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# A Generalized Ruscheweyh Derivatives Involving a General Fractional Derivative Operator Defined on a Class of Multivalent Functions II

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**Abstract**

In this paper, we introduce the class $\Sigma_p^{\lambda,\mu}(\gamma,\beta)$ by using the generalized Ruscheweyh derivatives involving a general fractional derivative operator. The aim of this paper is to study some interesting properties of this class, the coefficient bounds, radii of starlikeness and convexity and Quasi-Hadamard product are obtained, we have also several results in our paper.

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## 1. Introduction

Let $T(n,p)$ denote the class of functions of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \quad (n, p \in I\!N, a_k \geq 0) \tag{1.1}$$

where $f(z)$ is analytic and multivalent function in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Then the function $f(z) \in T(n,p)$ is said to be in the class $S(n,p,\alpha)$, if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in U, 0 \leq \alpha < p). \tag{1.2}$$

A function $f(z) \in S(n,p,\alpha)$ is called multivalent starlike of order $\alpha$. A function $f(z) \in T(n,p)$ is said to be in the class $C(n,p,\alpha)$, if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in U,\ 0 \leq \alpha < p). \tag{1.3}$$

A function $f(z) \in C(n,p,\alpha)$ is called multivalent convex function of order $\alpha$. It is observed that

$$f(z) \in C(n,p,\alpha) \text{ if and only if } zf'(z) \in S(n,p,\alpha) \ \ \forall \ \ n \in I\!N. \tag{1.4}$$

We recall here the fractional derivative operator given by Definition 1.1 below. (See, e.g. [4], [5]).

Let $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \cdots$. The Gaussian hypergeometric function ${}_2F_1$ is defined by

$${}_2F_1(z) \equiv {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{1.5}$$

where $(\lambda)_n$ is the pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in I\!N) \end{cases} \tag{1.6}$$

**Definition 1.1** : Let $0 \leq \lambda < 1$ and $\mu, \nu \in I\!R$. Then, in terms of familiar (Gauss's) hypergeometric function ${}_2F_1$, the generalized fractional derivative operator of order $\lambda$ for a function $f(z)$ is defined by:

$$J_{0,z}^{\lambda,\mu,\nu} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\{z^{\lambda-\mu}\int_0^z (z-\mathcal{E})^{-\lambda} f(\mathcal{E}) \cdot {}_2F_1(\mu-\lambda, 1-\nu; 1-\lambda; 1-\frac{\mathcal{E}}{z})d\mathcal{E}\} \\ \qquad (0 \leq \lambda < 1) \\ \frac{d^n}{dz^n} J_{0,z}^{\lambda-n,\mu,\nu} f(z), \quad (n \leq \lambda < n+1, n \in I\!N) \end{cases} \tag{1.7}$$

where the function $f(z)$ is analytic in a simply-connected region of the $z$-plane containing the origin, with the order

$$f(z) = O(|z|^{\epsilon}), \quad (z \to 0), \tag{1.8}$$

for $\epsilon > \max\{0, \mu-\nu\} - 1$, and the multiplicity of $(z-\mathcal{E})^{-\lambda}$ is removed by requiring $\log(z-\mathcal{E})$ to be real when $z - \mathcal{E} > 0$.

The fractional derivative of order $\lambda$ of a function $f(z)$ is defined by

$$D_z^\lambda\{f(z)\} = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\mathcal{E})}{(z-\mathcal{E})^\lambda}d\mathcal{E}, \quad 0 \le \lambda < 1, \tag{1.9}$$

where $f(z)$ is chosen as in (1.7), and the multiplicity of $(z-\mathcal{E})^{-\lambda}$ is removed by requiring $\log(z-\mathcal{E})$ to be real when $z-\mathcal{E}>0$.

By comparing (1.7) with (1.9), we find

$$J_{0,z}^{\lambda,\lambda,\nu} f(z) = D_z^\lambda\{f(z)\}, \quad (0 \le \lambda < 1). \tag{1.10}$$

In terms of gamma function, we have

$$J_{0,z}^{\lambda,\mu,\nu} z^k = \frac{\Gamma(k+1)\Gamma(1-\mu+\nu+k)}{\Gamma(1-\mu+k)\Gamma(1-\lambda+\nu+k)}\, z^{k-\mu} \tag{1.11}$$

$(0 \le \lambda < 1, \mu, \nu \in I\!R$ and $k > \max\{0, \mu-\nu\}-1)$.

