Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 3, 103 - 114

On Generalization of Starlike and Convexity Properties for Hypergeometric Functions Defined by Ruscheweyh Derivative

Waggas Galib Atshan

Department of Mathematics
College of Computer Sciences and Mathematics
University of Al-Qadisiya, Diwaniya, Iraq
waggashnd@yahoo.com

Abstract

We study the generalization of properties of starlike and convexity for the hypergeometric function $F(a,b;c;z) = \sum_{n=0}^{\infty}((a)_n(b)_n/(c)_n(1)_n)z^n$ defined by Ruscheweyh derivative through putting conditions on a, b, c, to ensure that $zF(a,b;c;z)$ will be in various subclasses of starlike and convex functions. We also obtain some results in this paper.

Mathematics Subject Classification: 30C45

Keywords: Starlike functions, Convex functions, Ruscheweyh Derivative, Hypergeometric function

1. Introduction

Let S be the class of fucntions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in S$ is said to be starlike of order $\alpha (0 \leq \alpha < 1)$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, (z \in U). \tag{1.2}$$

Denote the class of all starlike functions of order α in U by $S^*(\alpha)$. A functon $f \in S$ is said to be convex function of order α if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (0 \leq \alpha < 1, z \in U). \tag{1.3}$$

Denote the class of all convex functions of order α in U by $C(\alpha)$.

Let T be subclass of functions f in S of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0) \tag{1.4}$$

which are analytic and univalent in the unit disk U. This paper deals with the generalization of starlike and convexity properties for hypergeometric functions defined by Ruscheweyh derivative. H. Silverman [4] has studied starlike and cinvexity properties for hypergeometric function. Also E. S. Aqlan [1] has studied the generalization of starlike and convexity properties for hypergeometric functions defined by $F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \cdot \frac{z^n}{n!}$ where $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \cdots$.

Definition 1.1 : Let $S^*(\alpha, \beta, \mathcal{E}, \lambda)$ be a class of starlike functions of order α and type β defined by Ruscheweyh derivative that satisfies

$$\left| \frac{\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1}{2\mathcal{E}\left[\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - \alpha\right] - \left[\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1\right]} \right| < \beta \tag{1.5}$$

where $0 < \beta \leq 1, \frac{1}{2} \leq \mathcal{E} \leq 1, 0 \leq \alpha < \frac{1}{2\mathcal{E}}, \lambda > -1$ and $z \in U$. For $\mathcal{E} = \beta = 1$ and $\lambda = 0$ we get a particular case which we denote by starlike of order α. This has been studied in detail by S. R. Kulkarni [2].

Definition 1.2 : Let $C(\alpha, \beta, \mathcal{E}, \lambda)$ be a class of convex functions of order α and type β defined by Ruscheweyh derivative that satisfies

$$\left| \frac{\frac{z(D^\lambda f(z))''}{(D^\lambda f(z))'}}{2\mathcal{E}\left(1 - \alpha + \frac{z(D^\lambda f(z))''}{(D^\lambda f(z))'}\right) - \frac{z(D^\lambda f(z))''}{(D^\lambda f(z))'}} \right| < \beta \tag{1.6}$$

where $0 < \beta \leq 1, \frac{1}{2} \leq \mathcal{E} \leq 1, 0 \leq \alpha < \frac{1}{2\mathcal{E}}, \lambda > -1$ and $z \in U$. For $\mathcal{E} = \beta = 1$ and $\lambda = 0$, we get a particular case which we denote by convex function of order α.

Definition 1.3 : Let $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \cdots$. The Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \cdot \frac{z^n}{n!} \tag{1.7}$$

where $(\lambda)_n$ is the pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in I\!N) \end{cases} \tag{1.8}$$

Definition 1.4 [3], [7] : The Ruscheweyh derivative of order λ is denoted by $D^\lambda f$ and is defined as following: If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

then

$$D^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) = z + \sum_{n=2}^{\infty} B_n(\lambda) a_n z^n, \quad \lambda > -1, z \in U \tag{1.9}$$

where

$$B_n(\lambda) = \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)}{(n-1)!}. \tag{1.10}$$

2. Main Results

Theorem 2.1 : Let $f(z)$ be a function defined by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0)$$

is in $S^*(\alpha, \beta, \mathcal{E}, \lambda)$ if and only if

$$\sum_{n=2}^{\infty} [(n-1)(1-\beta) + 2\mathcal{E}\beta(n-\alpha)] B_n(\lambda) a_n \leq 2\mathcal{E}\beta(1-\alpha) \tag{2.1}$$

where $0 < \beta \leq 1, 0 \leq \alpha < \frac{1}{2\mathcal{E}}, \frac{1}{2} \leq \mathcal{E} \leq 1, \lambda > -1$, $n \geq 2, n \in I\!N$ and $B_n(\lambda)$ is given by (1.10).

