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A GENERALIZATION OF A CLASS OF p-VALENT HARMONIC FUNCTIONS INVOLVING A GENERALIZED RUSCHEWEYH TYPE OPERATOR

WAGGAS GALIB ATSHAN[1] AND RAFID HABIB BUTI[2]

Abstract. We introduce a generalization of a class of p-valent functions involving a generalized Ruscheweyh type operator of the form $f = h + \overline{g}$. We give next a characterization of this class by the means of coefficients and some of its properties concerning distorsion bounds, extreme points, convex combination.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply - connected domain $D \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f.

A necessary and sufficient condition for f to be locally univalent and sense - preserving in D is that $|h'(z)| > |g'(z)|$ in D.

Assume $\mathcal{H} = \{f : f = h + \overline{g}\}$, f is harmonic univalent and sense - preserving in open unit disk $U = \{z : |z| < 1\}$. So $f = h + \overline{g} \in \mathcal{H}$ is normalized by $f(0) = h(0) = f_z(0) = 1 = 0$.

Ahuja and Jahangiri [1] defined the class $\mathcal{H}(p)$ $(p \in \mathbb{N} = \{1, 2, 3, \cdots\})$ consisting of all p-valent harmonic functions $f = h + \overline{g}$ that are sense - preserving, and h, g are of the form:

$$h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \; g(z) = \sum_{n=p}^{\infty} b_n z^n, \; |b_p| < 1. \tag{1}$$

Let $R(p)$ denote the subclass of $\mathcal{H}(p)$ consisting of functions $f = h + \overline{g}$, where h and g are given by

$$h(z) = z^p - \sum_{n=p+1}^{\infty} |a_n| z^n, \; g(z) = -\sum_{n=p}^{\infty} |b_n| z^n, \; |b_p| < 1. \tag{2}$$

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We introduce here a class $\mathcal{H}_\lambda^k(p,\alpha,\beta,m)$ of harmonic functions of the form (1) that satisfy the inequality :

$$Re\left\{(1-\beta)\frac{D_\lambda^{k+p-1}f(z)}{z^p}+\beta(1-m)\frac{(D_\lambda^{k+p-1}f(z))'}{pz^{p-1}}+\beta m\frac{(D_\lambda^{k+p-1}f(z))''}{p(p-1)z^{p-2}}\right\}\geq\frac{\alpha}{p}, \tag{3}$$

where $0\leq\alpha<p, p\in\mathbb{N}=\{1,2,\cdots\},\ \beta\geq 0,\lambda\geq 0,k\in\mathbb{N}_0, 0\leq m\leq 1$ and

$$D_\lambda^{k+p-1}f(z)=D_\lambda^{k+p-1}h(z)+\overline{D_\lambda^{k+p-1}g(z)}. \tag{4}$$

The operator D_λ^{k+p-1} denotes the generalized Ruscheweyh derivative operator introduced in [2]. For h and g given by (1), we obtain

$$D_\lambda^{k+p-1}h(z)=z^p+\sum_{n=p+1}^{\infty}(1+\lambda(n-p))C(k,n,p)a_nz^n, \tag{5}$$

$$D_\lambda^{k+p-1}g(z)=\sum_{n=p}^{\infty}(1+\lambda(n-p))C(k,n,p)b_nz^n, \tag{6}$$

where

$$\lambda\geq 0,p\in\mathbb{N},\ k>-p \text{ and } C(k,n,p)=\begin{pmatrix} n+k-1\\ k+p-1\end{pmatrix}. \tag{7}$$

We deem it worthwhile to point here the relevance of the class $\mathcal{H}_\lambda^k(p,\alpha,\beta,m)$ with those classes which have been studied recently. Indeed, we observe that:

- **:** (i) $\mathcal{H}_0^0(1,\alpha,1,0)\equiv N_{\mathcal{H}}(\alpha)$ (Ahuja and Jahangiri [1]);
- **:** (ii) $\mathcal{H}_\lambda^k(p,\alpha,1,0)\equiv\mathcal{H}_\lambda^k(p,\alpha)$ (Al Shaqsi and Darus [2]);
- **:** (iii) $\mathcal{H}_\lambda^k(1,0,1,0)\equiv\mathcal{H}_\lambda^k$ (Darus and Al Shaqsi [5]);
- **:** (iv) $\mathcal{H}_0^0(1,0,1,0)\equiv S_{\mathcal{H}}^*$ Silverman[7];
- **:** (v) $\mathcal{H}_\lambda^0(1,0,1,0)\equiv\mathcal{H}P(0)$ (Yalcin and Öztürk [8]);
- **:** (vi) $\mathcal{H}_\lambda^k(p,\alpha,\beta,0)\equiv\mathcal{H}_\lambda^k(p,\alpha,\beta)$ (Atshan et al.[3]).

