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Application of Fractional Calculus on Certain Class of p-Valent Functions with Negative Coefficients Defined by Ruscheweyh Derivative

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Abstract

In the present paper, we apply the fractional calculus techniques for the subclass $WA(n, p, \beta, \gamma, \lambda)$. By using the definition of fractional derivative and integration, we obtain some theorems leading to distortion theorem and neighborhood properties. Also we get results about coefficient inequality and Hadamard product.

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1. Introduction

Let $S(n,p)$ be the class of functions $f(z)$ of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0, n, p \in I\!N = \{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic p-valent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $S^*WA(n,p,\beta)$ denote the subclass of $S(n,p)$ consisting of p-valent starlike functions of order $\beta, 0 \leq \beta < p$, if it also satisfies the inequality

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \quad z \in U. \tag{1.2}$$

And also, let $CWA(n,p,\beta)$ denote the subclass of $S(n,p)$ consisting of p-valent convex functions of order $\beta, 0 \leq \beta < p$, if it also satisfies the inequality

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta, \quad z \in U. \tag{1.3}$$

Then, we see that $f(z) \in CWA(n,p,\beta)$ if and only if $zf'(z) \in S^*WA(n,p,\beta)$.

Definition 1.1 [7] : The fractional integral of order δ $(0 < \delta)$ is defined by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(t)}{(z-t)^{1-\delta}} dt \tag{1.4}$$

where $f(z)$ is an analytic function in a simply connected region of z-plane containing the origin and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Definition 1.2 [7] : The fractional derivative of order δ $(0 \leq \delta < 1)$ is defined by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\delta}} dt \tag{1.5}$$

where $f(z)$ is as in Definition 1.1 and the multiplicity of $(z-t)^{-\delta}$ is removed like Definition 1.1.

Definition 1.3 [7] : [Under the conditions of Definition (1.2)] the fractional derivative of order $n+\delta$ $(n = 0, 1, 2, \cdots)$ is defined by

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z).$$

From Definitions 1.1 and 1.2 by applying a simple calculation we get

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(2+\delta)} z^{p+\delta} - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+p)}{\Gamma(k+p+\delta)} a_k z^{k+\delta}, \tag{1.6}$$

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(2-\delta)} z^{p-\delta} - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+p)}{\Gamma(k+p-\delta)} a_k z^{k-\delta}. \tag{1.7}$$

Also we note that the concept of δ-neighborhood $N_\delta(f)$ of analytic functions $f(z)$ was introduced by Ruscheweyh [5] and Goodman [2], but for meromorphic p-valent function studied by Liu and Srivastava [3].

Definition 1.4 [4], [7] : The Ruscheweyh derivative of order $\lambda + p - 1$ is denoted by $D^{\lambda+p-1} f$ and defined as following: If

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k,$$

then

$$D^{\lambda+p-1} f(z) = \frac{z^p}{(1-z)^{p+\lambda}} * f(z) = z^p - \sum_{k=n+p}^{\infty} \frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!} a_k z^k, \lambda > -p. \tag{1.8}$$

In particular when $p = 1$ we have $D^\lambda f(z) = z - \sum_{k=n+1}^{\infty} B_k(\lambda) a_k z^k$, where

$$B_k(\lambda) = \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+k-1)}{(k-1)!}, \quad \lambda > -1, \quad z \in U. \tag{1.9}$$

Definition 1.5 : A function $f(z) \in S(n,p), z \in U$ is said to be in the class $WA(n, p, \beta, \gamma, \lambda)$ if and only if it satisfies the inequality

$$Re\left\{\frac{D^{\lambda+p-1} f(z)}{(1-\gamma) z^2 (D^{\lambda+p-1} f(z))'' + \gamma z (D^{\lambda+p-1} f(z))' + D^{\lambda+p-1} f(z)}\right\} > \beta, \quad z \in U \tag{1.10}$$

where $0 \leq \gamma < \frac{1}{2}, 0 \leq \beta < \frac{1}{(1-\gamma)p(p-1)+\gamma p+1}, \lambda > -p, p \in I\!N$.

