Int. Journal of Math. Analysis, Vol. 7, 2013, no. 24, 1193 - 1206
HIKARI Ltd, www.m-hikari.com

Subclass of Multivalent Functions Defined by Hadamard Product Involving a Linear Operator

Waggas Galib Atshan

Department of Mathematics
College of Computer Science and Mathematics
University of Al-Qadisiya, Diwaniya, Iraq
waggashnd@gmail.com, waggas_hnd@yahoo.com

Hadi Jabber Mustafa and Emad Kadhim Mouajeeb

Department of Mathematics
College of Mathematics and Computer Science
University of Kufa, Najaf, Kufa, Iraq
eallamy@yahoo.com

Abstract

In the present paper, we introduce a subclass $R_p(\gamma, \beta, m, A, B, \alpha)$ of multivalent analytic functions in the open unit disc U. We study coefficient inequalities, closure theorem, neighborhood property and partial sums, radii of starlikeness, convexity and close-to-convexity. We also obtain weighted mean, arithmetic mean and linear combination.

Mathematics Subject Classification: 30C45, 30C50

Keywords: multivalent function, convolution, neighborhoods, partial sums, weighted mean, arithmetic mean, linear operator and linear combination

1 Introduction

Let $\mathcal{A}_p$ denote the class of all functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in N = \{1, 2, \cdots\}), \tag{1}$$

which are analytic and multivalent in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Let M_p denote the subclass of $\mathcal{A}_p$ containing of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad (a_k \geq 0, p \in N), \tag{2}$$

which are analytic and multivalent in the open unit disk U.
For the functions $f \in M_p$ given by (2) and $g \in M_p$ defined by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \qquad (b_k \geq 0, p \in N). \tag{3}$$

We define the convolution (or Hadamard product) of f and g by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k. \tag{4}$$

A function $f \in M_p$ is said to be p-valently starlike of order ρ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho, \qquad (0 \leq \rho < p; z \in U). \tag{5}$$

A function $f \in M_p$ is said to be p-valently convex of order ρ if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho, \qquad (0 \leq \rho < p; z \in U). \tag{6}$$

It follows from expression (5), (6) that f is convex if and only if $zf'(z)$ is starlike. A function $f \in M_p$ is close-to-convex of order ρ if

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \rho, \qquad (0 \leq \rho < p; z \in U). \tag{7}$$

Definition 1 [6] : Let $\gamma, \beta, m \in \mathcal{R}, \gamma \geq 0, \beta \geq 0, m \geq 0, p \in N$ and

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k.$$

Then, we define the linear operator

$$D_{p,m}^{\gamma,\beta} : \mathcal{A}_p \to \mathcal{A}_p \quad \text{by}$$

$$D_{p,m}^{\gamma,\beta} f(z) = z^p + \sum_{k=p+1}^{\infty} \left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m \quad a_k z^k, \text{‘} \quad z \in U. \tag{8}$$

Definition 2 : Let g be a fixed function defined by (3). The function $f \in M_p$ given by (2) is said to be in the class $R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$ if and only if

$$\left| \frac{z(D_{p,m}^{\gamma,\beta}(f*g)(z))' - p(D_{p,m}^{\gamma,\beta}(f*g)(z))}{\lambda z(D_{p,m}^{\gamma,\beta}(f*g)(z))' + (A+B)(D_{p,m}^{\gamma,\beta}(f*g)(z))} \right| < \alpha, \tag{9}$$

where

$0 < \lambda < 1, 0 < A < 1, 0 \leq B < 1, 0 < \alpha < 1, \ \gamma, \beta, m \in R, \ \gamma \geq 0, \ \beta \geq 0, \ m \geq 0, p \in N.$

Some of the following properties studied for other classes in [1], [2], [4] and [5].