**Definition 1.2** : Let $f(z) \in T(n,p)$ be given by (1.1), then for $0 \le \beta < p, \lambda > -1, p \in I\!N, 0 \le \gamma \le 1$, we define a new class of functions $\Sigma_p^{\lambda,\mu}(\gamma,\beta)$ consisting of functions $f$ in $T(n,p)$ satisfying the condition

$$Re\left\{\frac{z(\mathcal{J}_p^{\lambda,\mu}f(z))' + \gamma z^2(\mathcal{J}_p^{\lambda,\mu}f(z))''}{\gamma z(\mathcal{J}_p^{\lambda,\mu}f(z))' + (1-\gamma)(\mathcal{J}_p^{\lambda,\mu}f(z))}\right\} > \beta \tag{1.12}$$

where $\mathcal{J}_p^{\lambda,\mu}f$ is a generalized Ruscheweyh derivative defined by Goyal and Goyal [1, p. 442] as

$$\mathcal{J}_p^{\lambda,\mu}f(z) = \frac{\Gamma(\mu-\lambda+\nu+2)}{\Gamma(\nu+2)\Gamma(\mu+1)} z^p J_{0,z}^{\lambda,\mu,\nu}(z^{\mu-p}f(z)) = z^p - \sum_{k=n+p}^{\infty} a_k C_p^{\lambda,\mu}(k) z^k \tag{1.13}$$

where

$$C_p^{\lambda,\mu}(k) = \frac{\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}. \tag{1.14}$$

For $\mu = \lambda = \alpha + p - 1, \nu = 1$, the generalized Ruscheweyh derivatives get reduced to Ruscheweyh derivatives of $f(z)$ of order $\alpha + p - 1$ (see, e.g. [2]):

$$D^{\alpha+p-1}f(z) = \frac{z^p}{\Gamma(\alpha+p)} D^{\alpha+p-1}(z^{\alpha-1}f(z)) = z^p - \sum_{k=n+p}^{\infty} a_k C_k(\alpha) z^k \tag{1.15}$$

where

$$C_k(\alpha) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha+p)\Gamma(k-p+1)}. \tag{1.16}$$

For $p = 1$, (1.15) reduces to ordinary Ruscheweyh derivatives for univalent functions [3].

## 2. Coefficient Bounds

**Theorem 2.1** : Let $f(z) \in T(n,p)$. Then $f(z) \in \Sigma_p^{\lambda,\mu}(\gamma,\beta)$ if and only if

$$\sum_{k=n+p}^{\infty} [(\gamma k-\gamma+1)(k-\beta)]C_p^{\lambda,\mu}(k)a_k \leq (\gamma p-\gamma+1)(p-\beta) \tag{2.1}$$

where $0 \leq \beta < p, 0 \leq \gamma \leq 1, p \in I\!N$ and $C_p^{\lambda,\mu}(k)$ is given by (1.14).
**Proof** : Let $f \in \Sigma_p^{\lambda,\mu}(\gamma,\beta)$. By making use of (1.13), we get

$$(\mathcal{J}_p^{\lambda,\mu}f(z))' = pz^{p-1} - \sum_{k=n+p}^{\infty} ka_k C_p^{\lambda,\mu}(k)z^{k-1} \tag{2.2}$$

and

$$(\mathcal{J}_p^{\lambda,\mu}f(z))'' = p(p-1)z^{p-2} - \sum_{k=n+p}^{\infty} k(k-1)a_k C_p^{\lambda,\mu}(k)z^{k-2}. \tag{2.3}$$

