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New Classes of Multivalently Harmonic Functions

Waggas Galib Atshan

Department of Mathematics
University of Pune, Pune-411007, India
waggashnd@yahoo.com

S. R. Kulkarni

Department of Mathematics
Fergusson College, Pune-411004, India
kulkarni_ferg@yahoo.com

Abstract

New classes of multivalently harmonic functions are introduced. We give sufficient coefficient bounds for $f(z) \in \mathcal{H}W_{p,k}(\lambda, t, \alpha)$ and then we show that these sufficient coefficient conditions are also necessary for $f(z) \in \overline{\mathcal{H}W_{p,k}}(\lambda, t, \alpha)$. Furthermore, we determine extreme points, convex combination, distortion bounds and integral operator for these functions. Also we get new results in this paper.

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1. Introduction

Let u, v be real harmonic functions in the simply connected domain Ω, then the continuous function $f = u + iv$ defined in Ω is said to be harmonic in Ω. In any simply connected domain $\Omega \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where h and g are analytic in Ω. A necessary and sufficient condition for f to be locally univalent and sense-preserving in Ω is that $|h'(z)| > |g'(z)|$ in Ω (see Clunie and Sheil-Small [3]). Let $\mathcal{H}$ be the family of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}$ so that $f = h + \overline{g}$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Let $\mathcal{H}W_{p,k}$ for fixed p

and k $(p, k \in Z^{+})$ be the set of all harmonic multivalent and sense-preserving functions in U [1] of the form $f(z) = h(z) + \overline{g(z)}$, where

$$h(z) = z^p + \sum_{m=k+p}^{\infty} a_m z^m, \quad g(z) = \sum_{m=k+p-1}^{\infty} b_m z^m, \quad |b_{k+p-1}| < 1 \tag{1}$$

are analytic in U.

Let $\overline{\mathcal{H}W_{p,k}}$ be the subclass of $\mathcal{H}W_{p,k}$ consisting of functions of the form $f(z) = h(z) + \overline{g(z)}$, where

$$h(z) = z^p - \sum_{m=k+p}^{\infty} |a_m| z^m, \quad g(z) = \sum_{m=k+p-1}^{\infty} |b_m| z^m, \quad |b_{k+p-1}| < 1. \tag{2}$$

Definition 1 Let $f(z) = h(z) + \overline{g(z)}$, be the harmonic multivalent function of the form (1), then $f \in \mathcal{H}W_{p,k}(\lambda, t, \alpha)$ if and only if

$$Re\left\{(1-\lambda)(1-t)\frac{f(z)}{z^p} + (\lambda+t)\frac{f'(z)}{(z^p)'} + \lambda t\frac{f''(z)}{(z^p)''} - 2\lambda t\right\} \geq \frac{\alpha}{p} \tag{3}$$

where $0 \leq \alpha < p, \lambda \geq 0, 0 \leq t \leq 1$ and $z = re^{i\theta} \in U$,

$$(z^p)' = \frac{\partial}{\partial\theta}(z^p) = ipz^p, f'(z) = \frac{\partial}{\partial\theta}(f(re^{i\theta})) = izh' - \overline{izg'}, \tag{4}$$

$$(z^p)'' = \frac{\partial^2}{\partial\theta^2}(z^p) = -p^2 z^p, f''(z) = \frac{\partial^2}{\partial\theta^2}(f(re^{i\theta})) = -zh' - z^2h'' - \overline{zg'} - \overline{z^2g''}.$$

We also let $\overline{\mathcal{H}W_{p,k}}(\lambda, t, \alpha) = \mathcal{H}W_{p,k}(\lambda, t, \alpha) \cap \overline{\mathcal{H}W_{p,k}}$.