Proof : For $|z| = 1$, we have

$$\begin{aligned}
&|z(D^\lambda f(z))' - (D^\lambda f(z))| - \beta |2\mathcal{E}[z(D^\lambda f(z))' - \alpha D^\lambda f(z)] - [z(D^\lambda f(z))' - D^\lambda f(z)]| \\
&= |-\sum_{n=2}^{\infty}(n-1)B_n(\lambda)a_n z^n| - \beta |2\mathcal{E}(1-\alpha)z - \sum_{n=2}^{\infty}(2\mathcal{E}(n-\alpha) - (n-1))B_n(\lambda)a_n z^n| \\
&\leq \sum_{n=2}^{\infty}(n-1)B_n(\lambda)a_n - 2\mathcal{E}\beta(1-\alpha) + \sum_{n=2}^{\infty}(2\mathcal{E}\beta(n-\alpha) - \beta(n-1))B_n(\lambda)a_n \\
&= \sum_{n=2}^{\infty}[(n-1)(1-\beta) + 2\mathcal{E}\beta(n-\alpha)]B_n(\lambda)a_n - 2\mathcal{E}\beta(1-\alpha) \\
&\leq 0 \quad \text{by hypothesis.}
\end{aligned}$$

Thus by Maximum Modulus theorem $f \in S^*(\alpha, \beta, \mathcal{E}, \lambda)$.

Conversely, suppose that

$$\begin{aligned}
&\left| \frac{\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1}{2\mathcal{E}\left[\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - \alpha\right] - \left[\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1\right]} \right| \\
&= \left| \frac{z(D^\lambda f(z))' - D^\lambda f(z)}{2\mathcal{E}(z(D^\lambda f(z))' - \alpha(D^\lambda f(z))) - (z(D^\lambda f(z))' - D^\lambda f(z))} \right| \\
&= \left| \frac{-\sum_{n=2}^{\infty}(n-1)B_n(\lambda)a_n z^{n-1}}{2\mathcal{E}((1-\alpha) + \sum_{n=2}^{\infty}(\alpha-n)B_n(\lambda)a_n z^{n-1}) + \sum_{n=2}^{\infty}(n-1)B_n(\lambda)a_n z^{n-1}} \right| < \beta.
\end{aligned}$$

Since $|Re\{z\}| < |z|$ for all z, we have

$$Re\left\{\frac{\sum_{n=2}^{\infty}(n-1)B_n(\lambda)a_n z^{n-1}}{2\mathcal{E}((1-\alpha)+\sum_{n=2}^{\infty}(\alpha-n)B_n(\lambda)a_n z^{n-1})+\sum_{n=2}^{\infty}(n-1)B_n(\lambda)a_n z^{n-1}}\right\} < \beta \qquad (2.2)$$

we can choose value of z on the real axis so that $D^{\lambda}f(z)$ is real.

Let $z \to 1^{-}$, through real values, so we can write (2.2) as

$$\sum_{n=2}^{\infty}[(n-1)(1-\beta)+2\mathcal{E}\beta(n-\alpha)]B_n(\lambda)a_n \leq 2\mathcal{E}\beta(1-\alpha)$$

Remark : For $\lambda = 0$, we get the particular case which is introduced by Kulkarni [2].
Theorem 2.2 : A sufficient condition for a function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0),$$

to be in $S^*(\alpha, \beta, \mathcal{E}, \lambda)$ is that

$$\sum_{n=2}^{\infty}[(n-1)-\beta(n-1+2\alpha\mathcal{E}-2n\mathcal{E})]B_n(\lambda)a_n \leq 2\mathcal{E}\beta(1-\alpha) \qquad (2.3)$$