Now using (2.2) and (2.3) in (1.12), we find that

$$Re\left\{\frac{pz^p - \sum_{k=n+p}^{\infty} ka_kC_p^{\lambda,\mu}(k)z^k + \gamma p(p-1)z^p - \sum_{k=n+p}^{\infty}\gamma k(k-1)a_kC_p^{\lambda,\mu}(k)z^k}{\gamma pz^p - \sum_{k=n+p}^{\infty}\gamma ka_kC_p^{\lambda,\mu}(k)z^k + (1-\gamma)(z^p - \sum_{k=n+p}^{\infty}a_kC_p^{\lambda,\mu}(k)z^k)}\right\} > \beta$$

or

$$Re\left\{\frac{(\gamma p(p-1)+p)z^p - \sum_{k=n+p}^{\infty}(k+\gamma k(k-1))a_kC_p^{\lambda,\mu}(k)z^k}{(1-\gamma+\gamma p)z^p - \sum_{k=n+p}^{\infty}(\gamma k-\gamma+1)a_kC_p^{\lambda,\mu}(k)z^k}\right\} > \beta$$

By letting $z \to 1^-$ through real values, we have

$$(1-\gamma+\gamma p)\beta - \sum_{k=n+p}^{\infty}(\gamma k-\gamma+1)\beta a_kC_p^{\lambda,\mu}(k)$$
$$\leq (\gamma p(p-1)+p) - \sum_{k=n+p}^{\infty}(k+\gamma k(k-1))a_kC_p^{\lambda,\mu}(k).$$

Therefore,

$$\sum_{k=n+p}^{\infty} [(\gamma k-\gamma+1)(k-\beta)]C_p^{\lambda,\mu}(k)a_k \leq (\gamma p-\gamma+1)(p-\beta).$$

Conversely, assume that (2.1) holds. We will prove that (1.12) is satisfied and so $f \in \Sigma_p^{\lambda,\mu}(\gamma,\beta)$. Since $Re\{w\} > \beta$ if and only if $|w-(1+\beta)| < |w+(1-\beta)|$, it is sufficient to show that

$$\begin{aligned} Q &= \left| \frac{z(\mathcal{J}_p^{\lambda,\mu} f(z))' + \gamma z^2 (\mathcal{J}_p^{\lambda,\mu} f(z))''}{\gamma z(\mathcal{J}_p^{\lambda,\mu} f(z))' + (1-\gamma)(\mathcal{J}_p^{\lambda,\mu} f(z))} - 1 - \beta \right| \\ &< \left| \frac{z(\mathcal{J}_p^{\lambda,\mu} f(z))' + \gamma z^2 (\mathcal{J}_p^{\lambda,\mu} f(z))''}{\gamma z(\mathcal{J}_p^{\lambda,\mu} f(z))' + (1-\gamma)(\mathcal{J}_p^{\lambda,\mu} f(z))} + 1 - \beta \right| = T. \end{aligned}$$

Let $X = \gamma z(\mathcal{J}_p^{\lambda,\mu} f(z))' + (1-\gamma)(\mathcal{J}_p^{\lambda,\mu} f(z))$, then we have

$$Q = \frac{1}{|X|} |z(\mathcal{J}_p^{\lambda,\mu} f(z))' + \gamma z^2 (\mathcal{J}_p^{\lambda,\mu} f(z))'' - (1+\beta)X|.$$

From (1.13), (2.2) and (2.3), we get

$$\begin{aligned} Q &= \frac{1}{|X|} |[(p+\gamma p(p-1) - (1+\beta)\gamma p - (1+\beta)(1-\gamma)]z^p - \sum_{k=n+p}^{\infty} [(k+\gamma k(k-1)) \\ &\quad -(1+\beta)(\gamma k - \gamma + 1)] C_p^{\lambda,\mu}(k) a_k z^k| \\ &= \frac{1}{|X|} |[p(1-(1+\beta)\gamma) - (1+\beta)(1-\gamma) + \gamma p(p-1)]z^p \\ &\quad - \sum_{k=n+p}^{\infty} (\gamma k - \gamma + 1)(k-\beta-1) C_p^{\lambda,\mu}(k) a_k z^k| \\ &\quad < \frac{|z|^p}{|X|} [(\gamma p - \gamma + 1)(p-\beta-1) + \sum_{k=n+p}^{\infty} (\gamma k - \gamma + 1)(k-\beta-1) C_p^{\lambda,\mu}(k) a_k |z|^{k-p}] \end{aligned}$$