As λ changes from 0 to 1, the family $\mathcal{H}W_{p,k}(\lambda, t, \alpha)$ produces a passage from the class of harmonic functions $\mathcal{H}G_{p,k}(t, \alpha) \equiv \mathcal{H}W_{p,k}(0, t, \alpha)$ consisting of function f where

$$Re\left\{(1-t)\frac{f(z)}{z^p} + t\frac{f'(z)}{(z^p)'}\right\} \geq \frac{\alpha}{p}, \tag{5}$$

to the class of harmonic functions $\mathcal{H}A_{p,k}(t, \alpha) \equiv \mathcal{H}W_{p,k}(1, t, \alpha)$ consisting of functions f where

$$Re\left\{(1+t)\frac{f'(z)}{(z^p)'} + t\frac{f''(z)}{(z^p)''} - 2t\right\} \geq \frac{\alpha}{p}. \tag{6}$$

2. Coefficient Bounds

First we give the sufficient coefficient bounds for $f(z) \in \mathcal{H}W_{p,k}(\lambda, t, \alpha)$ and then show these sufficient coefficient conditions are also necessary for $f(z) \in \overline{\mathcal{H}W_{p,k}}(\lambda, t, \alpha)$.

Theorem 1 : Assume $f = h + \overline{g}, h$ and g be given by (1) and

$$\sum_{m=k+p}^{\infty}\left|(\lambda+t)m + p(1-\lambda-t+\lambda t) + \frac{m^2\lambda t}{p}\right| |a_m| + \sum_{m=k+p-1}^{\infty}\left|(\lambda+t)m - p(1-\lambda-t+\lambda t) - \frac{m^2\lambda t}{p}\right| |b_m| \leq p-\alpha. \tag{7}$$

Then $f(z) \in \mathcal{H}W_{p,k}(\lambda, t, \alpha)$.

Proof : Using the fact that Re $w \geq \alpha$ if and only if $|w+1-\alpha| \geq |w-1-\alpha|$ or equivalently $Re\ w \geq \frac{\alpha}{p}$ if and only if $|wp+p-\alpha| \geq |wp-p-\alpha|$, and letting

$$w = (1-\lambda)(1-t)\frac{f(z)}{z^p} + (\lambda+t)\frac{f'(z)}{(z^p)'} + \lambda t\frac{f''(z)}{(z^p)''} - 2\lambda t$$

it is enough to show that $|pw+p-\alpha| - |pw-p-\alpha| \geq 0$.

Now, we have

$$\begin{aligned} |pw+p-\alpha| \;=\; & \Bigg| p(1-\lambda)(1-t)\left(1 + \sum_{m=k+p}^{\infty} a_m z^{m-p} + \sum_{m=k+p-1}^{\infty} b_m(\overline{z})^{m-p}\right) \\ & + p(\lambda+t)\left(1 + \sum_{m=k+p}^{\infty}\frac{m}{p}a_m z^{m-p} - \sum_{m=k+p-1}^{\infty}\frac{m}{p}b_m(\overline{z})^{m-p}\right) \\ & + p\lambda t\Bigg(\frac{1}{p} + \sum_{m=k+p}^{\infty}\frac{m}{p^2}a_m z^{m-p} + \frac{p-1}{p} + \sum_{m=k+p}^{\infty}\frac{m(m-1)}{p^2}a_m z^{m-p} \\ & + \sum_{m=k+p-1}^{\infty}\frac{m}{p^2}b_m(\overline{z})^{m-p} + \sum_{m=k+p-1}^{\infty}\frac{m(m-1)}{p^2}b_m(\overline{z})^{m-p}\Bigg) - 2p\lambda t + p - \alpha \Bigg| \\ & \geq 2p - \alpha - \sum_{m=k+p}^{\infty}\left|p + (\lambda+t)(m-p) + p\lambda t + \frac{m^2\lambda t}{p}\right| |a_m| \left|\frac{z^m}{z^p}\right| \\ & - \sum_{m=k+p-1}^{\infty}\left|p - (\lambda+t)(m+p) + p\lambda t + \frac{m^2\lambda t}{p}\right| |b_m| \left|\frac{z^m}{z^p}\right| \quad \text{and} \end{aligned}$$

$$\begin{aligned} |pw-p-\alpha| \;\leq\; & \alpha + \sum_{m=k+p}^{\infty}\left|p + (\lambda+t)(m-p) + p\lambda t + \frac{m^2\lambda t}{p}\right| |a_m| \left|\frac{z^m}{z^p}\right| \\ & + \sum_{m=k+p-1}^{\infty}\left|p - (\lambda+t)(m+p) + p\lambda t + \frac{m^2\lambda t}{p}\right| |b_m| \left|\frac{z^m}{z^p}\right|. \end{aligned}$$