where $0 < \beta \leq 1, 0 \leq \alpha < \frac{1}{2\mathcal{E}}, \frac{1}{2} \leq \mathcal{E} \leq 1, \lambda > -1, n \geq 2, n \in I\!N$ and $B_n(\lambda)$ is given by (1.10).
Proof : In the same line of Theorem 2.1.
Theorem 2.3 : Let $f(z)$ be defined by (1.4), then $f(z)$ is in the class $C(\alpha, \beta, \mathcal{E}, \lambda)$ if and only if

$$\sum_{n=2}^{\infty} n[(n-1)(1-\beta+2\mathcal{E}\beta)+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)a_n \leq 2\mathcal{E}\beta(1-\alpha) \qquad (2.4)$$

where $\lambda > -1, 0 < \beta \leq 1, \frac{1}{2} \leq \mathcal{E} \leq 1, 0 \leq \alpha < \frac{1}{2\mathcal{E}}$ and $B_n(\lambda)$ is given by (1.10).
Proof : For $|z| = 1$, we have

$$\begin{aligned}
&|z(D^{\lambda}f(z))''| - \beta|2\mathcal{E}[(1-\alpha)(D^{\lambda}f(z))' + z(D^{\lambda}f(z))''] - z(D^{\lambda}f(z))''| \\
&= \left|-\sum_{n=2}^{\infty} n(n-1)B_n(\lambda)a_n z^{n-1}\right| - \beta\left|2\mathcal{E}(1-\alpha) - \sum_{n=2}^{\infty}[2\mathcal{E}n(n-\alpha)-n(n-1)]B_n(\lambda)a_n z^{n-1}\right| \\
&\leq \sum_{n=2}^{\infty} n(n-1)B_n(\lambda)a_n - 2\mathcal{E}\beta(1-\alpha) + \beta\sum_{n=2}^{\infty} n[2\mathcal{E}(n-\alpha)-(n-1)]B_n(\lambda)a_n \\
&= \sum_{n=2}^{\infty} n[(n-1)(1-\beta+2\mathcal{E}\beta)+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)a_n - 2\mathcal{E}\beta(1-\alpha) \\
&\leq 0 \quad \text{by hypothesis.}
\end{aligned}$$

Thus by Maximum Modulus Theorem $f \in C(\alpha,\beta,\mathcal{E},\lambda)$.

Conversely, assume that

$$\begin{aligned}
&\left|\frac{\frac{z(D^\lambda f(z))''}{(D^\lambda f(z))'}}{2\mathcal{E}\left((1-\alpha)+\frac{z(D^\lambda f(z))''}{(D^\lambda f(z))'}\right)-\frac{z(D^\lambda f(z))''}{(D^\lambda f(z))'}}\right| \\
&=\left|\frac{z(D^\lambda f(z))''}{2\mathcal{E}(1-\alpha)(D^\lambda f(z))'+2\mathcal{E}z(D^\lambda f(z))''-(D^\lambda f(z))''}\right| \\
&=\left|\frac{-\sum\limits_{n=2}^{\infty} n(n-1)B_n(\lambda)a_n z^{n-1}}{2\mathcal{E}(1-\alpha)(1-\sum\limits_{n=2}^{\infty} nB_n(\lambda)a_n z^{n-1})+\sum\limits_{n=2}^{\infty} n(n-1)[1-2\mathcal{E}]B_n(\lambda)a_n z^{n-1}}\right|<\beta.
\end{aligned}$$

Since $|Re(z)|<|z|$ for all z, we have

$$Re\left\{\frac{\sum\limits_{n=2}^{\infty} n(n-1)B_n(\lambda)a_n z^{n-1}}{2\mathcal{E}(1-\alpha)(1-\sum\limits_{n=2}^{\infty} nB_n(\lambda)a_n z^{n-1})+\sum\limits_{n=2}^{\infty} n(n-1)[1-2\mathcal{E}]B_n(\lambda)a_n z^{n-1}}\right\}<\beta. \quad (2.5)$$

We can choose the value of z on the real axis so that $(D^\lambda f(z))'$ is real.

Let $z\to 1^-$, through real values, so we can write (2.5) as

$$\sum_{n=2}^{\infty} n[(n-1)(1-\beta+2\mathcal{E}\beta)+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)a_n \le 2\mathcal{E}\beta(1-\alpha).$$

Theorem 2.4 : Let $f(z)$ be defined by (1.1), then a sufficient condition for $f(z)$ to be in $C(\alpha,\beta,\mathcal{E},\lambda)$ is that

$$\sum_{n=2}^{\infty} n[(n-1)(1+2\mathcal{E}\beta-\beta)+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)a_n \le 2\mathcal{E}\beta(1-\alpha).$$

Proof : In the same line of Theorem 2.3.