2. Main Results

In the following theorem, we derive the coefficient inequality for the class $WA(n,p,\beta,\gamma,\lambda)$.

Theorem 2.1 : Let $f(z) \in S(n,p)$. Then $f(z)$ is in the class $WA(n,p,\beta,\gamma,\lambda)$ if and only if

$$\sum_{k=n+p}^{\infty}(1-\beta(k(k-1)(1-\gamma)+\gamma k+1))\frac{\Gamma(\lambda+k)a_k}{\Gamma(\lambda+p)(k-p)!} < 1-\beta((1-\gamma)p(p-1)+\gamma p+1) \tag{2.1}$$

where $0 \leq \beta < \frac{1}{(1-\gamma)p(p-1)+\gamma p+1}, 0 \leq \gamma < \frac{1}{2}, \lambda > -p, a_k \geq 0, z \in U$ and $p, n \in I\!N = \{1, 2, 3, \cdots\}$.

The result (2.1) is sharp for the function $f(z)$ given by the following form

$$f(z) = z^p - \frac{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(\lambda+n+p)}z^{n+p} \tag{2.2}$$

for all β, γ, λ and p defined in Definition 1.5.

Proof : Assume $f(z) \in S(n,p)$. By Definition 1.5, $f(z) \in WA(n,p,\beta,\gamma,\lambda)$ if and only if $f(z)$ satisfies the inequality

$$Re\left\{\frac{D^{\lambda+p-1}f(z)}{(1-\gamma)z^2(D^{\lambda+p-1}f(z))'' + \gamma z(D^{\lambda+p-1}f(z))' + D^{\lambda+p-1}f(z)}\right\} > \beta, \ z \in U$$

where $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, 0 \leq \gamma < \frac{1}{2}, p \in I\!N, \lambda > -p, a_k \geq 0$ and $0 \leq \beta < \frac{1}{(1-\gamma)p(p-1)+\gamma p+1}$.

Thus (1.10) is equivalent to

$$|(1-\beta)D^{\lambda+p-1}f(z) - (1-\gamma)z^2\beta(D^{\lambda+p-1}f(z))'' - \gamma z\beta(D^{\lambda+p-1}f(z))'| > 0 \tag{2.3}$$

Hence

$$\Bigg|(1-\beta)\left(z^p-\sum_{k=n+p}^{\infty}\frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!}a_kz^k\right)-(1-\gamma)\beta$$
$$\times\left(p(p-1)z^p-\sum_{k=n+p}^{\infty}k(k-1)\frac{\Gamma(\lambda+k)a_kz^k}{\Gamma(\lambda+p)(k-p)!}\right)$$
$$-\gamma\beta\left(pz^p-\sum_{k=n+p}^{\infty}k\frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!}a_kz^k\right)\Bigg|>0$$

and hence

$$|(1-\beta((1-\gamma)p(p-1)+\gamma p+1))z^p$$
$$-\sum_{k=n+p}^{\infty}(1-\beta(k(k-1)(1-\gamma)+\gamma k+1))\frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!}a_kz^k\Bigg|>0.$$

Hence, we get

$$(1-\beta((1-\gamma)p(p-1)+\gamma p+1))|z|^p-$$
$$\sum_{k=n+p}^{\infty}(1-\beta(k(k-1)(1-\gamma)+\gamma k+1))\frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!}a_k|z|^k>0. \qquad (2.4)$$

Now, we letting $z\to 1^-$ through real values in (2.4), we obtain (2.1). Hence the result follows.