Also, we note that the analytic part of the class $\mathcal{H}_0^k(p,\alpha,1,0)$ was introduced and studied by Goel and Sohi [6].

We further denote by $R_\lambda^k(p,\alpha,\beta,m)$ the subclass of $\mathcal{H}_\lambda^k(p,\alpha,\beta,m)$ that satisfies the relation:

$$R_\lambda^k(p,\alpha,\beta,m)=R(p)\cap\mathcal{H}_\lambda^k(p,\alpha,\beta,m). \tag{8}$$

2. Coefficient Bounds

Here, we obtain the sufficient condition for $f=h+\overline{g}$ to be in the class $\mathcal{H}_\lambda^k(p,\alpha,\beta,m)$.

Theorem 2.1. *Let $f=h+\overline{g}$ (h and g being given by (1)). If*

$$\sum_{n=p+1}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|a_n|$$
$$+\sum_{n=p}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|b_n|\leq p-\alpha, \tag{9}$$

where $\lambda\geq 0,\beta\geq 0, 0\leq\alpha<p,p\in\mathbb{N}, 0\leq m\leq 1$ and $k\in\mathbb{N}_0$, then f is harmonic p-valent sense-preserving in U and $f\in\mathcal{H}_\lambda^k(p,\alpha,\beta,m)$.

Proof. Let

$$w(z) = (1-\beta)\frac{D_\lambda^{k+p-1}f(z)}{z^p} + \beta(1-m)\frac{(D_\lambda^{k+p-1}f(z))'}{pz^{p-1}} + \beta m\frac{(D_\lambda^{k+p-1}f(z))''}{p(p-1)z^{p-2}}.$$

To prove that $Re\{w\} \geq \frac{\alpha}{p}$, it is sufficient to show equivalently that $|p-\alpha+pw(z)| \geq |p+\alpha-pw(z)|$. Substituting for $w(z)$ and making use of (4) to (6), and resorting to simple calculations, we find that

$$|p-\alpha+pw(z)| \geq 2p-\alpha-\sum_{n=p+1}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right]$$
$$(1+\lambda(n-p))C(k,n,p)|a_n||z^{n-p}| - \sum_{n=p}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right]$$
$$(1+\lambda(n-p))C(k,n,p)|b_n||z^{n-p}| \ (10)$$

and

$$|p+\alpha-pw(z)| \leq \alpha+\sum_{n=p}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right]$$
$$(1+\lambda(n-p))C(k,n,p)|a_n||z^{n-p}| + \sum_{n=p}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right]$$
$$(1+\lambda(n-p))C(k,n,p)|b_n||z^{n-p}| \ (11)$$

where $C(k,n,p)$ is given by (7). Evidently, (10) and (11) in conjunction with (9) yields

$$|p-\alpha+pw(z)| - |p+\alpha-pw(z)| \geq 0.$$

The harmonic functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty}\frac{x_n}{\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)}z^n$$
$$+\sum_{n=p}^{\infty}\frac{\overline{y}_n}{\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)}(\overline{z})^n \ (12)$$

$\sum\limits_{n=p+1}^{\infty}|x_n| + \sum\limits_{n=p}^{\infty}|y_n| = p-\alpha$ show that the coefficient bound given by (9) is sharp.

The functions of the form (12) are in $\mathcal{H}_{\lambda}^{k}(p,\alpha,\beta,m)$ because in view of (9), we infer that

$$\sum_{n=p+1}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|a_n|$$
$$+\sum_{n=p}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|b_n|$$
$$=\sum_{n=p+1}^{\infty}|x_n|+\sum_{n=p}^{\infty}|y_n|=p-\alpha.$$

□

The restriction imposed in Theorem 2.1 on the moduli of the coefficients of $f=h+\overline{g}$ implies that for arbitrary rotation of the coefficients of f, the resulting functions would still be harmonic p-valent and $f\in\mathcal{H}_{\lambda}^{k}(p,\alpha,\beta,m)$.