Remark : For $\lambda=0$ and $\mathcal{E}=\beta=1$, we get the particular case which is introduced by Silverman [4].

Theorem 2.5 : Let a,b,c and $\alpha,\beta,\mathcal{E},\lambda$ satisfy the following condition such that $T_1(a,b,c,\alpha,\beta,\mathcal{E},\lambda)\le 2\mathcal{E}\beta(1-\alpha)$. $a,b>0, c>a+b+1, \beta\in(0,1], 0\le\alpha<\frac{1}{2\mathcal{E}}, \frac{1}{2}\le\mathcal{E}\le 1, \lambda>-1$ and

$$T_1(a,b,c,\alpha,\beta,\mathcal{E},\lambda)=\sum_{n=2}^{\infty}[(n-1)(1-\beta+2\mathcal{E}\beta)+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

where

$$\frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(a)\Gamma(b)}\sum_{n=2}^{\infty}\left[\frac{(n-1)(1-\beta+2\mathcal{E}\beta)}{2\mathcal{E}\beta(1-\alpha)}+1\right]\frac{\Gamma(\lambda+n)\Gamma(a+n-1)\Gamma(b+n-1)}{\Gamma(c+n-1)[(n-1)!]^2}\leq 1 \tag{2.6}$$

then $zF(a,b;c;z)\in S^*(\alpha,\beta,\mathcal{E},\lambda)$.

Proof :Clearly $zF(a,b;c;z)$ has the series representation of the form (1.1) where $a_n=\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$, hence we have

$$zF(a,b;c;z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n.$$

Hence it suffices to prove that

$$\sum_{n=2}^{\infty}[(n-1)(1-\beta)+2\mathcal{E}\beta(n-\alpha)]B_n(\lambda)a_n\leq 2\mathcal{E}\beta(1-\alpha)\quad \text{(by using Theorem 2.2).}$$

It is easy to see that

$$\begin{aligned}
&\sum_{n=2}^{\infty}[(n-1)(1-\beta)+2\mathcal{E}\beta(n-\alpha)]B_n(\lambda)a_n\\
&=\sum_{n=2}^{\infty}[(n-1)(1-\beta)+2\mathcal{E}\beta(n-\alpha)]B_n(\lambda)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\\
&=\sum_{n=1}^{\infty}[n(1-\beta)+2\mathcal{E}\beta(n+1-\alpha)]B_{n+1}(\lambda)\frac{(a)_n(b)_n}{(c)_n(1)_n}\\
&=\sum_{n=1}^{\infty}[n(1-\beta+2\mathcal{E}\beta)+2\mathcal{E}\beta(1-\alpha)]B_{n+1}(\lambda)\frac{(a)_n(b)_n}{(c)_n(1)_n}\\
&=\sum_{n=2}^{\infty}[(n-1)(1-\beta+2\mathcal{E}\beta)+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}.
\end{aligned}$$

By using hypothesis of the theorem we obtain the required result. Hence, we have (2.6) and hence $zF(a,b;c;z)\in S^*(\alpha,\beta,\mathcal{E},\lambda)$.

For $\beta=\mathcal{E}=1$, we get the next corollary.

Corollary 2.6 : Let a,b,c and α,λ satisfy the following condition such that $T_1(a,b,c,\alpha,1,1,\lambda)\leq 2(1-\alpha)$. $a,b>0, c>a+b+1, 0\leq\alpha<1$, $\lambda>-1$ and

$$T_1(a,b,c,\alpha,\beta,1,1,\lambda)=\sum_{n=2}^{\infty}[2(n-\alpha)]B_n(\lambda)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

where

$$\frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(a)\Gamma(b)}\sum_{n=2}^{\infty}\frac{(n-\alpha)\Gamma(\lambda+n)\Gamma(a+n-1)\Gamma(b+n-1)}{(1-\alpha)\Gamma(c+n-1)[(n-1)!]^2}\leq 1.$$

then $zF(a,b;c;z) \in S^*(\alpha,1,1,\lambda)$.