Conversely, let (2.1) hold. We will show that (1.10) is correct and then $f\in WA(n,p,\beta,\gamma,\lambda)$. By lemma (Re $w>\beta$ if and only if $|w-(1+\beta)|<|w+(1-\beta)|$) it is enough to show that $|w-(1+\beta)|<|w+(1-\beta)|$ where

$$w=\frac{D^{\lambda+p-1}f(z)}{(1-\gamma)z^2(D^{\lambda+p-1}f(z))''+\gamma z(D^{\lambda+p-1}f(z))'+D^{\lambda+p-1}f(z)}$$

or we show that

$$\begin{aligned}F\ &=\ \frac{1}{|\theta(z)|}|D^{\lambda+p-1}f(z)-(1+\beta)(1-\gamma)z^2(D^{\lambda+p-1}f(z))''-(1+\beta)\gamma z(D^{\lambda+p-1}f(z))'\\&\quad-(1+\beta)D^{\lambda+p-1}f(z)|<\frac{1}{|\theta(z)|}|D^{\lambda+p-1}f(z)+(1-\beta)(1-\gamma)z^2(D^{\lambda+p-1}f(z))''\\&\quad+(1-\beta)\gamma z(D^{\lambda+p-1}f(z))'+(1-\beta)D^{\lambda+p-1}f(z)|=G\end{aligned}$$

where $\theta(z)=(1-\gamma)z^2(D^{\lambda+p-1}f(z))''+\gamma z(D^{\lambda+p-1}f(z))'+D^{\lambda+p-1}f(z)$ and it is easy to verify that $G-F>0$ and so the proof is complete.

Corollary 2.2 : If $f(z) \in WA(n,p,\beta,\gamma,\lambda)$, then

$$a_k \leq \frac{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)(k-p)!}{(1-\beta(k(k-1)(1-\gamma)+\gamma k+1))\Gamma(\lambda+k)}, \quad k \geq n+p$$

where $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, a_k \geq 0, z \in U, n,p \in I\!N$.

The above mentioned corollary is entirely new and not found in the literature.

Theorem 2.3 : Let $f(z) \in WA(n,p,\beta,\gamma,\lambda)$, then

$$|D_z^{-\delta} f(z)| \leq \frac{|z|^{p+\delta}}{\Gamma(2+\delta)} \times \left[1 + \frac{\Gamma(n+2p)\Gamma(2+\delta)(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(n+2p+\delta)\Gamma(\lambda+n+p)}|z|^n\right] \quad (2.5)$$

$$|D_z^{-\delta} f(z)| \geq \frac{|z|^{p+\delta}}{\Gamma(2+\delta)} \times \left[1 - \frac{\Gamma(n+2p)\Gamma(2+\delta)(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(n+2p+\delta)\Gamma(\lambda+n+p)}|z|^n\right] \quad (2.6)$$

The result is sharp for the function $f(z)$ given by (2.2).

Proof : By (1.6), we have

$$\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z) = z^p - \sum_{k=n+p}^{\infty} \phi(k,\delta) a_k z^k,$$

such that

$$\phi(k,\delta) = \frac{\Gamma(k+p)\Gamma(2+\delta)}{\Gamma(k+p+\delta)}, \quad k \geq n+p.$$

Since for $k \geq n+p, \phi(k,\delta)$ is a decreasing function of k and $\frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!}$ is an increasing function of k, thus, we have $\phi(k,\delta) \leq \phi(n+p,\delta)$, then $\phi(k,\delta) \leq \frac{\Gamma(n+2p)\Gamma(2+\delta)}{\Gamma(n+2p+\delta)}$ and $\frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!} \geq \frac{\Gamma(n+p+\lambda)}{\Gamma(\lambda+p)n!}$. So we conclude that

$$|\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z)| \leq |z|^p + \frac{\Gamma(n+2p)\Gamma(2+\delta)}{\Gamma(n+2p+\delta)}|z|^{n+p} \sum_{k=n+p}^{\infty} a_k$$

$$\leq |z|^p + \frac{\Gamma(n+2p)\Gamma(2+\delta)(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!|z|^{n+p}}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(n+2p+\delta)\Gamma(\lambda+n+p)}.$$

Then

$$|D_z^{-\delta} f(z)| \leq \frac{|z|^{p+\delta}}{\Gamma(2+\delta)} \times \left[1 + \frac{\Gamma(n+2p)\Gamma(2+\delta)(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(n+2p+\delta)\Gamma(\lambda+n+p)}|z|^n\right].$$