2 Coefficient Inequalities

Theorem 1 : Let $f \in M_p$. Then $f \in R_p$ $(\gamma, \beta, m, \lambda, A, B, \alpha)$ if and only if

$$\sum_{k=p+1}^{\infty} \left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(\lambda k + (A+B))] a_k b_k \leq \alpha[p\lambda + (A+B)], \tag{10}$$

where

$0 < \lambda < 1, 0 < A < 1, 0 \leq B < 1, 0 < \alpha < 1, \ \gamma, \beta, m \in R, \ \gamma \geq 0, \ \beta \geq 0, \ m \geq 0, p \in N.$

The result is sharp for the function

$$f(z) = z^p + \frac{\alpha[p\lambda + (A+B)]}{\left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(\lambda k(A+B))] b_k} \ z^k. \tag{11}$$

Proof : Suppose that the inequality (10) holds true and $|z| = 1$. Then we have

$$\begin{aligned} &\left| \frac{z(D_{p,m}^{\gamma,\beta}(f*g)(z))' - p(D_{p,m}^{\gamma,\beta}(f*g)(z))}{\lambda z(D_{p,m}^{\gamma,\beta}(f*g)(z))' + (A+B)(D_{p,m}^{\gamma,\beta}(f*g)(z))} \right| \\ &= |z(D_{p,m}^{\gamma,\beta}(f*g)(z))' - p(D_{p,m}^{\gamma,\beta}(f*g)(z))| \\ &-\alpha|\lambda z(D_{p,m}^{\gamma,\beta}(f*g)(z))' + (A+B)(D_{p,m}^{\gamma,\beta}(f*g)(z))| \\ &= \left| \sum_{k=p+1}^{\infty} \left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m (k-p) a_k b_k z^k \right| \end{aligned}$$

$$-\alpha\left|\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m(\lambda k+(A+B))a_kb_kz^k+[p\lambda+(A+B)]z^p\right|$$
$$\leq \sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m(k-p)a_kb_k|z|^k$$
$$-\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m\alpha(\lambda k+(A+B)a_kb_k|z|^k-\alpha(p\lambda(p\lambda+(A+B))|z|^p$$
$$=\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m[(k-p)-\alpha(\lambda k+(A+B))]a_kb_k-\alpha(p\lambda+(A+B))\leq 0,$$

by hypothesis.
Hence, by maximum modulus principle, $f\in R_p(\gamma,\beta,m,\lambda,A,B,\alpha)$.
Conversely, suppose that $f\in R_p(\gamma,\beta,m,\lambda,A,B,\alpha)$. Then from (9), we have

$$\left|\frac{(D^{\gamma,\beta}_{p,m}(f*g)(z))'-p(D^{\gamma,\beta}_{p,m}(f*g)(z))}{\lambda z(D^{\gamma,\beta}_{p,m}(f*g)(z))'+(A+B)(D^{\gamma,\beta}_{p,m}(f*g)(z))}\right|$$
$$=\left|\frac{\sum\limits_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m(k-p)a_kb_kz^k}{\sum\limits_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m(\lambda k+(A+B))a_kb_kz^k+[p\lambda+(A+B)]z^p}\right|<\alpha.$$

Since $Re(z)\leq|z|$ for all z $(z\in U)$ we get

$$Re\left\{\frac{\sum\limits_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m(k-p)a_kb_kz^k}{\sum\limits_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m(\lambda k+(A+B))a_kb_kz^k+[p\lambda+(A+B)]z^p}\right\}<\alpha. \tag{12}$$

We choose the value of z on the real axis so that $(D^{\gamma,\beta}_{p,m}(f*g)(z))'$ is real. Letting $z\to 1^-$ through real values, we obtain inequality (10).
Finally, sharpness follows if we take

$$f(z)=z^p+\frac{\alpha[p\lambda+(A+B)]}{\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m[(k-p)-\alpha(\lambda k+(A+B))]b_k}\,z^k, \tag{13}$$

$k=p+1,p+2,\cdots$.
The proof is complete.
Corollary 1 : Let $f\in R_p(\gamma,\beta,m,\lambda,A,B,\alpha)$. Then

$$a_k\leq\frac{\alpha[p\lambda+(A+B)]}{\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m[(k-p)-\alpha(\lambda k+(A+B))]b_k},\ k=p+1,p+2,\cdots. \tag{14}$$