and

$$\begin{aligned} T &= \frac{1}{|X|} |[z(\mathcal{J}_p^{\lambda,\mu} f(z))' + \gamma z^2 (\mathcal{J}_p^{\lambda,\mu} f(z))'' + (1-\beta)X| \\ &= \frac{1}{|X|} |[p(1+(1-\beta)\gamma) + (1-\beta)(1-\gamma) + \gamma p(p-1)]z^p \\ &\quad - \sum_{k=n+p}^{\infty} (\gamma k - \gamma + 1)(k-\beta+1) C_p^{\lambda,\mu}(k) a_k z^k| \\ &\geq \frac{|z|^p}{|X|} [(\gamma p - \gamma + 1)(p-\beta+1) - \sum_{k=n+p}^{\infty} (\gamma k - \gamma + 1)(k-\beta+1) C_p^{\lambda,\mu}(k) a_k |z|^{k-p}] \end{aligned}$$

when $z \in \partial U = \{z : z \in \mathbb{C} \text{ and } |z| = 1\}$. It is easy to verify that $T - Q > 0$, if (2.1) holds and so the proof is complete.

**Remark 1** : The result (2.1) is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(\gamma p - \gamma + 1)(p-\beta)}{(\gamma(n+p-1)+1)(n+p-\beta) C_p^{\lambda,\mu}(n+p)} z^{n+p}, \tag{2.4}$$

where

$$C_p^{\lambda,\mu}(n+p) = \frac{\Gamma(n+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(n+\nu+2)}{\Gamma(n+1)\Gamma(n+\nu+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}.$$

**Corollary 1** : If $f(z) \in \Sigma_p^{\lambda,\mu}(\gamma,\beta)$, then

$$a_k \leq \frac{(\gamma p-\gamma+1)(p-\beta)}{(\gamma k-\gamma+1)(k-\beta)C_p^{\lambda,\mu}(k)}, \ (k \geq n+p, n \in I\!N) \tag{2.5}$$

where $C_p^{\lambda,\mu}(k)$ is given by (1.14).

## 3. Radii of Starlikeness and Convexity

Now we obtain radii of starlikeness and convexity.

**Theorem 3.1** : Let the function $f(z)$ defined by (1.1) be in the class $\Sigma_p^{\lambda,\mu}(\gamma,\beta)$, then $f(z)$ is starlike of order $\delta$ for $0 \leq \delta < p$ in $|z| < R_1$, where

$$R_1 = \inf_{k \geq n+p} \left[\frac{(p-\delta)(\gamma k-\gamma+1)(k-\beta)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{(k-\delta)(\gamma p-\gamma+1)(p-\beta)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}\right]^{\frac{1}{k-p}}. \tag{3.1}$$

The result is sharp.

**Proof** : To find the required result it is sufficient to prove that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p-\delta, \quad |z| \leq R_1. \tag{3.2}$$

But

$$\left|\frac{zf'(z)}{f(z)} - p\right| = \left|\frac{pz^p - \sum_{k=n+p}^{\infty} ka_k z^k}{z^p - \sum_{k=n+p}^{\infty} a_k z^k} - p\right| = \left|\frac{-\sum_{k=n+p}^{\infty}(k-p)a_k z^k}{z^p - \sum_{k=n+p}^{\infty} a_k z^k}\right|$$

$$\leq \frac{\sum_{k=n+p}^{\infty}(k-p)a_k|z|^{k-p}}{1-\sum_{k=n+p}^{\infty} a_k|z|^{k-p}} \leq p-\delta,$$

thus $\sum_{k=n+p}^{\infty} \left(\frac{k-\delta}{p-\delta}\right) a_k|z|^{k-p} \leq 1.$

Since $f(z) \in \Sigma_p^{\lambda,\mu}(\gamma,\beta)$ the last inequality holds if

$$|z| \leq \left[\frac{(p-\delta)(\gamma k-\gamma+1)(k-\beta)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{(k-\delta)(\gamma p-\gamma+1)(p-\beta)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}\right]^{\frac{1}{k-p}}.$$

This concludes the result.