So by using (7) we have

$$\begin{aligned}
&|pw+p-\alpha|-|pw-p-\alpha| \\
&\geq 2\left[p-\alpha-\sum_{m=k+p}^{\infty}\left|(\lambda+t)m+p(1-t-\lambda+\lambda t)+\frac{m^2\lambda t}{p}\right||a_m|\right. \\
&\left.-\sum_{m=k+p-1}^{\infty}\left|(\lambda+t)m-p(1-t-\lambda+\lambda t)-\frac{m^2\lambda t}{p}\right||b_m|\right]\geq 0,
\end{aligned}$$

and this completes the proof.

Remark (1) : The coefficient bound (7) in previous theorem is sharp for the function

$$\begin{aligned}
f(z) &= z^p+\sum_{m=k+p}^{\infty}\frac{u_m}{\left|(\lambda+t)m+p(1-\lambda-t+\lambda t)+\frac{m^2\lambda t}{p}\right|}z^m \\
&+\sum_{m=k+p-1}^{\infty}\frac{\overline{w}_m}{\left|(\lambda+t)m-p(1-\lambda-t+\lambda t)-\frac{m^2\lambda t}{p}\right|}(\overline{z})^m, \qquad (8)
\end{aligned}$$

where

$$\frac{1}{p-\alpha}\left(\sum_{m=k+p}^{\infty}|u_m|+\sum_{m=k+p-1}^{\infty}|w_m|\right)=1.$$

Theorem 2 : Let $f=h+\overline{g}\in\overline{\mathcal{H}W_{p,k}}$. Then $f(z)\in\overline{\mathcal{H}W_{p.k}}(\lambda,t,\alpha)$ if and only if

$$\begin{aligned}
&\sum_{m=k+p}^{\infty}\left|(\lambda+t)m+p(1-\lambda-t+\lambda t)+\frac{m^2\lambda t}{p}\right||a_m| \\
&+\sum_{m=k+p-1}^{\infty}\left|(\lambda+t)m-p(1-\lambda-t+\lambda t)-\frac{m^2\lambda t}{p}\right||b_m|\leq p-\alpha. \qquad (9)
\end{aligned}$$

Proof : From Theorem 1 and since $\overline{\mathcal{H}W_{p,k}}(\lambda,t,\alpha)\subset\mathcal{H}W_{p,k}(\lambda,t,\alpha)$ we conclude the "if" part.

For the "only if" part, assume that $f(z)\in\overline{\mathcal{H}W_{p,k}}(\lambda,t,\alpha)$. Therefore, for

$z = re^{i\theta} \in U$, we have

$$Re\left\{(1-\lambda)(1-t)\frac{f(z)}{z^p} + (\lambda+t)\frac{f'(z)}{(z^p)'} + \lambda t\frac{f''(z)}{(z^p)''} - 2\lambda t\right\}$$

$$= Re\left\{(1-\lambda)(1-t)\left(1 + \sum_{m=k+p}^{\infty} a_m z^{m-p} + \sum_{m=k+p-1}^{\infty} b_m(\overline{z})^{m-p}\right)\right.$$

$$+ \frac{(\lambda+t)}{p}\left(p + \sum_{m=k+p}^{\infty} ma_m z^{m-p} - \sum_{m=k+p-1}^{\infty} mb_m(\overline{z})^{m-p}\right)$$

$$+\frac{\lambda t}{p^2}\left(p + \sum_{m=k+p}^{\infty} ma_m z^{m-p} + p(p-1) + \sum_{m=k+p}^{\infty} m(m-1)a_m z^{m-p}\right.$$

$$\left.\left. + \sum_{m=k+p-1}^{\infty} mb_m(\overline{z})^{m-p} + \sum_{m=k+p-1}^{\infty} m(m-1)b_m(\overline{z})^{m-p}\right) - 2\lambda t\right\}$$