Also we have

$$|\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)| \geq |z|^p - \frac{\Gamma(n+2p)\Gamma(2+\delta)}{\Gamma(n+2p+\delta)}|z|^{n+p}\sum_{k=n+p}^{\infty} a_k$$

$$\geq |z|^p - \frac{\Gamma(n+2p)\Gamma(2+\delta)(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(n+2p+\delta)\Gamma(\lambda+n+p)}|z|^{n+p}.$$

Then

$$|D_z^{-\delta}f(z)| \geq \frac{|z|^{p+\delta}}{\Gamma(2+\delta)}$$
$$\times\left[1 - \frac{\Gamma(n+2p)\Gamma(2+\delta)((1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(n+2p+\delta)\Gamma(\lambda+n+p)}|z|^{n}\right].$$

Theorem 2.4 : Let $f(z) \in WA(n,p,\beta,\gamma,\lambda)$, then

$$|D_z^{\delta}f(z)| \leq \frac{|z|^{p-\delta}}{\Gamma(2-\delta)}$$
$$\times\left[1 + \frac{\Gamma(n+2p)\Gamma(2-\delta)(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(n+2p-\delta)\Gamma(\lambda+n+p)}|z|^{n}\right] \quad (2.7)$$

$$|D_z^{\delta}f(z)| \geq \frac{|z|^{p-\delta}}{\Gamma(2-\delta)}$$
$$\times\left[1 - \frac{\Gamma(n+2p)\Gamma(2-\delta)(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(n+2p-\delta)\Gamma(\lambda+n+p)}|z|^{n}\right]. \quad (2.8)$$

The inequalities (2.7) and (2.8) are sharp for the function $f(z)$ given by (2.2).

Proof : By (1.7), we have

$$\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z) = z^p - \sum_{k=n+p}^{\infty}\psi(k,\delta)a_k z^k,$$

such that

$$\psi(k,\delta) = \frac{\Gamma(k+p)\Gamma(2-\delta)}{\Gamma(k+p-\delta)}, \quad k \geq n+p.$$

For $k \geq n+p, \psi(k,\delta)$ is a decreasing function of k, then

$$\psi(k,\delta) \leq \psi(n+p,\delta) = \frac{\Gamma(n+2p)\Gamma(2-\delta)}{\Gamma(n+2p-\delta)}.$$

Also

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(\lambda+n+p)},$$

thus

$$|\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)| \leq |z|^p + \frac{\Gamma(n+2p)\Gamma(2-\delta)}{\Gamma(n+2p-\delta)}|z|^{n+p}\sum_{k=n+p}^{\infty} a_k$$

$$\leq |z|^p + \frac{\Gamma(n+2p)\Gamma(2-\delta)(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(n+2p-\delta)\Gamma(\lambda+n+p)}|z|^{n+p}.$$

Then

$$|D_z^{\delta}f(z)| \leq \frac{|z|^{p-\delta}}{\Gamma(2-\delta)}$$

$$\times\left[1+\frac{\Gamma(n+2p)\Gamma(2-\delta)(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(n+2p-\delta)\Gamma(\lambda+n+p)}|z|^n\right]$$

and by the same way we obtain

$$|D_z^{\delta}f(z)| \geq \frac{|z|^{p-\delta}}{\Gamma(2-\delta)}$$

$$\times\left[1-\frac{\Gamma(n+2p)\Gamma(2-\delta)(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(n+2p-\delta)\Gamma(\lambda+n+p)}|z|^n\right].$$

Corollary 2.5 : For every $f \in WA(n,p,\beta,\gamma,\lambda)$ we have

(i)
$$\frac{|z|^{p+1}}{2}\left[1-\frac{2(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!|z|^n}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))(n+2p)\Gamma(\lambda+n+p)}\right]$$

$$\leq |\int_0^z f(t)dt| \leq \frac{|z|^{p+1}}{2}$$

$$\times\left[1+\frac{2(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!|z|^n}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))(n+2p)\Gamma(\lambda+n+p)}\right]$$