Sharpness follows from the function defined by (2.4).

**Theorem 3.2** : Let the function $f(z)$ defined by (1.1) be in the class $\Sigma_p^{\lambda,\mu}(\gamma,\beta)$, then $f(z)$ is convex of order $\delta$ for $0 \leq \delta < p$ in $|z| < R_2$ where

$$R_2 = \inf_{k\geq n+p}\left[\frac{p(p-\delta)(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{k(k-\delta)(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}\right]^{\frac{1}{k-p}}. \tag{3.3}$$

The result is sharp.

**Proof** : The proof follows exactly on the same lines as above and sharpness follows from the function defined by (2.4).

**Theorem 3.3** : Let $f(z) \in \Sigma_p^{\lambda,\mu}(\gamma,\beta)$ be defined by (1.1) and $c$ be any real number such that $c > -p$, then the integral operator

$$G(z) = \frac{c+p}{z^c}\int_0^z s^{c-1}f(s)ds, \quad c > -p \tag{3.4}$$

also belongs to $\Sigma_p^{\lambda,\mu}(\gamma,\beta)$.

**Proof** : By virtue of (3.4) it follows from (1.1) that

$$\begin{aligned} G(z) &= \frac{c+p}{z^c}\int_0^z s^{c-1}\left[s^p - \sum_{k=n+p}^{\infty} a_k s^k\right]ds \\ &= \frac{c+p}{z^c}\int_0^z \left(s^{p+c-1} - \sum_{k=n+p}^{\infty} a_k s^{k+c-1}\right)ds \\ &= z^p - \sum_{k=n+p}^{\infty}\left(\frac{c+p}{c+k}\right)a_k z^k \\ &= z^p - \sum_{k=n+p}^{\infty} h_k z^k \quad \text{where} \quad h_k = \frac{c+p}{c+k}a_k. \end{aligned}$$

But

$$\begin{aligned} &\sum_{k=n+p}^{\infty}\frac{[(\gamma k-\gamma+1)(k-\beta)]\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}h_k \\ &= \sum_{k=n+p}^{\infty}\frac{[(\gamma k-\gamma+1)(k-\beta)]\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}\left(\frac{c+p}{c+k}\right)a_k. \end{aligned}$$

Since $\frac{c+p}{c+k} \leq 1$ and by (2.1) the last expression is less than or equal to $(\gamma p - \gamma + 1)(p-\beta)$, so the proof is complete.

**Theorem 3.4** : Let $c \in I\!R$ $(c > -p)$ if $G(z) \in \Sigma_p^{\lambda,\mu}(\gamma,\beta)$, then the function $f(z)$ defined by (3.4), is $p$-valent in $|z| < \gamma$ where

$$\gamma = \inf_{k \geq n+p} \left[ \frac{p(c+p)(\gamma k-\gamma+1)(k-\beta)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{k(c+k)(\gamma p-\gamma+1)(p-\beta)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)} \right]^{\frac{1}{k-p}}, \tag{3.5}$$

the result is sharp.

**Proof** : Let $G(z) = z^p - \sum_{k=n+p}^{\infty} h_k z^k = \frac{c+p}{z^c} \int_0^z s^{c-1} f(s) ds$ so

$$f(z) = z^p - \sum_{k=n+p}^{\infty} \left( \frac{c+k}{c+p} \right) h_k z^k, \ c > -p.$$

Thus it is enough to show that $\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p, \ |z| < r$. But

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| = \left| - \sum_{k=n+p}^{\infty} k \left( \frac{c+k}{c+p} \right) h_k z^{k-p} \right|,$$

then

$$\sum_{k=n+p}^{\infty} \frac{k}{p} \left( \frac{c+k}{c+p} \right) h_k |z|^{k-p} \leq 1. \tag{3.6}$$