3 Closure Theorem

Theorem 2 : Let the functions f_s defined by

$$f_s(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,s} z^k, \qquad (a_{k,s} \geq 0, p \in N, s = 1, 2, \cdots, l),$$

be in the class $R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$ for every $s = 1, 2, \cdots, l$. Then the function h defined by

$$h(z) = z^p + \sum_{k=p+1}^{\infty} e_k z^k, \qquad (e_k \geq 0, p \in N),$$

also belongs to the class $R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$, where

$$e_k = \frac{1}{l} \sum_{s=1}^{l} a_{k,s}, \quad (k \geq p+1).$$

Proof : Since $f_s \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$, then by Theorem 1, we have

$$\sum_{k=p+1}^{\infty} \left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(\lambda k + (A+B))] a_{k,s} b_k \leq \alpha[p\lambda + (A+B)], \quad (15)$$

for every $s = 1, 2, \cdots, l$. Hence

$$\begin{aligned}
&\sum_{k=p+1}^{\infty} \left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(\lambda k + (A+B))] e_k b_k \\
&= \sum_{k=p+1}^{\infty} \left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(\lambda k + (A+B))] b_k \left(\frac{1}{l} \sum_{s=1}^{l} a_{k,s}\right) \\
&= \frac{1}{l} \sum_{s=1}^{l} \left(\sum_{k=p+1}^{\infty} \left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(\lambda k + (A+B))] a_{k,s} b_k\right) \\
&\leq \alpha[p\lambda + (A+B)].
\end{aligned}$$

By Theorem 1, it follows that $h \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$.

4 Neighborhoods and Partial Sums

We define the (n, δ)-neighborhood of a function $f \in M_p$ by

$$N_{n,\delta}(f) = \left\{ g \in M_p : g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \text{ and } \sum_{k=p+1}^{\infty} k|a_k - b_k| \leq \delta, \ 0 \leq \delta < 1 \right\}. \quad (16)$$

For the identity function $e(z) = z, \quad (p \in N)$

$$N_{n,\delta}(e) = \left\{ g \in M_p : g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \text{ and } \sum_{k=p+1}^{\infty} k|b_k| \leq \delta, \ 0 \leq \delta < 1 \right\}. \tag{17}$$

The concept of neighborhoods was first introduced by Goodman [3] and then generalized by Ruscheweyh [7].

Definition 3 : A function $f \in M_p$ is said to be in the class $R_p^\eta(\gamma, \beta, m, \lambda, A, B, \alpha)$ if there exists a function $g \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < p - \eta \quad (z \in U, ` \ 0 \leq \eta < 1).$$

Theorem 3 : If $g \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$ and

$$\eta = p - \frac{\delta \left(1 + \frac{\gamma}{p+\beta}\right)^m [1 - \alpha(\lambda(p+1) + (A+B))] a_{p+1}}{(p+1)\left(1 + \frac{\gamma}{p+\beta}\right)^m [1 - \alpha(\lambda(p+1) + (A+B))] a_{p+1} - \alpha[p\lambda + (A+B)]}. \tag{18}$$

Then $N_{n,\delta}(g) \subset R_p^\eta(\gamma, \beta, m, \lambda, A, B, \alpha)$.