For $\alpha = 0$ and $\mathcal{E} = \frac{1+\alpha}{2}$ (that is $S^*(0,\beta,\frac{1+\alpha}{2},\lambda)$), we get

Corollary 2.7 : Let a,b,c and β,λ satisfy the following condition such that $T_1(a,b,c,0,\beta,\frac{1+\alpha}{2},\lambda) \le (1+\alpha)\beta$.

$a,b>0, c>a+b+1, \beta \in (0,1], 0 \le \alpha < 1, \lambda > -1$ and

$$T_1(a,b,c,0,\beta,\frac{1+\alpha}{2},\lambda) = \sum_{n=2}^{\infty}[(n-1)(1+\alpha\beta)+(1+\alpha\beta)]B_n(\lambda)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

where

$$\frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(a)\Gamma(b)}\sum_{n=2}^{\infty}\left[\frac{(n-1)(1+\alpha\beta)}{(1+\alpha)\beta}+1\right]\frac{\Gamma(\lambda+n)\Gamma(a+n-1)\Gamma(n+n-1)}{\Gamma(c+n-1)[(n-1)!]^2} \le 1.$$

then $zF(a,b;c;z) \in S^*(0,\beta,\frac{1+\alpha}{2},\lambda)$.

Theorem 2.8 : Let a,b,c and $\alpha,\beta,\mathcal{E},\lambda$ satisfy the following condition such that $T_2(a,b,c,\alpha,\beta,\mathcal{E},\lambda) \le 2\mathcal{E}\beta(1-\alpha)$.

$a,b>0, c>a+b+1, \beta \in (0,1], 0 \le \alpha < \frac{1}{2\mathcal{E}}, \frac{1}{2} \le \mathcal{E} \le 1, \lambda > -1$ and

$$T_2(a,b,c,\alpha,\beta,\mathcal{E},\lambda) = \sum_{n=2}^{\infty}[(n-1)^2(1+2\mathcal{E}\beta-\beta)+(n-1)(1-\beta+2\mathcal{E}\beta(2-\alpha))$$
$$+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)\times\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

where

$$\frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(a)\Gamma(b)}\sum_{n=2}^{\infty}\left[\frac{(n-1)^2(1+2\mathcal{E}\beta-\beta)+(n-1)(1-\beta+2\mathcal{E}\beta(2-\alpha))}{2\mathcal{E}\beta(1-\alpha)}+1\right]$$
$$\times\frac{\Gamma(\lambda+n)\Gamma(a+n-1)\Gamma(b+n-1)}{\Gamma(c+n-1)[(n-1)!]^2} \le 1 \qquad (2.7)$$

then $zF(a,b;c;z) \in C(\alpha,\beta,\mathcal{E},\lambda)$.

Proof :Clearly $zF(a,b;c;z)$ has the series representation of the form (1.1) where $a_n = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$, hence it suffices to prove that

$$\sum_{n=2}^{\infty}n[(n-1)(1+2\mathcal{E}\beta-\beta)+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)a_n \le 2\mathcal{E}\beta(1-\alpha) \quad \text{(by using Theorem 2.4).}$$

It is easy to see that

$$\sum_{n=2}^{\infty} n[(n-1)(1+2\mathcal{E}\beta-\beta)+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)a_n$$
$$=\sum_{n=2}^{\infty} n[(n-1)(1+2\mathcal{E}\beta-\beta)+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$
$$=\sum_{n=0}^{\infty}(n+2)[(n+1)(1+2\mathcal{E}\beta-\beta)+2\mathcal{E}\beta(1-\alpha)]B_{n+2}(\lambda)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$
$$=\sum_{n=0}^{\infty}[(n+1)^2(1+2\mathcal{E}\beta-\beta)+(n+1)(2\mathcal{E}\beta(1-\alpha)+1+2\mathcal{E}\beta-\beta)$$
$$+2\mathcal{E}\beta(1-\alpha)]B_{n+2}(\lambda)\times\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$
$$=\sum_{n=2}^{\infty}[(n-1)^2(1+2\mathcal{E}\beta-\beta)+(n-1)(1-\beta+2\mathcal{E}\beta(2-\alpha))+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)$$
$$\times\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}.$$

By using hypothesis of the theorem we obtain the required result. Hence, we have (2.7) and hence $zF(a,b;c;z)\in C(\alpha,\beta,\mathcal{E},\lambda)$.