$$\geq 1 - \frac{1}{p}\sum_{m=k+p}^{\infty}\left|(\lambda+t)m + p(1-\lambda-t+\lambda t) + \frac{m^2\lambda t}{p}\right| |a_m| r^{m-p}$$

$$-\frac{1}{p}\sum_{m=k+p-1}^{\infty}\left|(\lambda+t)m - p(1-\lambda-t+\lambda t) - \frac{m^2\lambda t}{p}\right| |b_m| r^{m-p} \geq \frac{\alpha}{p}.$$

The above inequality holds for all $z \in U$. So if $z = r \to 1$ we obtain the required result (9). Now the proof of Theorem 2 is complete.

As special cases of Theorem 2, we obtain the following two corollaries:

Corollary 1 : $f = h + \overline{g} \in \overline{\mathcal{H}G_{p,k}}(t,\alpha) \equiv \mathcal{H}G_{p,k}(t,\alpha) \cap \overline{\mathcal{H}W_{p,k}}$ if and only if

$$\sum_{m=k+p}^{\infty}\frac{|tm + p(1-t)|}{p-\alpha}|a_m| + \sum_{m=k+p-1}^{\infty}\frac{|tm - p(1-t)|}{p-\alpha}|b_m| \leq 1.$$

Corollary 2 : $f = h + \overline{g} \in \overline{\mathcal{H}A_{p,k}}(t,\alpha) \equiv \mathcal{H}A_{p,k}(t,\alpha) \cap \overline{\mathcal{H}W_{p,k}}$ if and only if

$$\sum_{m=k+p}^{\infty}\frac{\left|(1+t)m + \frac{m^2 t}{p}\right|}{p-\alpha}|a_m| + \sum_{m=k+p-1}^{\infty}\frac{\left|(1+t)m - \frac{m^2 t}{p}\right|}{p-\alpha}|b_m| \leq 1.$$

3. Extreme Points

In the following theorem, we introduce extreme points of $\overline{\mathcal{H}W_{p,k}}(\lambda,t,\alpha)$.

Theorem 3 : $f = h + \overline{g} \in \overline{\mathcal{H}W_{p,k}}(\lambda,t,\alpha)$ if and only if it can be expressed as

$$f(z) = X_p z^p + \sum_{m=k+p}^{\infty} X_m h_m(z) + \sum_{m=k+p-1}^{\infty} Y_m g_m(z), \quad z \in U \qquad (10)$$

where

$$h_m(z) = z^p - \frac{p-\alpha}{\left|(\lambda+t)m + p(1-\lambda-t+\lambda t) + \frac{m^2\lambda t}{p}\right|} z^m, \quad (m = k+p, k+p+1, \cdots), \tag{11}$$

$$g_m(z) = z^p + \frac{p-\alpha}{\left|(\lambda+t)m - p(1-\lambda-t+\lambda t) - \frac{m^2\lambda t}{p}\right|} (\overline{z})^m, \quad (m = k+p-1, k+p, \cdots), \tag{12}$$

$X_p \geq 0, Y_{k+p-1} \geq 0, X_p + \sum_{m=k+p}^{\infty} X_m + \sum_{m=k+p-1}^{\infty} Y_m = 1, X_m \geq 0, Y_m \geq 0$ for $m = k+p, k+p+1, \cdots$.

Proof : If $f(z)$ be given by (10), then

$$\begin{aligned} f(z) &= z^p - \sum_{m=k+p}^{\infty} \frac{p-\alpha}{\left|(\lambda+t)m + p(1-\lambda-t+\lambda t) + \frac{m^2\lambda t}{p}\right|} X_m z^m \\ &+ \sum_{m=k+p-1}^{\infty} \frac{p-\alpha}{\left|(\lambda+t)m - p(1-\lambda-t+\lambda t) - \frac{m^2\lambda t}{p}\right|} Y_m (\overline{z})^m. \end{aligned}$$

