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Differential Subordination of Meromorphically p-Valent Analytic Functions Associated with Mostafa Operator

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Abstract

In the present paper, we introduce a subclass $\Sigma_{p,\eta}^{\alpha}(\mu; a; h)$ of meromorphically p-valent analytic functions. We obtain main results of this subclass related with differential subordination. Also we study integral operator and convolution properties of our class.

Mathematics Subject Classification: 30C45

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1 Introduction

Let Σ_p be the class of p-valent meromorphic functions of the form,

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p}, \quad (p \in \mathbb{N} = \{1, 2, \cdots\}), \tag{1}$$

which are analytic in the punctured open unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. For functions $f \in \Sigma_p$ given by (1) and $g \in \Sigma_p$ defined by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p}, \quad (p \in \mathbb{N} = \{1, 2, \cdots\}),$$

we define the convolution (or Hadamard product) of f and g by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p}. \tag{2}$$

For functions f and g analytic in $U = U^* \cup \{0\}$, we say that g is subordinate to f in U, denote by $g \prec f$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $g(z) = f(w(z))$ $(z \in U)$. Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence relationship holds true (see [2] and [7]),

$$g(z) \prec f(z) \Leftrightarrow g(0) = f(0) \text{ and } g(U) \subset f(U).$$

Let A be the class of functions of the form,

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in U. A function $h \in A$ is said to be in the class $S^*(\beta)$ if,

$$Re\left\{\frac{zh'(z)}{h(z)}\right\} > \beta \quad (z \in U),$$

for some β $(\beta < 1)$. When $0 \leq \beta < 1, S^*(\beta)$ is the class starlike functions of order β in U. A function $h \in A$ is said to be prestarlike functions of order β in U if

$$\frac{z}{(1-z)^2(1-\beta)} * h(z) \in S^*(\beta) \quad (\beta < 1),$$

where the Symbol $*$ means the familiar Hadamard product (or convolution) of two analytic functions in U. We denote this class by $R(\beta)$ (see [8]). We note that a function $h \in A$ is in the class $R(0)$ if and only if h is convex univalent in U and $R(\frac{1}{2}) = S^*(\frac{1}{2})$.

Aqlan et. al. (see [1]) defined the operator $Q^{\alpha}_{\eta,p} f(z) : \Sigma_p \to \Sigma_p$ by:

$$Q^{\alpha}_{\eta,p} f(z) = \begin{cases} z^{-p} + \frac{\Gamma(\alpha+\eta)}{\Gamma(\eta)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\eta)}{\Gamma(k+\eta+\alpha)} a_{k-p} z^{k-p} & (\alpha > 0; \eta > -1; p \in N; f \in \Sigma_p) \\ f(z) & (\alpha = 0; \eta > -1; p \in N; f \in \Sigma_p). \end{cases} \tag{3}$$

Now, we define the operator $\mathcal{H}^{\alpha}_{p,\eta,\mu} : \Sigma_p \to \Sigma_p$ (see [4]) as follows: First; put

$$G^{\alpha}_{\eta,p}(z) = z^{-p} + \frac{\Gamma(\alpha+\eta)}{\Gamma(\eta)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\eta)}{\Gamma(k+\eta+\alpha)} z^{k-p} \quad (p \in N). \tag{4}$$

And let $G^{\alpha^*}_{\eta,p,\mu}$ be defined by

$$G^{\alpha}_{\eta,p}(z) * G^{\alpha^*}_{\eta,P,\mu}(z) = \frac{1}{z^p(1-z)^{\mu}} \quad (\mu > 0; p \in N). \tag{5}$$

Then

$$\mathcal{H}^{\alpha}_{P,\eta,\mu} f(z) = G^{\alpha^*}_{\eta,P,\mu}(z) * f(z) \ (f \in \Sigma_p). \tag{6}$$