Since $G(z) \in \Sigma_p^{\lambda,\mu}(\gamma,\beta)$ by (2.1) we have

$$\sum_{k=n+p}^{\infty} \frac{[(\gamma k-\gamma+1)(k-\beta)][\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{(\gamma p-\gamma+1)(p-\beta)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)} h_k \leq 1.$$

Therefore (3.6) will be true if

$$\frac{k}{p} \left( \frac{c+k}{c+p} \right) |z|^{k-p} \leq \frac{[(\gamma k-\gamma+1)(k-\beta)][\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{(\gamma p-\gamma+1)(p-\beta)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)},$$

or

$$|z| \leq \left[ \frac{p(c+p)(\gamma k-\gamma+1)(k-\beta)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{k(c+k)(\gamma p-\gamma+1)(p-\beta)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)} \right]^{\frac{1}{k-p}}, \quad k \geq n+p$$

and this proves the result. Sharpness of this theorem follows if we put

$$f(z) = z^p - \frac{(c+k)(\gamma p-\gamma+1)(p-\beta)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}{(c+p)(\gamma k-\gamma+1)(k-\beta)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)} z^k, \quad (k \geq n+p). \tag{3.7}$$

**Theorem 3.5** : Let $f(z) \in \Sigma_p^{\lambda,\mu}(\gamma,\beta)$, then the integral operator

$$F_\iota(z) = (1-\iota)z^p + \iota p \int_0^z \frac{f(s)}{s} ds \quad (\iota \geq 0, z \in U), \tag{3.8}$$

is also in $\Sigma_p^{\lambda,\mu}(\gamma,\beta)$ if $0 \leq \iota \leq \frac{n+p}{p}$.

**Proof** : If $f(z) = z^p - \sum\limits_{k=n+p}^{\infty} a_k z^k$, then

$$\begin{aligned} F_\iota(z) &= (1-\iota)z^p + \iota p \int_0^z \left( \frac{s^p - \sum\limits_{k=n+p}^{\infty} a_k s^k}{s} \right) ds \\ &= (1-\iota)z^p + \iota p \left[ \frac{1}{p} z^p - \sum_{k=n+p}^{\infty} \frac{a_k}{k} z^k \right] \\ &= z^p - \sum_{k=n+p}^{\infty} \frac{\iota p}{k} a_k z^k = z^p - \sum_{k=n+p}^{\infty} g_k z^k, \end{aligned}$$

where $g_k = \frac{\iota p}{k} a_k$. But

$$\begin{aligned} &\sum_{k=n+p}^{\infty} \frac{(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)} g_k \\ &= \sum_{k=n+p}^{\infty} \frac{(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)} \frac{\iota p}{k}\ a_k \\ &\leq \sum_{k=n+p}^{\infty} \frac{(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)} \frac{\iota p}{n+p} a_k \\ &\left(\frac{\iota p}{n+p} \leq 1\right) \leq \sum_{k=n+p}^{\infty} \frac{(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)} a_k \\ &\text{(by (2.1))} \quad \leq (p-\beta)(\gamma p-\gamma+1). \end{aligned}$$

So the proof is complete.

**Remark 2** : The radii of starlikeness and convexity of order $\sigma$ $(0 \leq \sigma < 1)$ for $G(z)$ and $F_\iota(z)$ respectively are as follows :

$$G(z) : \begin{cases} \text{(i)} \quad r_1 = \inf\limits_k \left\{ \frac{(c+k)(p-\sigma)(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{(c+p)(k-\sigma)(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)} \right\}^{\frac{1}{k-p}} \\ \text{(ii)} \quad r_2 = \inf\limits_k \left\{ \frac{p(c+k)(p-\sigma)(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{k(c+p)(k-\sigma)(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)} \right\}^{\frac{1}{k-p}} \end{cases} \tag{3.9}$$

$$F_\iota(z):\begin{cases}\text{(i)}\quad R_1=\inf_k\left\{\frac{k(p-\sigma)(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{\iota p(k-\sigma)(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}\right\}^{\frac{1}{k-p}}\\ \text{(ii)}\quad R_2=\inf_k\left\{\frac{(p-\sigma)(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{\iota(k-\sigma)(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}\right\}^{\frac{1}{k-p}}\end{cases} \tag{3.10}$$

