{ g \in \Sigma_p^* : g(z) = z^{-p} - \sum_{n=p}^{\infty} b_{nz^n} \text{ and } \sum_{n=p}^{\infty} n|a_n - b_n| \le \delta, 0 \le \delta < 1 \right\}. \tag{5.1}$$

Particulary for the identity function $e(z) = z^{-p}$, we have

$$N_\delta(e) = \left\{ g \in \Sigma_p^* : g(z) = z^{-p} - \sum_{n=p}^{\infty} b_{nz^n} \text{ and } \sum_{n=p}^{\infty} n|b_n| \le \delta \right\}. \tag{5.2}$$

Definition 5.1 : A function $f \in \Sigma_p^*$ is said to be in the class $\Sigma_{p,y}^*(q,s,\alpha_1,\sigma)$ if there exists function $g \in \Sigma_p^*(q,s,\alpha_1,\sigma)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - y, ‘ \quad (z \in U, 0 \le y < 1).$$

Theorem 5.1 : If $g \in \Sigma_p^*(q,s,\alpha_1,\sigma)$ and

$$y = 1 - \frac{p((p+1)-\sigma)(p-1)!}{[p+\sigma-1]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}}, \tag{5.3}$$

then $N_\delta(g) \subset \Sigma_{p,y}^*(q,s,\alpha_1,\sigma)$.

Proof : Let $f \in N_\delta(g)$. Then, we find from (5.1) that

$$\sum_{n=p}^{\infty} n|a_n - b_n| \le \delta,$$

which implies the coefficient inequality

$$\sum_{n=p}^{\infty} |a_n - b_n| \le \delta, \qquad (n \ge p).$$

Since $g \in \Sigma_p^*(q, s, \alpha_1, \sigma)$, then by using Theorem 2.1, we get

$$\sum_{n=p}^{\infty} b_n \le \frac{p((p+1)-\sigma)(p-1)!}{[p+\sigma-1]\frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n}},$$

so that

$$\begin{aligned} \left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum_{n=p}^{\infty} |a_n - b_n|}{1 - \sum_{n=p}^{\infty} b_n} \\ &\le \frac{\delta[p+\sigma-1]\frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n}}{[p+\sigma-1]\frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} - p((p+1)-\sigma)(p-1)!} = 1 - y. \end{aligned}$$

Hence, by Definition 5.1, $f \in \Sigma_{p,y}^*(q, s, \alpha_1, \sigma)$ for y given by (5.3).

This completes the proof.

Theorem 5.2 : If

$$\delta = \frac{p((p+1)-\sigma)}{[p+\sigma-1]\frac{(\alpha_1)_p \cdots (\alpha_q)_p}{(\beta_1)_p \cdots (\beta_s)_p}},$$

then $\Sigma_p^*(q, s, \alpha_1, \sigma) \subset N_\delta(e)$.

Proof : Let $f \in \Sigma_{p,y}^*(q, s, \alpha_1, \sigma)$. Then by using Theorem 2.1, we have

$$[p+\sigma-1]\frac{(\alpha_1)_p \cdots (\alpha_q)_p}{(\beta_1)_p \cdots (\beta_s)_p} \cdot \frac{1}{(p-1)!} \sum_{n=p}^{\infty} a_n \le p((p+1)-\sigma). \tag{5.4}$$

On the other hand, from (2.1) and (5.4) that

$$\begin{aligned} &\frac{(\alpha_1)_p \cdots (\alpha_q)_p}{(\beta_1)_p \cdots (\beta_s)_p} \sum_{n=p}^{\infty} n a_n \le p((p+1)-\sigma) - [p+\sigma-2]\frac{(\alpha_1)_p \cdots (\alpha_q)_p}{(\beta_1)_p \cdots (\beta_s)_p} \cdot \frac{1}{(p-1)!} \\ &\cdot \frac{p((p+1)-\sigma)(p-1)!}{[p+\sigma-1]\frac{(\alpha_1)_p \cdots (\alpha_q)_p}{(\beta_1)_p \cdots (\beta_s)_p}} \\ &= \frac{p((p+1)-\sigma)}{[p+\sigma-1]}. \end{aligned}$$