Using (4) - (6), we have

$$\mathcal{H}^{\alpha}_{p,\eta,\mu} f(z) = z^{-p} + \frac{\Gamma(\eta)}{\Gamma(\alpha+\eta)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\eta+\alpha)(\mu)_k}{\Gamma(k+\eta)(1)_k} \ a_{k-p}, \tag{7}$$

where $f \in \Sigma_p$ is in the form (1) and $(v)_n$ denote the Pochhammer symbol given by

$$(v)_n = \frac{\Gamma(v+\eta)}{\Gamma(v)} = \begin{cases} 1 & (n=0) \\ v(v+1)\cdots(v+n-1) & (n \in N) \end{cases}.$$

It is readily verified from (7) that

$$z(\mathcal{H}^{\alpha}_{p,\eta,\mu} f(z))' = (\alpha+\eta)\mathcal{H}^{\alpha+1}_{p,\eta,\mu} f(z) - (\alpha+\eta+p)\mathcal{H}^{\alpha}_{p,\eta,\mu} \tag{8}$$

and

$$z(\mathcal{H}^{\alpha}_{p,\eta,\mu} f(z))' = \mu\mathcal{H}^{\alpha}_{p,\eta,\mu+1} f(z) - (\mu+p)\mathcal{H}^{\alpha}_{p,\eta,\mu} f(z). \tag{9}$$

Definition 1 : A function $f \in \Sigma_p$ is said to be in the class $\Sigma^{\alpha}_{p,\eta}(\mu,\lambda;h)$, if it satisfies the subordination condition :

$$\frac{(1+\lambda)}{p(p+1)} z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu} f(z))'' + \frac{\lambda}{p(p+1)(p+2)} z^{p+3}(\mathcal{H}^{\alpha}_{p,\eta,\mu} f(z))''' \prec h(z). \tag{10}$$

In order to get the convolution results of the class of multivalent analytic functions class $\Sigma^{\alpha}_{p,\eta}(\mu,\lambda;h)$, it is necessary to put the following restrictions on the operator $\mathcal{H}^{\alpha}_{p,\eta,\mu}$:

$$\mathcal{H}^{\alpha}_{p,\eta,\mu}(f_1 * f_2) = (\mathcal{H}^{\alpha}_{p,\eta,\mu} f_1) * f_2 = f_1 * (\mathcal{H}^{\alpha}_{p,\eta,\mu} f_2), \tag{11}$$

where $f_1, f_2 \in \Sigma^{\alpha}_{p,\eta}(\mu,\lambda;h)$.

2 Some Lemmas

In order to prove our main results, we need the following lemmas.

Lemma 1 (see [6]) : Let $g(z)$ be analytic in U, and $h(z)$ be analytic and convex univalent in U with $h(0) = g(0)$. If

$$g(z) = \frac{1}{m} z g'(z) \prec h(z), \tag{12}$$

where $Re\{m\} \geq 0$ and $m \neq 0$, then

$$g(z) \prec \tilde{h}(z) = mz^{-m} \int_0^z t^{m-1} h(t)dt \prec h(z)$$

and $\tilde{h}(z)$ is the best dominant of (12).

Lemma 2 (see [9]) : Let $f(z) \prec F(z)$ $(z \in U)$ and $g(z) \prec G(z)$ $(z \in U)$ if the function $F(z)$ and $G(z)$ are convex in U. Then

$$(f * g)(z) \prec (F * G) \quad (z \in U). \tag{13}$$

Lemma 3 (see [8]) : Let $(\beta < 1), f(z) \in S^*(\beta)$ and $g(z) \in R(\beta)$. For any analytic function $F(z)$ in U, then

$$\frac{g * (fF)}{g * f}(U) \subset \overline{Co}(F(U)), \tag{14}$$

where $\overline{co}(F(U))$ denotes the convex hull of $F(U)$.