The following theorem shows the condition (9) is also necessary for function f to belong to $R_{\lambda}^{k}(p,\alpha,\beta,m)$.

Theorem 2.2. *Let $f=h+\overline{g}$ with h and g are given by (2). Then $f\in R_{\lambda}^{k}(p,\alpha,\beta,m)$ if and only if*

$$\sum_{n=p+1}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|a_n|$$
$$+\sum_{n=p}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|b_n|\le p-\alpha, \quad (13)$$

where $\lambda\ge0,\beta\ge0,0\le\alpha<p,p\in\mathbb{N},\ 0\le m\le1$ and $k\in\mathbb{N}_0$.

Proof. By noting that $R_{\lambda}^{k}(p,\alpha,\beta,m)\subset\mathcal{H}_{\lambda}^{k}(p,\alpha,\beta,m)$, the sufficiency part of Theorem 2.2 follows at once from Theorem 2.1. To prove the necessary part, let us assume that $f\in R_{\lambda}^{k}(p,\alpha,\beta,m)$. Using (3), we get

$$Re\left\{(1-\beta)\left(\frac{D_{\lambda}^{k+p-1}h(z)+\overline{D_{\lambda}^{k+p-1}g(z)}}{z^p}\right)+\beta(1-m)\right.$$
$$\left.\left(\frac{(D_{\lambda}^{k+p-1}h(z))'+\overline{(D_{\lambda}^{k+p-1}g(z))'}}{pz^{p-1}}\right)+\beta m\left(\frac{(D_{\lambda}^{k+p-1}h(z))''+\overline{(D_{\lambda}^{k+p-1}g(z))''}}{p(p-1)z^{p-2}}\right)\right\}$$
$$=Re\left\{1-\sum_{n=p+1}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p(p-1)}+(1-\beta)\right]\right.$$
$$(1+\lambda(n-p))C(k,n,p)|a_n|z^{n-p}-\sum_{n=p}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p(p-1)}+(1-\beta)\right]$$
$$\left.(1+\lambda(n-p))C(k,n,p)|b_n|(\overline{z})^{n-p}\right\}\ge\frac{\alpha}{p}.$$

If we choose z to be real and let $z \to 1^-$, we obtain

$$1-\sum_{n=p+1}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p(p-1)}+(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|a_n|$$
$$-\sum_{n=p}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p(p-1)}+(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|b_n| \geq \frac{\alpha}{p}.$$

Hence

$$\sum_{n=p+1}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|a_n|$$
$$+\sum_{n=p}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|b_n| \leq p-\alpha,$$

which completes the proof of Theorem 2.2. □

3. Distortion Bounds and Extreme Points

Here, we obtain the distortion bounds for functions belonging to the class $R_\lambda^k(p,\alpha,\beta,m)$ and also provide extreme points for the class $R_\lambda^k(p,\alpha,\beta,m)$.

Theorem 3.1. *Let $f \in R_\lambda^k(p,\alpha,\beta,m)$ for $\lambda \geq 0, \beta \geq 0, 0 \leq \alpha < p, p \in \mathbb{N}, 0 \leq m \leq 1, k \in \mathbb{N}_0$ and $|z| = r < 1$. Then*

$$|f(z)| \leq (1+|b_p|)r^p + \frac{(p-\alpha)-p|b_p|}{\left[\frac{(p+1)\beta(p+m-1)}{p-1}+p(1-\beta)\right](1+\lambda)(p+k)}r^{p+1}, \tag{14}$$

and

$$|f(z)| \geq (1-|b_p|)r^p - \frac{(p-\alpha)-p|b_p|}{\left[\frac{(p+1)\beta(p+m-1)}{p-1}+p(1-\beta)\right](1+\lambda)(p+k)}r^{p+1}. \tag{15}$$

Proof. We only prove the first inequality (14). The proof for the second inequality (15) is similar, and is hence omitted.