For $\beta=\mathcal{E}=1$, we get the next corollary.

Corollary 2.9 : Let a,b,c and α,λ satisfy the following condition such that $T_2(a,b,c,\alpha,1,1,\lambda)\le 2(1-\alpha)$. $a,b>0, c>a+b+1, 0\le\alpha<1, \lambda>-1$ and

$$T_2(a,b,c,\alpha,1,1,\lambda)=\sum_{n=2}^{\infty}2[(n-1)^2+(n-1)(2-\alpha)+(1-\alpha)]B_n(\lambda)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

where

$$\frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(a)\Gamma(b)}\sum_{n=2}^{\infty}\left[\frac{(n-1)((n-1)+(2-\alpha))}{(1-\alpha)}+1\right]\frac{\Gamma(\lambda+n)(a+n-1)\Gamma(b+n-1)}{\Gamma(c+n-1)[(n-1)!]^2}\le 1.$$

then $zF(a,b;c;z)\in C(\alpha,1,1,\lambda)$.

For $\alpha=0$ and $\mathcal{E}=\frac{1+\alpha}{2}$ (that is $c(0,\beta,\frac{1+\alpha}{2},\lambda)$), we get

Corolary 2.10 : Let a,b,c and β,λ satisfy the following condition such that $T_2(a,b,c,0,\beta,\frac{1+\alpha}{2},\lambda)\le(1+\alpha)\beta$. $a,b>0, c>a+b+1, 0<\beta\le 1, 0\le\alpha<1, \lambda>-1$ and

$$T_2(a,b,c,0,\beta,\frac{1+\alpha}{2},\lambda) = \sum_{n=2}^{\infty}[(n-1)^2(1+\alpha\beta)+(n-1)(1+\beta+2\alpha\beta)+(1+\alpha)\beta]B_n(\lambda)\times\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

where

$$\frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(a)\Gamma(b)}\sum_{n=2}^{\infty}\left[\frac{(n-1)[(n-1)(1+\alpha\beta)+(1+\beta+2\alpha\beta)}{(1+\alpha)\beta}+1\right]$$
$$\times\frac{\Gamma(\lambda+n)\Gamma(a+n-1)\Gamma(b+n-1)}{\Gamma(c+n-1)[(n-1)!]^2}\leq 1.$$

then $zF(a,b;c;z)\in C(0,\beta,\frac{1+\alpha}{2},\lambda)$.

Theorem 2.11 : Let a,b,c and $\alpha,\beta,\mathcal{E},\lambda$ satisfy the following condition such that $T_3(a,b,c,\alpha,\beta,\mathcal{E},\lambda)\leq 2\mathcal{E}\beta\left|\frac{c}{ab}\right|(1-\alpha)$. $a,b>-1, c>0, ab<0, \beta\in(0,1], 0\leq\alpha<\frac{1}{2\mathcal{E}}, \frac{1}{2}\leq\mathcal{E}\leq 1, \lambda>-1$ and

$$T_3(a,b,c,\alpha,\beta,\mathcal{E},\lambda)=\sum_{n=2}^{\infty}[(n-1)(1-\beta+2\mathcal{E}\beta)+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

where

$$\left|\frac{ab}{c}\right|\frac{\Gamma(c+1)}{\Gamma(\lambda)\Gamma(a+1)\Gamma(b+1)}\sum_{n=2}^{\infty}\left[\frac{(n-1)(1-\beta+2\mathcal{E}\beta)}{2\mathcal{E}\beta(1-\alpha)}+1\right]$$
$$\times\frac{\Gamma(\lambda+n)\Gamma(a+n-1)\Gamma(b+n-1)}{\Gamma(c+n-1)[(n-1)!]^2}\leq 1. \tag{2.8}$$

then $zF(a,b;c;z)\in S^*(\alpha,\beta,\mathcal{E},\lambda)$.