Proof : Let $f \in N_{n,\delta}(g)$. We want to find from (16) that

$$\sum_{k=p+1}^{\infty} |a_k - b_k| \leq \delta,$$

which readily implies the following coefficient inequality

$$\sum_{k=p+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{p+1}.$$

Next, since $g \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$, we have from Theorem 1

$$\sum_{k=p+1}^{\infty} b_k \leq \frac{\alpha[p\lambda + (A+B)]}{\left(1 + \frac{\gamma}{p+\beta}\right)^m [1 - \alpha(\lambda(p+1) + (A+B))] a_{p+1}}.$$

So that

$$\left| \frac{f(z)}{g(z)} - 1 \right| \leq \frac{\sum_{k=p+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=p+1}^{\infty} b_k}$$

$$\leq \frac{\delta}{p+1} \cdot \frac{\left(1 + \frac{\gamma}{p+\beta}\right)^m [1 - \alpha(\lambda(p+1) + (A+B))] a_{p+1}}{\left(1 + \frac{\gamma}{p+\beta}\right)^m [1 - \alpha(\lambda(p+1) + (A+B))] a_{p+1} - \alpha[p\lambda + (A+B)]} = p - \eta \tag{19}$$

Thus by Definition 3, $f \in R_p^{\eta}(\gamma, \beta, m, \lambda, A, B, \alpha)$ for η given by (18). This completes the proof.
Now, we introduce the partial sums and the same property has been for other class in [8].
Theorem 4 : Let $f \in M_p$ be given by (2) and define the partial sums $s_1(z)$ and $s_q(z)$ by

$$s_1(z) = z^p$$

and

$$s_q(z) = z^p + \sum_{k=p+1}^{p+q-1} a_k z^k, \quad q > p+1. \tag{20}$$

Suppose also that

$$\sum_{k=p+1}^{\infty} d_k a_k \leq 1,$$

$$\left(d_k = \frac{\left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(\lambda k + (A+B))]b_k}{\alpha[p\lambda + (A+B)]} \right). \tag{21}$$

Thus, we have

$$Re\left\{\frac{f(z)}{s_q(z)}\right\} > 1 - \frac{1}{d_k} \tag{22}$$

and

$$Re\left\{\frac{s_q(z)}{f(z)}\right\} > 1 - \frac{d_k}{1+d_k} \tag{23}$$

Each of the bounds in (22) and (23) is the best possible for $p \in N$.
Proof : For the coefficients d_k given by (21), it is not difficult to verify that

$$d_{k+1} > d_k > 1, \quad k = p+1, p+2, \cdots.$$

Therefore, by using the hypothesis (20), we have

$$\sum_{k=p+1}^{p+q-1} a_k + d_k \sum_{k=p+q}^{\infty} a_k \leq \sum_{k=p+1}^{\infty} d_k a_k \leq 1. \tag{24}$$

By setting

$$g_1 = d_k \left(\frac{f(z)}{s_q(z)} - \left(1 - \frac{1}{d_k}\right)\right) = 1 + \frac{d_k \sum_{k=p+q}^{\infty} a_k z^{k-p}}{1 + \sum_{k=p+q}^{p+q-1} a_k z^{k-p}} \tag{25}$$

and applying (24) we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \le \frac{d_k \sum\limits_{k=p+q}^{\infty} a_k}{2-2\sum\limits_{k=p+1}^{p+q-1} a_k - d_k \sum\limits_{k=p+q}^{\infty} a_k} \le 1.$$

This prove (22). Therefore, $Re(g_1(z)) > 0$ and we obtain

$$Re\left\{\frac{f(z)}{s_q(z)}\right\} > 1 - \frac{1}{d_k}.$$

Now, in the same manner, we can prove the assertion (23), by setting

$$g_2(z) = (1+d_k)\left(\frac{s_q(z)}{f(z)} - \frac{d_k}{1+d_k}\right).$$

This completes the proof.

5 Radii of Starlikeness, Convexity and Close-to-Convexity

Using the inequalities (5), (6) and (7) and Theorem1, we can compute the radii of starlikeness, convexity and close-to-convexity.
Theorem 5 : If $f \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$, then $f(z)$ is p-valently starlike of order ρ $(0 \le \rho < p)$ in the disc $|z| < r_1$, where

$$r_1(\gamma,\beta,m,\lambda,A,B,\alpha,\rho) = \inf_k \left[\frac{(p-\rho)\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p)-\alpha(\lambda k+(A+B))]}{\alpha(k-\rho)[p\lambda+(A+B)]}\right]^{\frac{1}{k-p}}.$$