Assume $f \in R^k_\lambda(p,\alpha,\beta,m)$. Using (1) and (9) of Theorem 2.1, we find that

$$|f(z)| \leq (1+|b_p|)r^p + \sum_{n=p+1}^{\infty}(|a_n|+|b_n|)r^n$$

$$\leq (1+|b_p|)r^p + r^{p+1}\sum_{n=p+1}^{\infty}(|a_n|+|b_n|)$$

$$= (1+|b_p|)r^p + \frac{1}{\left[\frac{(p+1)\beta(p+m-1)}{p-1}+p(1-\beta)\right](1+\lambda)(p+k)}$$

$$\sum_{n=p+1}^{\infty}\left[\frac{(p+1)\beta(p+m-1)}{p-1}+p(1-\beta)\right](1+\lambda)(p+k)(|a_n|+|b_n|)r^{p+1}$$

$$\leq (1+|b_p|)r^p + \frac{1}{\left[\frac{(p+1)\beta(p+m-1)}{p-1}+p(1-\beta)\right](1+\lambda)(p+k)}$$

$$\sum_{n=p+1}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)(|a_n|+|b_n|)r^{p+1}$$

$$\leq (1+|b_p|)r^p + \frac{1}{\left[\frac{(p+1)\beta(p+m-1)}{p-1}+p(1-\beta)\right](1+\lambda)(p+k)}[(p-\alpha)-p|b_p|]r^{p+1}.$$

□

The bounds given in Theorem 3.1 (for the functions $f = h+\overline{g}$ of the form (2)) also hold for functions of the form (1) if the coefficient condition (9) is satisfied.

The functions

$$f(z) = z^p + |b_p|(\overline{z})^p + \frac{(p-\alpha)-p|b_p|}{\left[\frac{(p+1)\beta(p+m-1)}{p-1}+p(1-\beta)\right](1+\lambda)(p+k)}(\overline{z})^{p+1} \tag{16}$$

and

$$f(z) = z^p - |b_p|(\overline{z})^p - \frac{(p-\alpha)-p|b_p|}{\left[\frac{(p+1)\beta(p+m-1)}{p-1}+p(1-\beta)\right](1+\lambda)(p+k)}(\overline{z})^{p+1} \tag{17}$$

for $|b_p| < 1$ show that the bounds given by Theorem 3.1 are sharp.

The covering result given below in Corollary 3.2 follows from the inequality (15) of Theorem 3.1.

Corollary 3.2. *Let $f \in R^k_\lambda(p,\alpha,\beta,m)$. Then*

$$\left\{w : |w| < (1-|b_p|) - \frac{(p-\alpha)-p|b_p|}{\left[\frac{(p+1)\beta(p+m-1)}{p-1}+p(1-\beta)\right](1+\lambda)(p+k)}\right\} \subset f(U). \tag{18}$$

The next theorem gives the extreme points of the closed convex hulls of $R^k_\lambda(p,\alpha,\beta,m)$, denoted by $R^k_\lambda(p,\alpha,\beta,m)$.

Theorem 3.3. *$f \in clco\ R^k_\lambda(p,\alpha,\beta,m)$ if and only if*

$$f(z) = \sum_{n=p}^{\infty}(\sigma_n h_n + \epsilon_n g_n) \tag{19}$$

where $z \in U, h_p(z) = z^p$,

$$h_n(z) = z^p - \frac{p-\alpha}{\left[\frac{n\beta(p+m(n-p)-1)}{p-1} + p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)} z^n, \tag{20}$$

$(n = p+1, p+2, \cdots)$

$$g_n(z) = z^p - \frac{p-\alpha}{\left[\frac{n\beta(p+m(n-p)-1)}{p-1} + p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)} (\overline{z})^n, \tag{21}$$

$(n = p, p+1, \cdots)$ *and* $\sum\limits_{n=p}^{\infty}(\sigma_n + \epsilon_n) = 1$ $(\sigma_n \geq 0,\ \epsilon_n \geq 0)$.