Since by (9), we have

$$\begin{aligned} &\sum_{m=k+p}^{\infty} \left|(\lambda+t)m + p(1-\lambda-t+\lambda t) + \frac{m^2\lambda t}{p}\right| \cdot \frac{p-\alpha}{\left|(\lambda+t)m + p(1-\lambda-t+\lambda t) + \frac{m^2\lambda t}{p}\right|} |X_m| \\ &+ \sum_{m=k+p-1}^{\infty} \left|(\lambda+t)m - p(1-\lambda-t+\lambda t) - \frac{m^2\lambda t}{p}\right| \cdot \frac{p-\alpha}{\left|(\lambda+t)m - p(1-\lambda-t+\lambda t) - \frac{m^2\lambda t}{p}\right|} |Y_m| \\ &= (p-\alpha)\left(\sum_{m=k+p}^{\infty} |X_m| + \sum_{m=k+p-1}^{\infty} |Y_m|\right) = (p-\alpha)(1-X_p) \leq p-\alpha. \end{aligned}$$

So $f(z) \in \overline{\mathcal{H}W_{p,k}}(\lambda, t, \alpha)$. Conversely, assume $f(z) \in \overline{\mathcal{H}W_{p,k}}(\lambda, t, \alpha)$, by putting

$$X_p = 1 - \left(\sum_{m=k+p}^{\infty} X_m + \sum_{m=k+p-1}^{\infty} Y_m\right),$$

where

$$X_m = \frac{\left|(\lambda+t)m + p(1-\lambda-t+\lambda t) + \frac{m^2\lambda t}{p}\right|}{p-\alpha} |a_m|,$$

$$Y_m = \frac{\left|(\lambda+t)m - p(1-\lambda-t+\lambda t) - \frac{m^2\lambda t}{p}\right|}{p-\alpha} |b_m|,$$

we obtain

$$\begin{aligned} f(z) &= z^p - \sum_{m=k+p}^{\infty} |a_m| z^m + \sum_{m=k+p-1}^{\infty} |b_m| (\overline{z})^m \\ &= z^p - \sum_{m=k+p}^{\infty} \frac{(p-\alpha)X_m}{\left|(\lambda+t)m + p(1-\lambda-t+\lambda t) + \frac{m^2\lambda t}{p}\right|} z^m \\ &\quad + \sum_{m=k+p-1}^{\infty} \frac{(p-\alpha)Y_m}{\left|(\lambda+t)m - p(1-\lambda-t+\lambda t) - \frac{m^2\lambda t}{p}\right|} (\overline{z})^m \\ &= z^p - \sum_{m=k+p}^{\infty} (z^p - h_m(z))X_m - \sum_{m=k+p-1}^{\infty} (z^p - g_m(z))Y_m \\ &= \left[1 - \left(\sum_{m=k+p}^{\infty} X_m + \sum_{m=k+p-1}^{\infty} Y_m\right)\right] z^p + \sum_{m=k+p}^{\infty} h_m(z)X_m + \sum_{m=k+p-1}^{\infty} g_m(z)Y_m \\ &= X_p z^p + \sum_{m=k+p}^{\infty} X_m h_m(z) + \sum_{m=k+p-1}^{\infty} Y_m g_m(z), \end{aligned}$$

that is the required representation.

4. Convex Combination

Now we show $\overline{\mathcal{H}W_{p,k}}(\lambda, t, \alpha)$ is closed under convex combination.

Theorem 4 : If $f_j (j = 1, 2, \cdots)$ belongs to $\overline{\mathcal{H}W_{p,k}}(\lambda, t, \alpha)$, then the function $Q(z) = \sum_{j=1}^{\infty} \sigma_j f_j(z)$ is also in $\overline{\mathcal{H}W_{p,k}}(\lambda, t, \alpha)$, where $f_j(z)$ is defined by

$$f_j(z) = z^p - \sum_{m=k+p}^{\infty} a_{m,j} z^m + \sum_{m=k+p-1}^{\infty} b_{m,j} (\overline{z})^m (j = 1, 2, \cdots, 0 \le \sigma_j < 1, \sum_{j=1}^{\infty} \sigma_j = 1). \tag{13}$$