Proof : It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le p-\rho \quad (0 \le \rho < p),$$

for

$$|z| < r_1(\gamma,\beta,m,\lambda,A,B,\alpha,\rho),$$

we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le \frac{\sum\limits_{k=p+1}^{\infty} (k-p)a_k|z|^{k-p}}{1-\sum\limits_{k=p+1}^{\infty} a_k|z|^{k-p}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \rho,$$

if

$$\sum_{k=p+1}^{\infty} \frac{(k-\rho)a_k|z|^{k-p}}{(p-\rho)} \leq 1. \quad (26)$$

Hence, by Theorem 1, (26) will be true if

$$\frac{(k-\rho)a_k|z|^{k-p}}{(p-\rho)} \leq \frac{\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p)-\alpha(\lambda k+(A+B))]}{\alpha[p\lambda+(A+B)]},$$

and hence

$$|z| \leq \left[\frac{(p-\rho)\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p)-\alpha(\lambda k+(A+B))]}{\alpha(k-\rho)[p\lambda+(A+B)]}\right]^{\frac{1}{k-p}}.$$

Setting $|z| = r_1$, we get the desired result.
Theorem 6 : If $f(z) \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$. Then $f(z)$ is p-valently convex of order ρ $(0 \leq \rho < p)$ in the disc $|z| < r_2$, where

$$r_2(\gamma,\beta,m,\lambda,A,B,\alpha,\rho) = \inf_k \left[\frac{(p-\rho)\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p)-\alpha(\lambda k+(A+B))]}{\alpha k[p\lambda+(A+B)]}\right]^{\frac{1}{k-p}}.$$

Proof : It is sufficient to show that

$$\left|1+\frac{f''(z)}{f'(z)} - p\right| \leq p-\rho, \quad (0 \leq \rho < p),$$

for

$$|z| < r_2(\gamma,\beta,m,\lambda,A,B,\alpha,\rho),$$

we have

$$\left|1+\frac{zf''(z)}{f'(z)} - p\right| = \frac{\sum\limits_{k=p+1}^{\infty} k(k-p)a_k|z|^{k-p}}{1-\sum\limits_{k=p+1}^{\infty} ka_k|z|^{k-p}}.$$

Thus

$$\left|1+\frac{zf''(z)}{f'(z)} - p\right| \leq p-\rho,$$

if

$$\sum_{k=p+1}^{\infty} \frac{k(k-\rho)a_k|z|^{k-p}}{(p-\rho)} \leq 1. \quad (27)$$

Hence, by Theorem 1, (27) will be true if

$$\frac{k(k-\rho)a_k|z|^{k-p}}{(p-\rho)} \leq \frac{\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p)-\alpha(\lambda k+(A+B))]}{\alpha[p\lambda+(A+B)]},$$

and hence,

$$|z| \leq \left[\frac{(p-\rho)\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p)-\alpha(\lambda k+(A+B))]}{\alpha k(k-\rho)[p\lambda+(A+B)]}\right]^{\frac{1}{k-p}}.$$

Setting $|z| = r_2$, we get the desired result.

Theorem 7 : Let a function $f(z) \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$. Then $f(z)$ is p-valently close-to-convex of order ρ $(0 \leq \rho < p)$ in the disc $|z| < r_3$, where

$$r_3(\gamma, \beta, m, \lambda, A, B, \alpha, \rho) = \inf_k \left[\frac{(p-\rho)\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p)-\alpha(\lambda k+(A+B))]}{\alpha k[p\lambda+(A+B)]}\right]^{\frac{1}{k-p}}.$$