In particular, the extreme points of $R^k_\lambda(p,\alpha,\beta,m)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Suppose f is of the form (19). Using (20) and (21), we get

$$\begin{aligned}
f(z) &= \sum_{n=p}^{\infty}(\sigma_n h_n + \epsilon_n g_n) \\
&= \sum_{n=p}^{\infty}(\sigma_n + \epsilon_n) z^p \\
&- \sum_{n=p+1}^{\infty} \frac{p-\alpha}{\left[\frac{n\beta(p+m(n-p)-1)}{p-1} + p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)} \sigma_n z^n \\
&- \sum_{n=p}^{\infty} \frac{p-\alpha}{\left[\frac{n\beta(p+m(n-p)-1)}{p-1} + p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)} \epsilon_n (\overline{z})^n \\
&= z^p - \sum_{n=p+1}^{\infty} \frac{p-\alpha}{\left[\frac{n\beta(p+m(n-p)-1)}{p-1} + p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)} \sigma_n z^n \\
&- \sum_{n=p}^{\infty} \frac{p-\alpha}{\left[\frac{n\beta(p+m(n-p)-1)}{p-1} + p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)} \epsilon_n (\overline{z})^n.
\end{aligned}$$

Then

$$\sum_{n=p+1}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)$$
$$\frac{p-\alpha}{\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)}\sigma_n$$
$$+\sum_{n=p}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)$$
$$\frac{p-\alpha}{\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)}\epsilon_n$$
$$=(p-\alpha)\left(\sum_{n=p}^{\infty}(\sigma_n+\epsilon_n)-\sigma_p\right)=(p-\alpha)(1-\sigma_p)\leq p-\alpha,$$

which implies that $f\in R_\lambda^k(p,\alpha,\beta,m)$.

Conversely, assume that $f\in R_\lambda^k(p,\alpha,\beta,m)$. Putting

$$\sigma_n=\frac{\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)}{p-\alpha}|a_n|,(n=p+1,p+2,\cdots),$$

$$\epsilon_n=\frac{\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)}{p-\alpha}|b_n|,(n=p,p+1,p+2,\cdots),$$

we get

$$f(z)=\sum_{n=p}^{\infty}(\sigma_n h_n+\epsilon_n g_n),$$

and this completes the proof of Theorem 3.3. □

4. Convex Combination

In the following theorem, we determine the convex combination properties of the members of $R_\lambda^k(p,\alpha,\beta,m)$.

Theorem 4.1. *The class $R_\lambda^k(p,\alpha,\beta,m)$ is closed under convex combination.*

Proof. Suppose $f_i\in R_\lambda^k(p,\alpha,\beta,m)(i=1,2,3,\cdots)$, are defined by

$$f_i(z)=z^p-\sum_{n=p+1}^{\infty}|a_{n,i}|z^n-\sum_{n=p}^{\infty}|b_{n,i}|(\overline{z})^n.$$

From (13), we have

$$\sum_{n=p+1}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|a_{n,i}|$$
$$+\sum_{n=p}^{\infty}\left[\frac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|b_{n,i}|$$
$$\leq p-\alpha. \tag{22}$$

For $\sum\limits_{i=1}^{\infty}\sigma_i=1, 0\leq\sigma_i\leq 1$, we can write the convex combination of f_i as

$$\sum_{i=1}^{\infty}\sigma_i f_i(z)=z^p-\sum_{n=p+1}^{\infty}\left[\sum_{i=1}^{\infty}\sigma_i|a_{n,i}|\right]z^n-\sum_{n=p}^{\infty}\left[\sum_{i=1}^{\infty}\sigma_i|b_{n,i}|\right](\overline{z})^n,$$

from (22) we have

$$\sum_{n=p+1}^{\infty}\left[\tfrac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)\left(\sum_{i=1}^{\infty}\sigma_i|a_{n,i}|\right)$$
$$+\sum_{n=p}^{\infty}\left[\tfrac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)\left(\sum_{i=1}^{\infty}\sigma_i|b_{n,i}|\right)$$
$$=\sum_{i=1}^{\infty}\sigma_i\left\{\sum_{n=p+1}^{\infty}\left[\tfrac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|a_{n,i}|\right.$$
$$\left.+\sum_{n=p}^{\infty}\left[\tfrac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|b_{n,i}|\right\}$$
$$\leq\sum_{i=1}^{\infty}\sigma_i(p-\alpha)=p-\alpha.$$

Then $\sum\limits_{i=1}^{\infty}\sigma_i f_i(z)\in R_{\lambda}^{k}(p,\alpha,\beta,m)$. □