3 Main Results

Theorem 3.1 : If the function $f(z)$ belongs to the class $\Sigma^{\alpha}_{p,\eta}(\mu, \lambda; h)$, then

$$g(z) = \frac{z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu} f(z))''}{p(p+1)} \prec h(z) \tag{15}$$

and if $\lambda > 0$, then $g(z) \prec \tilde{h}(z)$, where

$$\tilde{h}(z) = \frac{(p+2)}{\lambda} z^{-\frac{(p+2)}{\lambda}} \int_0^z t^{\frac{(p+2)}{\lambda}-1} h(t)dt \prec h(z), \quad (z \in U) \tag{16}$$

the function $h(z)$ is convex univalent in U and $\tilde{h}(z)$ is the best dominant of subordination

$$g(z) \prec \tilde{h}(z) \quad (z \in U).$$

Proof :

(i) if $\lambda = 0$, the proof is trivial.

(ii) if $\lambda > 0$, let $f(z) \in \Sigma^{\alpha}_{p,\eta}(\mu, \lambda; h)$, then

$$\frac{(1+\lambda)}{p(p+1)} z^{p+2}(\mathcal{H}^{\alpha}_{P,\eta,\mu} f(z))'' + \frac{\lambda}{p(p+1)(p+2)} z^{p+3}(\mathcal{H}^{\alpha}_{p,\eta,\mu} f(z))''' \prec h(z). \tag{17}$$

Then by (10) and (15), we have

$$g(z) + \frac{\lambda}{(p+2)} zg'(z) \prec h(z) \quad (z \in U). \tag{18}$$

Using Lemma (1) in (18) with $m = \frac{(p+2)}{\lambda}$ and $\lambda > 0$, we get

$$g(z) \prec \tilde{h}(z) = \frac{(p+2)}{\lambda} z^{-\frac{(p+2)}{\lambda}} \int_0^z t^{\frac{(p+2)}{\lambda}-1} h(t)dt \prec h(z),$$

where $g(z)$ is given by (15).

The proof of Theorem 3.1 is complete.

Theorem 3.2 : Let $0 \le \lambda_2 < \lambda_1$. Then $\Sigma^{\alpha}_{p,\eta}(\mu, \lambda_1; h) \subset \Sigma^{\alpha}_{p,\eta}(\mu, \lambda_2; h)$.

Proof : Suppose that $f(z) \in \Sigma^{\alpha}_{p,\eta}(\mu, \lambda_1; h)$. A simple computation

$$\begin{aligned}
&\frac{(1+\lambda_2)}{p(p+1)} z^{p+2}(\mathcal{H}^{\alpha}_{P,\eta,\mu} f(z))'' + \frac{\lambda_2}{p(p+1)(p+2)} z^{p+3}(\mathcal{H}^{\alpha}_{P,\eta,\mu} f(z))''' \\
&= \left[1 - \frac{\lambda_2}{\lambda_1}\right] \frac{z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu} f(z))''}{p(p+1)} \\
&+ \frac{\lambda_2}{\lambda_1}\left[\frac{(1+\lambda_1)}{p(p+1)} z^{p+2}(\mathcal{H}^{\alpha}_{P,\eta,\mu} f(z))'' + \frac{\lambda_1}{p(p+1)(p+2)} z^{p+3}(\mathcal{H}^{\alpha}_{p,\eta,\mu} f(z))'''\right] \qquad (19)
\end{aligned}$$

and since $h(z)$ is a convex set. We can write (19) as follows :

$$\begin{aligned}
&\left[\frac{(1+\lambda_2)}{p(p+1)} z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu} f(z))'' + \frac{\lambda_2}{p(p+1)(p+2)} z^{p+3}(\mathcal{H}^{\alpha}_{p,\eta,\mu} f(z))'''\right] \\
&= \left[1 - \frac{\lambda_2}{\lambda_1}\right] g_1(z) + \frac{\lambda_2}{\lambda_1} g_2(z) = \phi(z),
\end{aligned}$$

where $g_1(z) \prec h(z), g_2(z) \prec h(z)$, by using definition of convex set and by Theorem 3.1, since $f(z) \in \Sigma^{\alpha}_{p,\eta}(\mu, \lambda_1; h)$, we get $\phi(z) \prec h(z)$. Therefore $f(z) \in \Sigma^{\alpha}_{p,\eta}(\mu, \lambda_1; h)$.