Proof : From (1.7) and as $ab<0$, we get

$$zF(a,b;c;z)=z-\left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n.$$

From Theorem 2.1, we have to show that

$$\sum_{n=2}^{\infty}[(n-1-\beta(n-1)-2\mathcal{E}\beta(\alpha-n))]B_n(\lambda)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\leq 2\mathcal{E}\beta\left|\frac{c}{ab}\right|(1-\alpha).$$

It is easy to see that

$$\sum_{n=2}^{\infty}[(n-1-\beta(n-1)-2\mathcal{E}\beta(\alpha-n))]B_n(\lambda)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$
$$=\sum_{n=0}^{\infty}[(n+2)(1-\beta+2\beta\mathcal{E})-1+\beta-2\mathcal{E}\beta\alpha]B_{n+2}(\lambda)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$
$$=\sum_{n=0}^{\infty}[(n+1)(1-\beta+2\mathcal{E}\beta)+2\mathcal{E}\beta(1-\alpha)]B_{n+2}(\lambda)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$
$$=\sum_{n=2}^{\infty}[(n-1)(1-\beta+2\mathcal{E}\beta)+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}.$$

By using hypothesis of the theorem we obtain the required result. Hence, we have (2.8) and hence $zF(a,b;c;z) \in S^*(\alpha,\beta,\mathcal{E},\lambda)$.

For $\alpha = 0$ and $\mathcal{E} = \frac{1+\alpha}{2}$, we get

Corollary 2.12 : Let a, b, c and β, λ satisfy the following condition such that $T_3(a,b,c,0,\beta,\frac{1+\alpha}{2},\lambda) \leq |\frac{c}{ab}|(1+\alpha)\beta$. $a, b > -1, c > 0, ab < 0, 0 < \beta \leq 1, 0 \leq \alpha < 1, \lambda > -1$ and

$$T_3(a,b,c,0,\beta,\frac{1+\alpha}{2},\lambda) = \sum_{n=2}^{\infty}[(n-1)(1+\alpha\beta)+(1+\alpha)\beta]B_n(\lambda)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

where

$$\left|\frac{ab}{c}\right|\frac{\Gamma(c+1)}{\Gamma(\lambda)\Gamma(a+1)\Gamma(b+1)}\sum_{n=2}^{\infty}\left[\frac{(n-1)(1+\alpha\beta)}{(1+\alpha)\beta}+1\right]\frac{\Gamma(\lambda+n)\Gamma(a+n-1)\Gamma(b+n-1)}{\Gamma(c+n-1)[(n-1)!]^2} \leq 1.$$

then $zF(a,b;c;z) \in S^*(0,\beta,\frac{1+\alpha}{2},\lambda)$.

Theorem 2.13 : Let a, b, c and $\alpha, \beta, \mathcal{E}, \lambda$ satisfy the following condition such that $T_4(a,b,c,\alpha,\beta,\mathcal{E},\lambda) \leq 2\mathcal{E}\beta\frac{c}{ab}(1-\alpha)$. $a, b > -1, c > 0, ab < 0, \beta \in (0,1], 0 \leq \alpha < \frac{1}{2\mathcal{E}}, \frac{1}{2} \leq \mathcal{E} \leq 1, \lambda > -1$ and

$$T_4(a,b,c,\alpha,\beta,\mathcal{E},\lambda) = \sum_{n=2}^{\infty}[(n-1)^2(1+2\mathcal{E}\beta-\beta)+(n-1)(1-\beta+2\mathcal{E}\beta(2-\alpha))$$
$$+2\mathcal{E}(1-\alpha)]B_n(\lambda)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

where

$$\frac{|ab|}{c}\frac{\Gamma(c+1)}{\Gamma(\lambda)\Gamma(a+1)\Gamma(b+1)}\sum_{n=2}^{\infty}\left[\frac{(n-1)^2(1+2\mathcal{E}\beta-\beta)+(n-1)(1-\beta+2\mathcal{E}\beta(2-\alpha)}{2\mathcal{E}\beta(2-\alpha)}+1\right]$$
$$\times\frac{\Gamma(\lambda+n)\Gamma(a+n-1)\Gamma(b+n-1)}{\Gamma(c+n-1)[(n-1)!]^2} \leq 1 \qquad (2.9)$$

then $zF(a,b;c;z) \in C(\alpha,\beta,\mathcal{E},\lambda)$.