Theorem 4.2. *The class $R_{\lambda}^{k}(p,\alpha,\beta,m)$ is a convex set.*

Proof. Let f_1, f_2 be the arbitrary elements of $R_{\lambda}^{k}(p,\alpha,\beta,m)$. Then for every t $(0<t<1)$, we show that $(1-t)f_1+tf_2\in R_{\lambda}^{k}(p,\alpha,\beta,m)$, thus we have

$$(1-t)f_1(z)+tf_2(z)=z^p-\sum_{n=p+1}^{\infty}[(1-t)|a_{n,1}|+t|a_{n,2}|]z^n-\sum_{n=p}^{\infty}[(1-t)|b_{n,1}|+t|b_{n,2}|](\overline{z})^n$$

and

$$\sum_{n=p+1}^{\infty}\left[\tfrac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)[(1-t)|a_{n,1}|+t|a_{n,2}|]$$
$$+\sum_{n=p}^{\infty}\left[\tfrac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)[(1-t)|b_{n,1}|+t|b_{n,2}|]$$
$$=(1-t)\left[\sum_{n=p+1}^{\infty}\left[\tfrac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))\right.$$
$$C(k,n,p)|a_{n,1}|+\sum_{n=p}^{\infty}\left[\tfrac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right]$$
$$(1+\lambda(n-p))C(k,n,p)|b_{n,1}|]$$
$$+t\left[\sum_{n=p+1}^{\infty}\left[\tfrac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|a_{n,2}|\right.$$
$$\left.+\sum_{n=p+1}^{\infty}\left[\tfrac{n\beta(p+m(n-p)-1)}{p-1}+p(1-\beta)\right](1+\lambda(n-p))C(k,n,p)|b_{n,2}|\right]$$
$$\leq(1-t)(p-\alpha)+t(p-\alpha)=p-\alpha.$$
□

References

[1] O. Ahuja and J. Jahangiri, Noshiro - type harmonic univalent functions, Sci. Math. Japan, 56 (2002), 293-299.

[2] K. Al Shaqsi and M. Darus, A new class of multivalent harmonic functions, General Math., 14 (2006), 37-46.

[3] W. G. Atshan , S. R. Kulkarni and R. K. Rania, A class of multivalent harmonic functions involving a generalized Ruscheweyh Type operator, Matematicki Vesnik, 60(3)(2008), 207-213.

[4] J. Clunie and T. Sheil - Small, Harmonic univalent functions, Ann, Acad. Aci. Fenn. Ser. AI Math., 9(1984), 3 -25.

[5] M. Darus and K. Al Shaqsi, On harmonic univalent functions defined by a generalized Ruscheweyh derivatives operator, Labachevskii J. Math., 22 (2006), 19-26.

[6] R. Goel and N. Sohi, New criteria for p-valence, Indian J. pure Appl. Math., 7 (2004), 55-61.

[7] H. Silverman, Harmonic univalent functions with negative coefficient, Proc. Amer. Math. Soc., 51 (1998), 283-289.

[8] S. Yalçin and M. Öztürk, A new subclass of complex harmonic functions, Math. Inequal. Appl., 7 (2004), 55-61.

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NOTE FROM THE EDITORS

APOLOGY ON PLAGIARISM PAPER

It has come to the attention of the Editors of An. Univ. Oradea, Fasc. Mat. by Professor Francisco J. Aragon ARTACHO, from "Departamento Estadística e Investigación Operativa, Universidad de Alicante, Spain", that the paper entitled "Convergence of the proximal point algorithm variational inequalities with regular mappings", by Corina L. CHIRIAC, appeared in An. Univ. Oradea, Fasc. Mat., vol. 17(2010), No. 1, 65-69, is a plagiarism of the paper [F. J. Aragon Artacho, A.L. Dontchev and M.H. Geoffroy, Convergence of the proximal point method for metrically regular mappings, in : CSVAA 2004-ESAIM Proceedings, 17(2007), 1-8 ; MR2362687 (2008j:90164)].

A review pointing out accordingly this plagiarism, was submitted by Professor ARTACHO to Mathematical Reviews and appeared under the number MR2676410.

Also, we informed the Editorial Office of Zentralblatt fur Mathematik about this plagiarism (see Zbl pre05730216).

We apologize to the authors J. Aragon Artacho, A.L. Dontchev and M.H. Geoffroy, to the readers and to the international mathematical community for this awkward situation.

Because it is the most severe violation of the scientific publishing practice, we have decided to ban Corina L. CHIRIAC from publishing in our journal.

THE EDITORS

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