That is,

$$\sum_{n=p}^{\infty} n a_n \leq \frac{p((p+1)-\sigma)}{[p+\sigma-1]\frac{(\alpha_1)_p\cdots(\alpha_q)_p}{(\beta_1)_p\cdots(\beta_s)_p}} = \delta.$$

Thus, by the definition given by (5.2), $f \in N_\delta(e)$.

This completes the proof.

6 Radii of Starlikeness and Convexity

In the following theorems, we discuss the radii of starlikeness and convexity.

Theorem 6.1 : If $f \in \Sigma_p^*(q, s, \alpha_1, \sigma)$, then f is multivalent meromorphic starlike of order φ $(0 \leq \varphi < p)$ in the disk $|z| < r_1$, where

$$r_1 = \inf_n \left\{ \frac{(p-\varphi)[n+\sigma-1]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}}{(n-\varphi+2p)p((p+1)-\sigma)(n-1)!} \right\}^{\frac{1}{n+p}}, \quad n \geq p.$$

The result is sharp for the function f given by (2.2).

Proof : It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} + p \right| \leq p - \varphi \quad \text{for} \quad |z| < r_1. \tag{6.1}$$

But

$$\left| \frac{zf'(z) + pf(z)}{f(z)} \right| \leq \frac{\sum_{n=p}^{\infty} (n+p) a_n |z|^{n+p}}{1 - \sum_{n=p}^{\infty} a_n |z|^{n+p}}.$$

Thus, (6.1) will be satisfied if

$$\frac{\sum_{n=p}^{\infty} (n+p) a_n |z|^{n+p}}{1 - \sum_{n=p}^{\infty} a_n |z|^{n+p}} \leq p - \varphi,$$

or if

$$\sum_{n=p}^{\infty} \frac{(n-\varphi+2p)a_n}{p-\varphi} \ |z|^{n+p} \leq 1. \tag{6.2}$$

Since $f \in \Sigma_p^*(q, s, \alpha_1, \sigma)$, we have

$$\sum_{n=p}^{\infty} \frac{[n+\sigma-1]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}}{p((p+1)-\sigma)(n-1)!} \ a_n \leq 1.$$

Hence, (6.2) will be true if

$$\frac{(n-\varphi+2p)}{p-\varphi}|z|^{n+p} \leq \frac{[n+\sigma-1]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}}{p((p+1)-\sigma)(n-1)!},$$

or equivalently

$$|z| \leq \left\{\frac{(p-\varphi)[n+\sigma-1]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}}{(n-\varphi+2p)p((p+1)-\sigma)(n-1)!}\right\}^{\frac{1}{n+p}}, \quad n \geq p$$

which follows the result.

Theorem 6.2 : If $f \in \Sigma_p^*(q, s, \alpha_1, \sigma)$, then f is multivalent meromorphic convex of order φ $(0 \leq \varphi < p)$ in the disk $|z| < r_2$, where

$$r_2 = \inf_n \left\{\frac{(p-\varphi)[n+\sigma-1]\frac{(\alpha_1)_p\cdots(\alpha_q)_p}{(\beta_1)_p\cdots(\beta_s)_p}}{n(n-\varphi+2p)((p+1-\sigma)(n-1)!}\right\}^{\frac{1}{n+p}}.$$

The result is sharp for the function f given by (2.2).