Proof : Since $f_j(z) \in \overline{\mathcal{H}W_{p,k}}(\lambda, t, \alpha)$, by (9) we have

$$\begin{aligned} &\sum_{m=k+p}^{\infty} \left|(\lambda+t)m + p(1-\lambda-t+\lambda t) + \frac{m^2\lambda t}{p}\right| |a_{m,j}| \\ &+ \sum_{m=k+p-1}^{\infty} \left|(\lambda+t)m - p(1-\lambda-t+\lambda t) - \frac{m^2\lambda t}{p}\right| |b_{m,j}| \le p - \alpha, (j = 1, 2, \cdots). \end{aligned}$$

Also

$$Q(z) = \sum_{j=1}^{\infty} \sigma_j f_j(z) = z^p - \sum_{m=k+p}^{\infty} \left(\sum_{j=1}^{\infty} \sigma_j a_{m,j}\right) z^m + \sum_{m=k+p-1}^{\infty} \left(\sum_{j=1}^{\infty} \sigma_j b_{m,j}\right) (\overline{z})^m.$$

Now according to Theorem 2 we have

$$\sum_{m=k+p}^{\infty}\left|(\lambda+t)m+p(1-\lambda-t+\lambda t)+\frac{m^2\lambda t}{p}\right|\left|\sum_{j=1}^{\infty}\sigma_j a_{m,j}\right|$$
$$+\sum_{m=k+p-1}^{\infty}\left|(\lambda+t)m-p(1-\lambda-t+\lambda t)-\frac{m^2\lambda t}{p}\right|\left|\sum_{j=1}^{\infty}\sigma_j b_{m,j}\right|$$
$$=\sum_{j=1}^{\infty}\left\{\sum_{m=k+p}^{\infty}\left|(\lambda+t)m+p(1-\lambda-t+\lambda t)+\frac{m^2\lambda t}{p}\right||a_{m,j}|\right.$$
$$\left.+\sum_{m=k+p-1}^{\infty}\left|(\lambda+t)m-p(1-\lambda-t+\lambda t)-\frac{m^2\lambda t}{p}\right||b_{m,j}|\right\}\sigma_j$$
$$\leq(p-\alpha)\sum_{j=1}^{\infty}\sigma_j=p-\alpha.$$

Thus $Q(z)\in\overline{\mathcal{H}W_{p,k}}(\lambda,t,\alpha)$.
Remark 2 : We note that $\overline{\mathcal{H}W_{p,k}}(\lambda,t,\alpha)$ is a convex set.

5. Distortion Bounds

In the next theorem, we obtain the distortion bounds for $f\in\overline{\mathcal{H}W_{p,k}}(\lambda,t,\alpha)$.
Theorem 5 : If $f=h+\overline{g}\in\overline{\mathcal{H}W_{p,k}}(\lambda,t,\alpha),\lambda\geq 1,|z|=r<1$, then

$$|f(z)|\geq(1-|b_{k+p-1}|r^{k-1})r^p-\left(\frac{p-\alpha}{(\lambda+t)k+p(1+\lambda t)+\frac{(k+p)^2\lambda t}{p}}\right.$$
$$\left.-\frac{(\lambda+t)(k+2p-1)-p(1+\lambda t)-\frac{(k+p-1)^2\lambda t}{p}}{(\lambda+t)k+p(1+\lambda t)+\frac{(k+p)^2\lambda t}{p}}|b_{k+p-1}|\right)r^{k+p}. \tag{14}$$

and

$$|f(z)|\leq(1+|b_{k+p-1}|r^{k-1})r^p+\left(\frac{p-\alpha}{(\lambda+t)k+p(1+\lambda t)+\frac{(k+p)^2\lambda t}{p}}\right.$$
$$\left.-\frac{(\lambda+t)(k+2p-1)-p(1+\lambda t)-\frac{(k+p-1)^2\lambda t}{p}}{(\lambda+t)k+p(1+\lambda t)+\frac{(k+p)^2\lambda t}{p}}|b_{k+p-1}|\right)r^{k+p}. \tag{15}$$