Proof : It is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p-\rho, \quad (0 \leq \rho < p),$$

for

$$|z| < r_3(\gamma, \beta, , m, \lambda, , A, B, \alpha, \rho),$$

we have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{k=p+1}^{\infty} k a_k |z|^{k-p}.$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p-\rho,$$

if

$$\sum_{k=p+1}^{\infty} \frac{k a_k |z|^{k-p}}{(p-\rho)} \leq 1. \qquad (28)$$

Hence, by Theorem 1, (28) will be true if

$$\frac{k|z|^{k-p}}{(p-\rho)} \leq \frac{\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p)-\alpha(\lambda k+(A+B))]}{\alpha[p\lambda+(A+B)]},$$

and hence

$$|z| \leq \left[\frac{(p-\rho)\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p)-\alpha(\lambda k+(A+B))]}{\alpha k[p\lambda+(A+B)]}\right]^{\frac{1}{k-p}}.$$

Setting $|z| = r_3$, we get the desired result.

6 Weighted Mean and Arithmetic Mean

Definition 4 : Let f_1 and f_2 be in the class $R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$. Then the weighted mean w_q of f_1 and f_2 is given by

$$w_q(z) = \frac{1}{2}[(1-q)f_1(z) + (1+q)f_2(z)], \quad 0 < q < 1.$$

Theorem 8 : Let f_1 and f_2 be in the class $R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$. Then the weighted mean w_q of f_1 and f_2 is also in the class $R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$.
Proof : By Definition 4, we have

$$\begin{aligned} w_q(z) &= \tfrac{1}{2}[(1-q)f_1(z) + (1+q)f_2(z)] \\ &= \tfrac{1}{2}\left[(1-q)\left(z^p + \sum_{k=p+1}^{\infty} a_{k,1}z^k\right) + (1+q)\left(z^p + \sum_{k=p+1}^{\infty} a_{k,2}z^k\right)\right] \\ &= z^p + \sum_{k=p+1}^{\infty} \tfrac{1}{2}[(1-q)a_{k,1} + (1+q)a_{k,2}]z^k. \end{aligned} \tag{29}$$

Since f_1 and f_2 are in the class $R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$ so by Theorem 1, we get

$$\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(\lambda k + (A+B))]a_{k,1}b_k \le \alpha[p\lambda + (A+B)].$$

And

$$\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(\lambda k + (A+B))]a_{k,2}b_k \le \alpha[p\lambda + (A+B)].$$

Hence

$$\begin{aligned} &\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(\lambda k + (A+B))] \times \\ &\left(\frac{1}{2}[(1-q)a_{k,1} + (1+q)a_{k,2}]\right) b_k z^k \\ &= \frac{1}{2}(1-q)\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(\lambda k + (A+B))]a_{k,1}b_k \\ &+\frac{1}{2}(1+q)\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(\lambda k + (A+B))]a_{k,2}b_k \\ &\le \frac{1}{2}(1-q)\alpha[p\lambda + (A+B)] + \frac{1}{2}(1+q)\alpha[p\lambda + (A+B)] = \alpha[p\lambda + (A+B)]. \end{aligned}$$

Therefore $w_q \in R_p(\gamma,\beta,m,\lambda,A,B,\alpha)$.
The proof is complete.
Theorem 9 : Let $f_1(z), f_2(z), \cdots, f_l(z)$ defined by

$$f_i(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,i} z^k, \quad (a_{k,i} \geq 0, i = 1,2,\cdots,l, \ k \geq p+1) \tag{30}$$

be in the class $R_p(\gamma,\beta,m,\lambda,A,B,\alpha)$. Then the arithmetic mean of $f_i(z)$ $(i = 1,2,\cdots,l)$ defined by

$$h(z) = \frac{1}{l}\sum_{i=1}^{l} f_i(z) \tag{31}$$

is also in the class $R_p(\gamma,\beta,m,\lambda,A,B,\alpha)$.
Proof : By (30), (31), we can write

$$\begin{aligned} h(z) &= \frac{1}{l}\sum_{i=1}^{l}\left(z^p + \sum_{k=p+1}^{\infty} a_{k,i} z^k\right) \\ &= z^p + \sum_{k=p+1}^{\infty}\left(\frac{1}{l}\sum_{i=1}^{l} a_{k,i}\right) z^k. \end{aligned}$$