**Theorem 3.6** : Assume that $f(z)=z^p-\sum_{k=n+p}^{\infty}a_kz^k$, $g(z)=z^p-\sum_{k=n+p}^{\infty}b_kz^k$ be in $\Sigma_p^{\lambda,\mu}(\gamma,\beta)$, then the function $h(z)=z^p-\sum_{k=n+p}^{\infty}(a_k^m+b_k^m)z^k, m\in I\!N$ is also in $\Sigma_p^{\lambda,\mu}(\gamma,\beta_1)$, where

$$\begin{aligned}&\beta_1<\inf_k\{[p[(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)]^{m-1}(k-\beta)^m\\&-2k[(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)]^{m-1}(p-\beta)^m]\\&/[[(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)]^{m-1}(k-\beta)^m\\&-2[(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)]^{m-1}(p-\beta)^m]\}\end{aligned} \tag{3.11}$$

**Proof** : Since $f,g$ belong to $\Sigma_p^{\lambda,\mu}(\gamma,\beta)$, then we have

$$\sum_{k=n+p}^{\infty}\left[\frac{(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}\right]^m a_k^m$$
$$\le\left[\sum_{k=n+p}^{\infty}\frac{(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}a_k\right]^m<1,$$
$$\sum_{k=n+p}^{\infty}\left[\frac{(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}\right]^m b_k^m$$
$$\le\left[\sum_{k=n+p}^{\infty}\frac{(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}b_k\right]^m<1.$$

Consequently

$$\frac{1}{2}\sum_{k=n+p}^{\infty}\left[\frac{(k-\beta)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{(p-\beta)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}\right]^m(a_k^m+b_k^m)<1.$$

Now we must show

$$\sum_{k=n+p}^{\infty}\frac{(k-\beta_1)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{(p-\beta_1)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}(a_k^m+b_k^m)<1.$$

But the last inequality holds true if

$$\frac{(k-\beta_1)(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{(p-\beta_1)(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}$$
$$\le\frac{1}{2}\left[\frac{(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}\right]^m\left(\frac{k-\beta}{p-\beta}\right)^m$$

or

$$\frac{k-\beta_1}{p-\beta_1} \le \frac{1}{2}\left[\frac{(\gamma k-\gamma+1)\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{(\gamma p-\gamma+1)\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}\right]^{m-1}\left(\frac{k-\beta}{p-\beta}\right)^{m}$$

or $\beta_1 \le \frac{Wp-k}{W-1}$, where $W$ is the right hand side of the last inequality and this gives the result.

## 4. Quasi-Hadamard Product

**Definition 4.1** : If the functions $f_j(z)(j=1,\cdots,m)$ be in the class $\Sigma_p^{\lambda,\mu}(\gamma,\beta)$ defined by

$$f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k \quad (j=1,\cdots,m;\ n,p \in I\!N = \{1,2,\cdots\}),$$

then the Quasi-Hadamard product of the functions $f_j(z)$ denoted by $(f_1 * f_2 * \cdots * f_m)(z)$, defined by

$$(f_1 * f_2 * \cdots * f_m)(z) = z^p - \sum_{k=n+p}^{\infty} (a_{k,1}a_{k,2}\cdots a_{k,m})z^k. \tag{4.1}$$

**Theorem 4.1** : Let $f_j(z) \in \Sigma_p^{\lambda,\mu}(\gamma,\beta_j)$. Then $(f_1 * f_2 * \cdots * f_m)(z) \in \Sigma_p^{\lambda,\mu}(\gamma,w)$, where

$$0 < w \le p - \frac{n}{H(n+p,m)-1} \tag{4.2}$$

and

$$H(n+p,m) = \left[\frac{(\gamma(n+p-1)-1)C_p^{\lambda,\mu}(n+p)(n+p-w)}{(\gamma p-\gamma+1)(p-w)}\right]^{m-1}\prod_{j=1}^{m}\left(\frac{n+p-\beta_j}{p-\beta_j}\right). \tag{4.3}$$