Proof : Assume $f(z) \in \overline{\mathcal{H}W_{p,k}}(\lambda, t, \alpha)$, then by (9), we have

$$|f(z)| = \left| z^p - \sum_{m=k+p}^{\infty} |a_m| z^m + \sum_{m=k+p-1}^{\infty} |b_m| (\overline{z})^m \right|$$

$$= \left| z^p + |b_{k+p-1}| (\overline{z})^{k+p-1} - \sum_{m=k+p}^{\infty} (|a_m| z^m - |b_m| (\overline{z})^m) \right|$$

$$\geq r^p - |b_{k+p-1}| r^{k+p-1} - \frac{p-\alpha}{(\lambda+t)k + p(1+\lambda t) + \frac{(k+p)^2 \lambda t}{p}} \times$$

$$\sum_{m=k+p}^{\infty} \left(\frac{(\lambda+t)k + p(1+\lambda t) + \frac{(k+p)^2 \lambda t}{p}}{p-\alpha} |a_m| + \frac{(\lambda+t)k + p(1+\lambda t) + \frac{(k+p)^2 \lambda t}{p}}{p-\alpha} |b_m| \right) r^m$$

$$\geq r^p - |b_{k+p-1}| r^{k+p-1} - \frac{p-\alpha}{(\lambda+t)k + p(1+\lambda t) + \frac{(k+p)^2 \lambda t}{p}} \times$$

$$\sum_{m=k+p}^{\infty} \left(\frac{(\lambda+t)(m-p) + p(1+\lambda t) + \frac{m^2 \lambda t}{p}}{p-\alpha} |a_m| + \frac{(\lambda+t)(m+p) - p(1+\lambda t) - \frac{m^2 \lambda t}{p}}{p-\alpha} |b_m| \right) r^m$$

$$\geq r^p - |b_{k+p-1}| r^{k+p-1} - \frac{p-\alpha}{(\lambda+t)k + p(1+\lambda t) + \frac{(k+p)^2 \lambda t}{p}} \times$$

$$\left(1 - \frac{(\lambda+t)(k+2p-1) - p(1+\lambda t) - \frac{(k+p-1)^2 \lambda t}{p}}{p-\alpha} |b_{k+p-1}| \right) r^{k+p}$$

$$= r^p - |b_{k+p-1}| r^{k+p-1}$$

$$- \left(\frac{p-\alpha}{(\lambda+t)k + p(1+\lambda t) + \frac{(k+p)^2 \lambda t}{p}} - \frac{(\lambda+t)(k+2p-1) - p(1+\lambda t) - \frac{(k+p-1)^2 \lambda t}{p}}{(\lambda+t)k + p(1+\lambda t) + \frac{(k+p)^2 \lambda t}{p}} |b_{k+p-1}| \right) r^{k+p}.$$

Relation (15) can be proved by using the similar statements. So the proof is complete.

Corollary 3 : If $f \in \overline{\mathcal{H}G_{p,k}}(t, \alpha)$, then

$$|f(z)| \geq (1 - |b_{k+p-1}| r^{k-1}) r^p - \left(\frac{p-\alpha}{tk+p} - \frac{t(k+2p-1) - p}{tk+p} |b_{k+p-1}| \right) r^{k+p} \quad \text{and}$$

$$|f(z)| \leq (1 + |b_{k+p-1}| r^{k-1}) r^p + \left(\frac{p-\alpha}{tk+p} - \frac{t(k+2p-1) - p}{tk+p} |b_{k+p-1}| \right) r^{k+p}.$$