Since $f_i \in R_p(\gamma,\beta,m,\lambda,A,B,\alpha)$ for every $(i = 1,2,\cdots,l)$ so by using Theorem 1, we prove that

$$\begin{aligned} &\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p)-\alpha(\lambda k+(A+B))]\left(\frac{1}{l}\sum_{i=1}^{l} a_{k,i}\right) b_k \\ = &\frac{1}{l}\sum_{i=1}^{l}\left(\sum_{k=p+1}^{\infty}\left(1+\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p)-\alpha(\lambda k+(A+B))] a_{k,i} b_k\right) \\ &\leq \frac{1}{l}\sum_{i=1}^{l}\alpha[p\lambda+(A+B)] \\ &= \alpha[p\lambda+(A+B)]. \end{aligned}$$

7 Linear Combination

In the theorem below, we prove a linear combination for the class $R_p(\gamma,\beta,m,\lambda,A,B,\alpha)$.
Theorem 10 : Let

$$f_i(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,i} z^k, \quad (a_{k,i} \geq 0, i = 1,2,\cdots,l, k \geq p+1)$$

belong to the class $R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$. Then

$$F(z) = \sum_{i=1}^{l} c_i f_i(z) \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha),$$

where

$$\sum_{i=1}^{l} c_i = 1.$$

Proof : By Theorem 1, we can write for every $i \in \{1, 2, \cdots, l\}$

$$\sum_{k=p+1}^{\infty} \frac{\left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(\lambda k + (A+B))]}{\alpha[p\lambda + (A+B)]} \ a_{k,i} b_k \leq 1.$$

Therefore

$$\begin{aligned} F(z) &= \sum_{i=1}^{l} c_i \left(z^p + \sum_{k=p+1}^{\infty} a_{k,i} z^k \right) \\ &= z^p + \sum_{k=p+1}^{\infty} \left(\sum_{i=1}^{l} c_i a_{k,i} \right) z^k. \end{aligned}$$

Hence

$$\sum_{k=p+1}^{\infty} \frac{\left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(\lambda k + (A+B))]}{\alpha[p\lambda + (A+B)]} \left(\sum_{i=1}^{l} c_i a_{k,i} \right) b_k$$

$$\sum_{i=1}^{l} c_i \left[\sum_{k=p+1}^{\infty} \frac{\left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(\lambda k + (A+B))]}{\alpha[p\lambda + (A+B)]} \ a_{k,i} b_k \right] \leq 1.$$

Then $F(z) \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$. So the proof is complete.

References

[1] W. G. Atshan and A. S. Joudah, *Subclass of meromorphic univalent functions defined by Hadamard product with multiplier transformation*, Int. Math. Forum, 6(46)(2011), 2279 . 2292.

[2] W. G. Atshan and S. R. Kulkarni, *On application of Differential subordination for certain subclass of meromorphically p-valent functions with positive coefficients defined by linear operator*, J. Ineq. Pure Appl. Math., 10(2)(2009), Article 53, 11 pages.

[3] A. W. Goodman, *Univalent functions and non-analytic curves*, Proc. Amer. Math. Soc., 8(3) (1957), 598-601.

[4] S. M. Khairnar and M. More, *Some analytic and multivalent functions defined by subordination property*, Gen. Math. 17(3)(2009), 105-124.

[5] H. Mahzoon, *New subclasses of multivalent functions defined by differential Subordination*, Appl. Math. Sciences, 6(95)(2012), 4701 . 4708.

[6] H. Mahzoon and S. Latha, *Neighborhoods of multivalent functions*, Int. Journal of Math. Analysis, 3(30)(2009), 1501-1507.

[7] S. Ruscheweyh, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc., 81(1981), 521-527.

[8] Ming -Sheng Liu and Nian -Sheng Song, *Two new subclasses of meromorphically multivalent functions associated with generalized hypergeometric function*, Southeast Asian Bulletin of Mathematics, 34 (2010), 705-727.

Received: January 5, 2013