(ii)
$$|z|^p\left[1-\frac{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(\lambda+n+p)}|z|^n\right] \leq |f(z)|$$

$$\leq |z|^p\left[1+\frac{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(\lambda+n+p)}|z|^n\right].$$

Proof : (i) By Definition 1.1 and Theorem 2.3 for $\delta = 1$, we have $D_z^{-1}f(z) = \int_0^z f(t)dt$, the result is true.

(ii) By Definition 1.2 and Theorem 2.4 for $\delta = 0$, we have $D_z^0 f(z) = \frac{d}{dz}\int_0^z f(t)dt = f(z)$, the result is true.

Corollary 2.6 : $D_z^{-\delta}f(z)$ and $D_z^{\delta}f(z)$ are included in the disks with center at the origin and radii

$$\frac{1}{\Gamma(2+\delta)}\left[1+\frac{\Gamma(n+2p)\Gamma(2+\delta)(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(n+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(n+2p+\delta)\Gamma(\lambda+n+p)}\right],$$

$$\frac{1}{\Gamma(2-\delta)}\left[1+\frac{\Gamma(n+2p)\Gamma(2-\delta)(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(n+p)n!}{(1-\beta((n+p)(n+p-1)(1-\gamma)+\gamma(n+p)+1))\Gamma(n+2p-\delta)\Gamma(\lambda+n+p)}\right],$$

respectively.

The above mentioned corollaries are entirely new and not found in the literature.

Theorem 2.7 : If $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k$ belong to $WA(n, p, \beta, \gamma, \lambda)$, then the Hadamard product of f and g given by $(f * g)(z) = z^p - \sum_{k=n+p}^{\infty} a_k b_k z^k$ belongs to $WA(n, p, \beta, \gamma, \lambda_1)$ where

$$\lambda_1 < \inf_k \left\{ \frac{[1-\beta(k(k-1)(1-\gamma)+\gamma k+1)] \left[\frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!}\right]^2}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))} - p \right\}.$$

Proof : Since f and $g \in WA(n, p, \beta, \gamma, \lambda)$, we have

$$\sum_{k=n+p}^{\infty} \left[\frac{[1-\beta(k(k-1)(1-\gamma)+\gamma k+1)]\Gamma(\lambda+k)}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)(k-p)!}\right] a_k < 1$$

and

$$\sum_{k=n+p}^{\infty} \left[\frac{[1-\beta(k(k-1)(1-\gamma)+\gamma k+1)]\Gamma(\lambda+k)}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)(k-p)!}\right] b_k < 1$$

and by applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\sum_{k=n+p}^{\infty} \left[\frac{[1-\beta(k(k-1)(1-\gamma)+\gamma k+1)]\Gamma(\lambda+k)}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)(k-p)!}\right] \sqrt{a_k b_k} \\ &\leq \left(\sum_{k=n+p}^{\infty} \left[\frac{[1-\beta(k(k-1)(1-\gamma)+\gamma k+1)]\Gamma(\lambda+k)}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)(k-p)!}\right] a_k\right)^{1/2} \\ &\times \left(\sum_{k=n+p}^{\infty} \left[\frac{[1-\beta(k(k-1)(1-\gamma)+\gamma k+1)]\Gamma(\lambda+k)}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)(k-p)!}\right] b_k\right)^{1/2}. \end{aligned}$$

However, we obtain

$$\sum_{k=n+p}^{\infty} \left[\frac{[1-\beta(k(k-1)(1-\gamma)+\gamma k+1)]\Gamma(\lambda+k)}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)(k-p)!}\right] \sqrt{a_k b_k} < 1. \quad (2.9)$$

Now we want to prove

$$\sum_{k=n+p}^{\infty} \left[\frac{[1-\beta(k(k-1)(1-\gamma)+\gamma k+1)]\Gamma(\lambda_1+k)}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda_1+p)(k-p)!}\right] a_k b_k < 1. \quad (2.10)$$