Corollary 4 : If $f \in \overline{\mathcal{H}A_{p.k}}(t, \alpha)$, then

$$|f(z)| \geq (1 - |b_{k+p-1}| r^{k-1}) r^p$$
$$- \left(\frac{p-\alpha}{(1+t)(k+p) + \frac{(k+p)^2 t}{p}} - \frac{(1+t)(k+p-1) - \frac{(k+p-1)^2 t}{p}}{(1+t)(k+p) + \frac{(k+p)^2 t}{p}} |b_{k+p-1}| \right) r^{k+p}$$

and

$$|f(z)| \leq (1 + |b_{k+p-1}| r^{k-1}) r^p$$
$$+ \left(\frac{p-\alpha}{(1+t)(k+p) + \frac{(k+p)^2 t}{p}} - \frac{(1+t)(k+p-1) - \frac{(k+p-1)^2 t}{p}}{(1+t)(k+p) + \frac{(k+p)^2 t}{p}} |b_{k+p-1}| \right) r^{k+p}.$$

6. Integral Operator

Definition 2 : The Jung-Kim-Srivastava integral operator [4] is defined by

$$J^{\sigma}k(z) = \frac{(p+1)^{\sigma}}{2\Gamma(\sigma)} \int_0^z (\log \frac{z}{t})^{\sigma+1} k(t)dt, \quad \sigma > 0. \tag{16}$$

If $k(z) = z^p + \sum_{m=k+p}^{\infty} c_m z^m$, then

$$J^{\sigma}k(z) = z^p + \sum_{m=k+p}^{\infty} \left(\frac{p+1}{k+1}\right)^{\sigma} c_m z^m, \tag{17}$$

also J^{σ} is a linear operator.
Remark 3 : If $f(z) = h(z) + \overline{g(z)}$, where

$$h(z) = z^p - \sum_{m=k+p}^{\infty} |a_m| z^m, \qquad g(z) = \sum_{m=k+p-1}^{\infty} |b_m| z^m, |b_{k+p-1}| < 1,$$

then

$$J^{\sigma}f(z) = J^{\sigma}h(z) + \overline{J^{\sigma}g(z)}. \tag{18}$$

Theorem 6 : If $f(z) \in \overline{\mathcal{H}W_{p,k}}(\lambda, t, \alpha)$ and $p < k$, then $J^{\sigma}f(z)$ is also in $\overline{\mathcal{H}W_{p,k}}(\lambda, t, \alpha)$.
Proof : By (17) and (18), we obtain

$$\begin{aligned} J^{\sigma}f(z) &= J^{\sigma}\left(z^p - \sum_{m=k+p}^{\infty} a_m z^m + \sum_{m=k+p-1}^{\infty} b_m (\overline{z})^m\right) \\ &= z^p - \sum_{m=k+p}^{\infty} \left(\frac{p+1}{k+1}\right)^{\sigma} a_m z^m + \sum_{m=k+p-1}^{\infty} \left(\frac{p+1}{k+1}\right)^{\sigma} b_m (\overline{z})^m. \end{aligned}$$

Since $f(z) \in \overline{\mathcal{H}W_{p,k}}(\lambda, t, \alpha)$, then by Theorem 2 we have

$$\begin{aligned} &\sum_{m=k+p}^{\infty} \left|(\lambda+t)m + p(1-\lambda-t+\lambda t) + \frac{m^2\lambda t}{p}\right| |a_m| \\ &+ \sum_{m=k+p-1}^{\infty} \left|(\lambda+t)m - p(1-\lambda-t+\lambda t) - \frac{m^2\lambda t}{p}\right| |b_m| \leq p-\alpha, \end{aligned} \tag{19}$$

we must show

$$\begin{aligned} &\sum_{m=k+p}^{\infty} \left|(\lambda+t)m + p(1-\lambda-t+\lambda t) + \frac{m^2\lambda t}{p}\right| |a_m| \left(\frac{p+1}{k+1}\right)^{\sigma} \\ &+ \sum_{m=k+p-1}^{\infty} \left|(\lambda+t)m - p(1-\lambda-t+\lambda t) - \frac{m^2\lambda t}{p}|b_m|\right] \left(\frac{p+1}{k+1}\right)^{\sigma} \leq p-\alpha. \end{aligned} \tag{20}$$

But in view of (19) the inequality in (20) holds true if $\left(\frac{p+1}{k+1}\right)^{\sigma} < 1$, since $\sigma > 0$, therefore (20) holds true if $p < k$, and this gives the result.

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