The result is sharp for the functions $f_j(z)(j=1,2,\cdots,m)$ given by

$$f_j(z) = z^p - \frac{(\gamma p-\gamma+1)(p-\beta_j)}{(\gamma(n+p-1)+1)(n+p-\beta_j)C_p^{\lambda,\mu}(n+p)}\, z^{n+p}. \tag{4.4}$$

**Proof** : We prove this theorem by induction. For $m=1$, we see that $w=\beta_1$. For $m=2$, the inequality (2.1) gives

$$\sum_{k=n+p}^{\infty}\frac{[(\gamma k-\gamma+1)(k-\beta_j)]C_p^{\lambda,\mu}(k)}{(\gamma p-\gamma+1)(p-\beta_j)}a_{k,j} \le 1 \quad (j=1,2; n,p \in I\!N),$$

thus

$$\sum_{k=n+p}^{\infty}\frac{(\gamma k-\gamma+1)}{(\gamma p-\gamma+1)}C_p^{\lambda,\mu}(k)\sqrt{\prod_{j=1}^{2}\frac{(k-\beta_j)}{(p-\beta_j)}a_{k,j}} \le 1. \tag{4.5}$$

Now, we must find the largest $w$ such that

$$\sum_{k=n+p}^{\infty} \frac{(\gamma k-\gamma+1)}{(\gamma p-\gamma+1)} C_p^{\lambda,\mu}(k)\ a_{k,1}a_{k,2} \leq 1$$

or such that

$$\frac{k-w}{p-w}\sqrt{a_{k,1}a_{k,2}} \leq \sqrt{\prod_{j=1}^{2}\frac{(k-\beta_j)}{(p-\beta_j)}}\ , \quad (k \geq n+p).$$

Besides, by (4.5), we need to find the largest $w$ such that

$$\frac{k-w}{p-w} \leq \frac{(\gamma k-\gamma+1)}{(p\gamma-\gamma+1)} C_p^{\lambda,\mu}(k)\prod_{j=1}^{2}\left(\frac{k-\beta_j}{p-\beta_j}\right)$$

or such that $\frac{k-w}{p-w} \leq Y(k)$, where

$$Y(k) = \frac{(\gamma k-\gamma+1)}{(p\gamma-\gamma+1)} C_p^{\lambda,\mu}(k)\prod_{j=1}^{2}\left(\frac{k-\beta_j}{p-\beta_j}\right)$$

or $k-w \leq (p-w)Y(k)$ or $0 < w \leq p - \frac{n}{Y(k)-1}$.

Now, we define the function $\psi(k)$ by $\psi(k) = p - \frac{n}{Y(k)-1}$, $(k \geq n+p)$. Since $\psi'(k) = \frac{nY'(k)}{(Y(k)-1)^2} \geq 0$ for $k \geq n+p$, therefore $\psi(k)$ is an increasing function and so

$$0 < w \leq \psi(n+p) \leq p - \frac{n}{Y(n+p)-1},$$

where

$$Y(n+p) = \frac{((n+p)\gamma-\gamma+1)}{(p\gamma-\gamma+1)} C_p^{\lambda,\mu}(n+p)\prod_{j=1}^{2}\left(\frac{(n+p)-\beta_j}{p-\beta_j}\right),$$

therefore, the result is true for $m = 2$.

Now, assume the result is true for fixed natural number $m$. We must show that the result is true for $m+1$ natural number, that is $(f_1 * f_2 * \cdots * f_m * f_{m+1})(z) \in \Sigma_p^{\lambda,\mu}(\gamma,\eta)$, where $\eta$ satisfies the condition

$$0 < \eta < p - \frac{n}{H-1} \tag{4.6}$$

and $H = \frac{((n+p)\gamma-\gamma+1)C_p^{\lambda,\mu}(n+p)(n+p-w)(n+p-\beta_{m+1})}{(p\gamma-\gamma+1)(p-w)(p-\beta_{m+1})}$, also $w$ is given by (4.2). From (4.6) we have $0 < \eta < p - \frac{n}{H(n+p,m+1)-1}$. This shows that the result holds true for $m+1$. Therefore, using the induction, we obtain that for any positive integer $m$ the result is true.

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