Proof : By using (1.7) and Theorem 2.3, we must show that

$$\sum_{n=2}^{\infty}n[(n-1)(1+2\mathcal{E}\beta-\beta)+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$
$$\leq 2\frac{c}{|ab|}\mathcal{E}\beta(1-\alpha).$$

It is easy to see that

$$\sum_{n=2}^{\infty} n[(n-1)(1+2\mathcal{E}\beta-\beta)+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$=\sum_{n=0}^{\infty}[(n+1)^2(1+2\mathcal{E}\beta-\beta)+(n+1)(1-\beta+2\mathcal{E}\beta(2-\alpha))+2\mathcal{E}\beta(1-\alpha)]B_{n+2}(\lambda)\times$$

$$\times\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$=\sum_{n=2}^{\infty}[(n-1)^2(1+2\mathcal{E}\beta-\beta)+(n-1)(1-\beta+2\mathcal{E}\beta(2-\alpha))+2\mathcal{E}\beta(1-\alpha)]B_n(\lambda)$$

$$\times\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}.$$

By using hypothesis of the theorem we obtain the required result. Hence we have (2.9) and hence $zF(a,b;c;z)\in C(\alpha,\beta,\mathcal{E},\lambda)$.

For $\mathcal{E}=\beta=1$, we get

Corollary 2.14 : Let a,b,c and α,λ satisfy the following condition such that $T_4(a,b,c,\alpha,1,1,\lambda)\le 2\frac{c}{|ab|}(1-\alpha)$. $a,b>-1, c>0, ab<0, 0\le\alpha<1, \lambda>-1$ and

$$T_4(a,b,c,\alpha,1,1,\lambda)=\sum_{n=2}^{\infty}2[(n-1)^2+(n-1)(2-\alpha)+(1-\alpha)]B_n(\lambda)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

where

$$\frac{|ab|}{c}\frac{\Gamma(c+1)}{\Gamma(\lambda)\Gamma(a+1)\Gamma(b+1)}\sum_{n=2}^{\infty}\left[\frac{(n-1)(n-\alpha+1)}{(1-\alpha)}+1\right]\frac{\Gamma(\lambda+n)\Gamma(a+n-1)\Gamma(b+n-1)}{\Gamma(c+n-1)[(n-1)!]^2}\le 1.$$

then $zF(a,b;c;z)\in C(\alpha,1,1,\lambda)$.

For $\alpha=0$ and $\mathcal{E}=\frac{1+\alpha}{2}$, we get

Corollary 2.15 : Let a,b,c and β,λ satisfy the following condition such that $T_4(a,b,c,0,\beta,\frac{1+\alpha}{2},\lambda)\le\frac{c}{|ab|}(1+\alpha)\beta$. $a,b>-1, c>0, ab<0, 0<\beta\le 1, 0\le\alpha<1, \lambda>-1$ and

$$T_4(a,b,c,0,\beta,\frac{1+\alpha}{2},\lambda)=\sum_{n=2}^{\infty}2[(n-1)^2(1+\alpha\beta)+(n-1)(1+\beta+2\alpha\beta)+(1+\alpha)\beta]B_n(\lambda)\times$$

$$\times\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

where

$$\frac{|ab|}{c}\frac{\Gamma(c+1)}{\Gamma(\lambda)\Gamma(a+1)\Gamma(b+1)}\sum_{n=2}^{\infty}\left[\frac{(n-1)((n-1)(1+\alpha\beta)+(1+\beta+2\alpha\beta))}{(1+\alpha)\beta}+1\right]$$

$$\times\frac{\Gamma(\lambda+n)\Gamma(a+n-1)\Gamma(b+n-1)}{\Gamma(c+n-1)[(n-1)!]^2}\le 1.$$

then $zF(a,b;c;z) \in C(0,\beta,\frac{1+\alpha}{2},\lambda)$.

REFERENCES

[1] E. S. Aqlan, Some problems connected with Geometric Function theory, Ph.D. Thesis (2004), Pune university, Pune (unpublished).

[2] S. R. Kulkarni, Some problems connected with univalent functions, Ph.D. thesis (1981), Shivaji University, Kolhapur (unpublished).

[3] S. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975), 109-115.

[4] H. Silverman, Starlike and convexity properties for hypergeometric functions, J. Math. Anal. Appl. 172, 574-581, (1993).

[5] _______________, Subclasses of starlike functions, Rev. Roumaine Math. Pures Appl. 3 (1978), 1093-1099.

[6] _______________, Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51 (1975), 109-116.

[7] H. M. Srivastava and S. Owa (Eds.), Current topics in Analytic Function Theory, World Scientific, Singapore, 1992.

[8] _____________, A note on certain subclasses of starlike functions, Rend. Sem. Math. Univ. Padova 77 (1987), 115-124.

Received: May, 2009