4 Integral Operator

Theorem 4.1 : Let the functions $f(z) \in \Sigma_p$ and $F(z)$ defined by

$$F(z) = \frac{(\sigma - p)}{z^{\sigma}} \int_0^z t^{\sigma-1} f(t)dt \qquad (Re\{\sigma\} > -p),$$

if

$$\left(1 + \frac{\gamma}{p}\right) \frac{z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu} F(z))''}{p(p+1)} - \gamma \frac{z^{p+3}(\mathcal{H}^{\alpha}_{p,\eta,\mu} f(z))'''}{p(p+1)(p+2)} \prec h(z). \qquad (20)$$

Then the function $F(z) \in \Sigma^{\alpha}_{p,\eta}(\mu, 0; \tilde{h})$, where

$$\tilde{h}(z) = \frac{(\mu - p)}{\gamma} z^{-\frac{(\mu-p)}{\gamma}} \int_0^z t^{\frac{(\mu-p)}{\gamma}-1} h(t)dt \prec h(z). \qquad (21)$$

Proof : Let us define

$$G(z)=\frac{z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu}F(z))''}{p(p+1)}, \tag{22}$$

then $G(z)$ is analytic in $U, G(0)=1$ and

$$\frac{zG'(z)}{(p+2)}=G(z)+\frac{z^{p+3}(\mathcal{H}^{\alpha}_{p,\eta,\mu}F(z))'''}{p(p+1)(p+2)}. \tag{23}$$

Making use of (20), (22), (23) and by

$$(\sigma p-p)f(z)=\sigma(p+2)F''(z)+zF'''(z).$$

Then

$$\begin{aligned}
&\left(1+\frac{\gamma}{p}\right)\frac{z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu}F(z))''}{p(p+1)}-\gamma\frac{z^{p+3}(\mathcal{H}^{\alpha}_{p,\eta,\mu}F(z))'''}{p(p+1)(p+2)}\\
&=\left(1+\frac{\gamma}{p}\right)\frac{z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu}F(z))''}{p(p+1)}\\
&-\frac{\gamma}{(\sigma p-p)}\left[\frac{z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu}F(z))''}{p(p+1)}+\frac{z^{p+3}(\mathcal{H}^{\alpha}_{p,\eta,\mu}F(z))'''}{p(p+1)(p+2)}\right]\\
&=G(z)-\frac{\gamma}{(\sigma p-p)(p+2)}zG'(z)\\
&=G(z)+\frac{-\gamma}{(\sigma p-p)(p+2)}zG'(z)\prec h(z).
\end{aligned}$$

Then $G(z)\prec\tilde{h}(z)$, where $\tilde{h}(z)$ is given by (21), and thus $F(z)\in\Sigma^{\alpha}_{p,\eta}(\mu,0;\tilde{h})$.
The proof is complete.

5 Convolution Properties

Theorem 5.1 : If $f_j(z)\in\Sigma^{\alpha}_{p,\eta}\left(\mu,\lambda;\frac{1+A_jz}{1+B_jz}\right)$ $(j=1,2)$ and let the operator $\mathcal{H}^{\alpha}_{p,\eta,\mu}$ satisfy the condition (11). Then each of the following inclusion relationship holds true:

$$\begin{aligned}
G(z) &= \frac{(1+\lambda)}{p^2(p+1)^2}z^2(\mathcal{H}^{\alpha}_{p,\eta,\mu}(f_1''*f_2'')(z))+\frac{\lambda}{p^2(p+1)^2(p+2)^2}z^3\times\\
&\quad(\mathcal{H}^{\alpha}_{p,\eta,\mu}(f_1'''*f_2''')(z))\in\Sigma^{\alpha}_{p,\eta}\left(\mu,\lambda;\left(\frac{1+A_1z}{1+B_1z}\right)*\left(\frac{1+A_2z}{1+B_2z}\right)\right)
\end{aligned} \tag{24}$$

$$h(z)=\frac{z^2(\mathcal{H}^{\alpha}_{P,\eta,\mu}(f_1''*f_2'')(z))}{p^2(p+1)^2}\in\Sigma^{\alpha}_{p,\eta}\left(\mu,\lambda;\left(\frac{1+A_1z}{1+B_1z}\right)*\left(\frac{1+A_2z}{1+B_2z}\right)\right) \tag{25}$$

and

$$\frac{z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu}h(z))''}{p(p+1)} \prec \left(\left(\frac{1+A_1z}{1+B_1z}\right) * \left(\frac{1+A_2z}{1+B_2z}\right)\right). \tag{26}$$

Proof : Since $f_1(z) \in \Sigma^{\alpha}_{p,\eta}\left(\mu, \lambda; \left(\frac{1+A_1z}{1+B_1z}\right)\right)$ and $f_2(z) \in \Sigma^{\alpha}_{p,\eta}\left(\mu, \lambda; \left(\frac{1+A_2z}{1+B_2z}\right)\right)$. Then

$$\frac{(1+\lambda)}{p(p+1)}z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu}f_1(z))'' + \frac{\lambda}{p(p+1)(p+2)}z^{p+3}(\mathcal{H}^{\alpha}_{p,\eta,\mu}f_1(z))''' \prec \left(\frac{1+A_1z}{1+B_1z}\right) \tag{27}$$

and

$$\frac{(1+\lambda)}{p(p+1)}z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu}f_2(z))'' + \frac{\lambda}{p(p+1)(p+2)}z^{p+3}(\mathcal{H}^{\alpha}_{p,\eta,\mu}f_2(z))''' \prec \left(\frac{1+A_2z}{1+B_2z}\right) \tag{28}$$

and from (27), (28) and Theorem 3.1, we have

$$\frac{z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu}f_1(z))''}{p(p+1)} \prec \left(\frac{1+A_1z}{1+B_1z}\right)$$

and

$$\frac{z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu}f_2(z))''}{p(p+1)} \prec \left(\frac{1+A_2z}{1+B_2z}\right),$$

by using (11), (27), (28) and Lemma 2, in conjuction with the technique used before, we have

$$\begin{aligned}
&\frac{(1+\lambda)}{p(p+1)}z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu}G(z))'' + \frac{\lambda}{p(p+1)(p+2)}z^{p+3}(\mathcal{H}^{\alpha}_{p,\eta,\mu}G(z))''' \\
&= \frac{(1+\lambda)}{p(p+1)}z^{p+2}\left(\mathcal{H}^{\alpha}_{p,\eta,\mu}\left[\frac{1+\lambda}{p^2(p+1)^2}z^2(\mathcal{H}^{\alpha}_{p,\eta,\mu}(f_1''*f_2'')(z))\right.\right. \\
&\left.\left.+\frac{\lambda}{p^2(p+1)^2(p+2)^2}z^3(\mathcal{H}^{\alpha}_{p,\eta,\mu}(f_1'''*f_2''')(z))\right]\right)'' \\
&+\frac{\lambda}{p(p+1)(p+2)}z^{p+3}\left(\mathcal{H}^{\alpha}_{p,\eta,\mu}\left[\frac{(1+\lambda)}{p^2(p+1)^2}z^2(\mathcal{H}^{\alpha}_{p,\eta,\mu}(f_1''*f_2'')(Z))\right.\right. \\
&\left.\left.+\frac{\lambda}{p^2(p+1)^2(p+2)^2}z^3(\mathcal{H}^{\alpha}_{p,\eta,\mu}(f_1'''*f_2''')(z))\right]\right)''' \\
&\prec \left(\left(\frac{1+A_1z}{1+B_1z}\right) * \left(\frac{1+A_2z}{1+B_2z}\right)\right).
\end{aligned}$$