Proof : It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}+1+p\right| \leq p-\varphi \quad \text{for‘} \quad |z| < r_2. \tag{6.3}$$

But

$$\left|\frac{zf''(z)}{f'(z)}+1+p\right| = \left|\frac{zf''(z)+(p+1)f'(z)}{f'(z)}\right| \leq \frac{\sum\limits_{n=p}^{\infty} n(n+p)a_n|z|^{n+p}}{p-\sum\limits_{n=p}^{\infty} na_n|z|^{n+p}}.$$

Thus, (6.3) will be satisfied if

$$\frac{\sum\limits_{n=p}^{\infty} n(n+p)a_n|z|^{n+p}}{p-\sum\limits_{n=p}^{\infty} a_n|z|^{n+p}} \leq p-\varphi,$$

or if

$$\sum_{n=p}^{n} \frac{n(n-\varphi+2p)}{p(p-\varphi)}a_n|z|^{n+p} \leq 1. \tag{6.4}$$

Since $f \in \Sigma_p^*(q, s, \alpha_1, \sigma)$, we have

$$\sum_{n=p}^{\infty} \frac{[n+\sigma-1]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}}{p((p+1)-\sigma)(n-1)!}\, a_n \leq 1.$$

Hence, (6.4) will be true if

$$\frac{n(n-\varphi+2p)}{p(p-\varphi)}|z|^{n+p} \leq \frac{[n+\sigma-1]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}}{p((p+1)-\sigma)(n-1)!},$$

or equivalently

$$|z| \leq \left\{\frac{(p-\varphi)[n+\sigma-1]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}}{n(n-\varphi+2p)((p+1)-\sigma)(n-1)!}\right\}^{\frac{1}{n+p}}, \quad n \geq p,$$

which follows the result .

7 Integral Operator

Theorem 7.1 : Let the function f be given by (1.1) in the class $\Sigma_p^*(q,s,\alpha_1,\sigma)$. Then, the integral operator

$$w(z) = \mu\int_0^1 u^\mu f(uz)du, \quad (0 < u \leq 1, 0 < \mu < \infty), \tag{7.1}$$

is in the class $\Sigma_p^*(q,s,\alpha_1,\tau)$, where

$$\tau = \frac{(p+1)(\mu+p+1)[p+\sigma-1]-\mu[p-1]((p+1)-\sigma)}{\mu((p+1)-\sigma)+(\mu+p+1)[p+\sigma-1]}.$$

The result is sharp for the function f given by (3.2).

Proof : Let $f(z) = \frac{1}{z^p} + \sum\limits_{n=p}^{\infty} a_n z^n$, in the class $\Sigma_p^*(q,s,\alpha_1,\sigma)$. Then

$$\begin{aligned} w(z) &= \mu\int_0^1 u^\mu f(uz)du \\ &= \mu\int_0^1 \left(\frac{u^{\mu-1}}{z^n} - \sum_{n=p}^{\infty} u^{n+\mu} a_n z^n\right) du \\ &= \frac{1}{z^p} - \sum_{n=p}^{\infty} \frac{\mu}{\mu+n+1} a_n z^n. \end{aligned}$$

It is enough to show that

$$\sum_{n=p}^{\infty} \frac{\mu[n+\tau-1]\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}}{(\mu+n+1)p((p+1)-\tau)(n-1)!} a_n \leq 1. \tag{7.2}$$

Since $f \in \Sigma_p^*(q, s, \alpha_1, \sigma)$, then by Theorem 2.1, we get

$$\sum_{n=p}^{\infty} \frac{[n+\sigma-1]\frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n}}{p((p+1)-\sigma)(n-1)!} a_n \leq 1.$$

Note that (7.2) is satisfied if

$$\frac{\mu[n+\tau-1]\frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n}}{(\mu+n+1)p((p+1)-\tau)(n-1)!} \leq \frac{[n+\sigma-1]\frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n}}{p((p+1)-\sigma)(n-1)!}$$

or equivalently

$$\tau \leq \frac{(p+1)(\mu+n+1)[n+\sigma-1] - \mu[n-1]((p+1)-\sigma)}{\mu((p+1)-\sigma) + (\mu+n+1)[n+\sigma-1]} = w(n).$$

A simple computation will show that $w(n)$ is increasing function of n.

This means that $w(n) \geq w(p)$. Using this, we obtain the result.

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