Let (2.10) holds true, then we have

$$\sum_{k=n+p}^{\infty}\left[\frac{[1-\beta(k(k-1)(1-\gamma)+\gamma k+1)]}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))}\right]\sqrt{a_k b_k}\frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!}\sqrt{a_k b_k}\frac{\frac{\Gamma(\lambda_1+k)}{\Gamma(\lambda_1+p)(k-p)!}}{\frac{\Gamma(\lambda+p)}{\Gamma(\lambda+p)(k-p)!}}\leq 1. \tag{2.11}$$

Therefore (2.11) (consequently (2.10)) holds true if

$$\sqrt{a_k b_k}<\frac{\frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!}}{\frac{\Gamma(\lambda_1+k)}{\Gamma(\lambda_1+p)(k-p)!}} \tag{2.12}$$

But from (2.9) we conclude that

$$\sqrt{a_k b_k}<\frac{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)(k-p)!}{[1-\beta(k(k-1)(1-\gamma)+\gamma k+1)]\Gamma(\lambda+k)}. \tag{2.13}$$

In view of (2.13) the inequality (2.12) holds true if

$$\frac{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))\Gamma(\lambda+p)(k-p)!}{[1-\beta(k(k-1)(1-\gamma)+\gamma k+1)]\Gamma(\lambda+k)}<\frac{\frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!}}{\frac{\Gamma(\lambda_1+k)}{\Gamma(\lambda_1+p)(k-p)!}}$$

$$\frac{\Gamma(\lambda_1+k)}{\Gamma(\lambda_1+p)(k-p)!}<\frac{[1-\beta(k(k-1)(1-\gamma)+\gamma k+1)]\left[\frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!}\right]^2}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))}.$$

But we have

$$\lambda_1+p\leq\frac{\Gamma(\lambda_1+n+p)}{\Gamma(\lambda_1+p)(k-p)!}\leq\frac{\Gamma(\lambda_1+k)}{\Gamma(\lambda_1+p)(k-p)}.$$

Hence

$$\lambda_1+p<\frac{[1-\beta(k(k-1)(1-\gamma)+\gamma k+1)]\left[\frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!}\right]^2}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))}$$

and this inequality gives the required result.

Therefore, with respect to Theorem 2.3 and 2.4 and Corollaries 2.5 and 2.6 we can obtain the similar theorems and corollaries by replacing λ with λ_1.

3. Neighborhoods

We deal the neighborhood concepts for functions in the class $WA(n,p,\beta,\gamma,\lambda)$. This concept has been investigated by several authors e.g. O. Altintas and S.

Owa [1] studied neighborhoods of analytic functions with negative coefficients. Also J. Lin Liu and H. M. Srivastava [3], (see also [7]) extended the concept of neighborhoods for a subclass of meromorphically p-valent functions and finally St. Ruscheweyh [5] considered this concept for subclasses of analytic functions.

Definition 3.1 : Let

$$0 \leq \gamma < \frac{1}{2}, 0 \leq \beta < \frac{1}{(1-\gamma)p(p-1)+\gamma p+1}, \lambda > -p, \delta \geq 0,$$

we define the δ-neighborhood of a function $f \in S(n,p)$ and denote by $N_\delta(f)$ consisting of all functions $g(z) = z^p - \sum\limits_{k=n+p}^{\infty} a_k z^k \in S(n,p)$ satisfying

$$\sum_{k=n+p}^{\infty} \left[\frac{(1-\beta(k(k-1)(1-\gamma)+\gamma k+1))}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))}\right] \frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!}|a_k - b_k| \leq \delta. \tag{3.1}$$