Hence

$$G(z) \in \Sigma^{\alpha}_{p,\eta}\left(\mu, \lambda; \left(\frac{1+A_1z}{1+B_1z}\right) * \left(\frac{1+A_2z}{1+B_2z}\right)\right),$$

and

$$h(z) \in \Sigma_{p,\eta}^{\alpha}\left(\mu, \lambda; \left(\frac{1+A_1 z}{1+B_1 z}\right) * \left(\frac{1+A_2 z}{1+B_2 z}\right)\right).$$

The proof of (24) and (25) is complete, we again proceed in a similar manner and apply Lemma 2, we obtain

$$\frac{(1+\lambda)}{p(p+1)} z^{p+2}(\mathcal{H}_{p,\eta,\mu}^{\alpha} h(z))'' + \frac{\lambda}{p(p+1)(p+2)} z^{p+3}(\mathcal{H}_{p,\eta,\mu}^{\alpha} h(z))''' \prec \left(\frac{1+A_1 z}{1+B_1 z}\right) * \left(\frac{1+A_2 z}{1+B_2 z}\right), \tag{29}$$

where $h(z)$ given by (25) and from (29) and Theorem 3.1 we have the proof of (26) and the proof is complete.

Theorem 5.2 : Let $f(z) \in \Sigma_{p,\eta}^{q}(\mu, \lambda; h), w(z) \in \Sigma_p$ and

$$Re\{z^p w(z)\} \geq \frac{1}{2} \quad (z \in U). \tag{30}$$

Then $(f * w)(z) \in \Sigma_{p,\eta}^{\alpha}(\mu, \lambda; h)$.

Proof : Let $f(z) \in \Sigma_{p,\eta}^{\alpha}(\mu, \lambda; h)$ be given (1) and $w(z) \in \Sigma_p$, we have

$$\begin{aligned} &\frac{(1+\lambda)}{p(p+1)} z^{p+2}(\mathcal{H}_{P,\eta,\mu}^{\alpha}(f * w)(z))'' + \frac{\lambda}{p(p+1)(p+2)} z^{p+3}(\mathcal{H}_{p,\eta,\mu}^{\alpha}(f * w)(z))''' \\ &= \frac{(1+\lambda)}{p(p+1)}\left[z^p w(z) * (z^{p+2}(\mathcal{H}_{p,\eta,\mu}^{\alpha} f(z))'')\right] \\ &+ \frac{\lambda}{p(p+1)(p+2)}\left[z^p w(z) * (z^{p+3}(\mathcal{H}_{p,\eta,\mu}^{\alpha} f(z))''')\right] \\ &= \{z^p w(z)\} * H(z), \end{aligned} \tag{31}$$

where

$$H(z) = \frac{1+\lambda}{p(p+1)} z^{p+2}\left(\mathcal{H}_{p,\eta,\mu}^{\alpha} f(z)\right)'' + \frac{(\lambda)}{p(p+1)(p+2)} z^{p+3}(\mathcal{H}_{p,\eta,\mu}^{\alpha} f(z))'''. \tag{32}$$

In view of (30), the function $z^p w(z)$ has the Herglotz representation

$$z^p w(z) = \int_{|x|=1} \frac{dp(x)}{1-xz} \quad (z \in U), \tag{33}$$

where $p(x)$ is a probability measure defined on the unit circle $|x| = 1$ and $\int_{|x|=1} dp(x) = 1$. Since $h(z)$ is convex univalent in U, it follows from (31) to (33) that

$$\begin{aligned} &\frac{(1+\lambda)}{p(p+1)} z^{p+2}(\mathcal{H}_{p,\eta,\mu}^{\alpha}(f * w)(z))'' + \frac{\lambda}{p(p+1)(p+2)} z^{p+3}(\mathcal{H}_{p,\eta,\mu}^{\alpha}(f * w)(z))''' \\ &= \int_{|x|=1} H(xz)dp(z) \prec h(z). \end{aligned}$$

Then $(f*w)(z) \in \Sigma^{\alpha}_{p,\eta}(\mu,\lambda;h)$.
Corollary 5.1 : Let $f(z) \in \Sigma^{\alpha}_{p,\eta}(\mu,\lambda;h)$ be given by (1) and let

$$Re\left\{1+\sum_{k=1}^{\infty}\frac{\delta}{\delta+k}z^k\right\} > \frac{1}{2}.$$

Then

$$F_{P,\delta}(f)(z) = \frac{\delta}{z^{p+\delta}}\int_0^z t^{p+\delta-1}f(t)dt \qquad (\delta > -p),$$

is also in the class $\Sigma^{\alpha}_{p,\eta}(\mu,\lambda;h)$.
Proof : Let $f(z) \in \Sigma^{\alpha}_{p,\eta}(\mu,\lambda;h)$ be defined in (1). Then,

$$\begin{aligned} F_{P,\delta}(f)(z) &= \frac{\delta}{z^{p+\delta}}\int_0^z t^{p+\delta-1}f(t)dt = z^{-p}+\sum_{k=1}^{\infty}\frac{\delta}{\delta+k}a_{k-p}z^{k-p} \\ &= \left(z^{-p}+\sum_{k=1}^{\infty}a_{k-p}z^{k-p}\right)*\left(z^{-p}+\sum_{k=1}^{\infty}\frac{\delta}{\delta+k}z^{k-p}\right) \\ &= (f*F)(z), \end{aligned} \tag{34}$$

where

$$F(z) = z^{-p}+\sum_{k=1}^{\infty}\frac{\delta}{\delta+k}z^{k-p} \qquad (\delta > -p),$$

and belong to Σ_p. We note that

$$Re\{z^pF(z)\} = Re\left\{1+\sum_{k=1}^{\infty}\frac{\delta}{\delta+k}z^k\right\} > \frac{1}{2} \tag{35}$$

by (34), (35) and by Theorem 5.2, thus $F_{P,\delta}(f)(z) \in \Sigma^{\alpha}_{p,\eta}(\mu,\lambda;h)$.
Theorem 5.3 : Let $f(z) \in \Sigma^{\alpha}_{p,\eta}(\mu,\lambda;h)$ be defined in (1), $w(z) \in \Sigma_p$ and $z^{p+1}w(z) \in R(\beta)$ $(\beta < 1)$. Then $(f*w) \in \Sigma^{\alpha}_{p,\eta}(\mu,\lambda;h)$.
Proof : Let $f(z) \in \Sigma^{\alpha}_{p,\eta}(\mu,\lambda;h)$ and $w(z) \in \Sigma_p$, from (31), we have

$$\begin{aligned} &\frac{(1+\lambda)}{p(p+1)}z^{p+2}(\mathcal{H}^{\alpha}_{p,\eta,\mu}(f*w)(z))''+\frac{\lambda}{p(p+1)(p+2)}z^{p+3}(\mathcal{H}^{\alpha}_{p,\eta,\mu}(f*w)(z))''' \\ &\qquad = \frac{(z^{p+1}w(z))*(zH(z))}{(z^{p+1}w(z))*z} \qquad (z \in U), \end{aligned} \tag{36}$$

where $H(z)$ is defined as in (32).

Since $h(z)$ is convex univalent in U.

$$H(z) \prec h(z),\ z^{p+1}w(z) \in R(\beta),\quad z \in S^*(\beta) \quad (\beta < 1).$$

It follows from (36) and Lemma 3, we get the proof.

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