Lemma 3.1 : Let $f \in WA(n,p,\beta,\gamma,\lambda)$. Then for every complex number μ with $|\mu| < \delta$ $(0 \leq \delta)$, we have

$$\frac{f(z)+\mu z^p}{1+\mu} \in WA(n,p,\beta,\gamma,\lambda).$$

Proof :

$$\begin{aligned} \frac{f(z)+\mu z^p}{1+\mu} &= \frac{z^p(1+\mu) - \sum\limits_{k=n+p}^{\infty} \frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!} a_k z^k}{1+\mu} \\ &= z^p - \sum_{k=n+p}^{\infty} \frac{\frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!} a_k}{1+\mu} z^k. \end{aligned}$$

By (2.1) we have

$$\begin{aligned} &\sum_{k=n+p}^{\infty} \frac{a_k}{1+\mu} \frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!} \left[\frac{(1-\beta(k(k-1)(1-\gamma)+\gamma k+1))}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))}\right] \\ &= \frac{1}{1+\mu} \sum_{k=n+p}^{\infty} a_k \frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!} \left[\frac{(1-\beta(k(k-1)(1-\gamma)+\gamma k+1))}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))}\right] < \frac{1}{1-\mu} < 1. \end{aligned}$$

this completes the proof of the Lemma.

Theorem 3.1 : Let $f \in WA(n,p,\beta,\gamma,\lambda)$. If for every complex number μ with $|\mu| < \delta$ $(0 \leq \delta)$ we have $\frac{f(z)+\mu z^p}{1+\mu} \in WA(n,p,\beta,\gamma,\lambda)$. Then $N_\delta(f) \subset WA(n,p,\beta,\gamma,\lambda)$.

Proof : We have $f \in WA(n,p,\beta,\gamma,\lambda)$ if and only if
$\frac{(f*\psi)(z)}{z^p} \neq 0, \quad z \in U - \{0\}$ where $\psi(z) = z^p - \sum\limits_{k=n+p}^{\infty} e_k z^k$ and

$$|e_k| \leq \frac{(1-\beta(k(k-1)(1-\gamma)+\gamma k+1))}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))} \frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!},$$

since $\frac{f(z)+\mu z^p}{1+\mu} \in WA(n,p,\beta,\gamma,\lambda)$. So $z^{-p}\left(\frac{f(z)+\mu z^p}{1+\mu} * \psi(z)\right) \neq 0$. Therefore

$$z^{-p}\left(\frac{f(z)*\psi(z)}{1+\mu} + \frac{\mu z^p}{1+\mu} * \psi(z)\right) = \frac{f(z)*\psi(z)}{(1+\mu)z^p} + \frac{\mu}{1+\mu} \neq 0.$$

Assume $\left|\frac{(f*\psi)(z)}{z^p}\right| > \delta$, then by above we must have

$$\left|\frac{(f*\psi)(z)}{z^p(1+\mu)} + \frac{\mu}{1+\mu}\right| \geq \frac{|\mu|}{|1+\mu|} - \frac{1}{|1+\mu|}\left|\frac{(f*\psi)(z)}{z^p}\right| > \frac{|\mu|-\delta}{|1+\mu|} \geq 0$$

which is a contradiction by $|\mu| < \delta$. However, we have $\left|\frac{(f*\psi)(z)}{z^p}\right| < \delta$. If $g(z) = z^p - \sum\limits_{k=n+p}^{\infty} b_k z^k \in N_\delta(f)$, then

$$\delta - \left|\frac{(g*\psi)(z)}{z^p}\right| \leq \left|\frac{((f-g)*\psi)(z)}{z^p}\right| \leq \sum_{k=n+p}^{\infty} |a_k - b_k||e_k||z^k|$$

$$\leq \sum_{k=n+p}^{\infty}\left[\frac{(1-\beta(k(k-1)(1-\gamma)+\gamma k+1))}{(1-\beta((1-\gamma)p(p-1)+\gamma p+1))}\right]|a_k - b_k|\frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)(k-p)!} < \delta.$$

So we conclude $\frac{(g*\psi)(z)}{z^p} \neq 0$ and finally $g \in WA(n,p,\beta,\gamma,\lambda)$.

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