A subclass of multivalent analytic functions with fixed argument of coefficients involving Hohlov operator

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Abstract

The main purpose of this paper is to drive some important results for a subclass of analytic functions with fixed argument of its coefficients involving Hohlov operator , such as coefficients estimates ,quasi Hadamard product and integral means for functions belonging to the defined class are obtained. **Keywords: multivalent function, Hohlov operator, subordination, quasi Hadamard product , integral means.**

1 Introduction

Let M_p denote the class of functions of the form :

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad ,(a_n \in C, p \in N = 1, 2, ...), \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in C \quad and \quad |z| < 1\}$ and its subclass is denoted by M^{ϑ} whose members are of the form :

$$f(z) = z^p + e^{i\vartheta} \sum_{n=p+1}^{\infty} |a_n| \, z^n \qquad ,(p \in N) \tag{2}$$

where ϑ is the fixed argument of $a_n \neq 0$
A function $f \in M_p$ is called multivalent starlike of order $\alpha \quad (0 \leq \alpha < p)$ if f is satisfies the conditions

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad (z \in U) \tag{3}$$

and

$$\int_0^{2\pi} Re\left\{\frac{zf'(z)}{f(z)}\right\} d\vartheta = 2p\pi, \qquad (z \in U) \tag{4}$$

we denote by $S(p, \alpha)$ the class of multivalent starlike functions of order α. Also a function $f \in M_p$ is called multivalent convex of order $\alpha(0 \leq \alpha < p)$ if f is satisfies the following conditions

$$Re\left\{\frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad (z \in U) \tag{5}$$

and

$$\int_0^{2\pi} Re\left\{\frac{zf''(z)}{f'(z)}\right\} d\vartheta = 2p\pi, \qquad (z \in U) \tag{6}$$

we denote by $K(p,\alpha)$ the class of multivalent convex functions of order α.
Definition1.([4]) The Gaussian hypergeometric function denoted by ${}_2F_1$ and is defined by

$$ {}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n, \qquad |z|<1 $$

where $(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$, $c>b>0$. It is well known (see[1]) that under the conditions $c>b>0$ and $c>a+b$, we have

$$ \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{7} $$

Definition 2. Let $f\in M^{\vartheta}$ be of the form (2) , then the Hohlov operator $F(a,b,c)$ is defined by means of Hadamard product below

$$ \begin{aligned} F(a,b,c)f(z) &= z^p \quad {}_2F_1(a,b;c;z)*f(z) \\ &= z^p+e^{i\vartheta}\sum_{n=p+1}^{\infty}\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(n-p)!}\left|a_n\right|z^n, \end{aligned} \tag{8} $$

$(a,b,c\in N_0, c\neq Z_0^-=\{0,-1,-2,\ldots\}\,; z\in U)$. The same operator have been studied by W. G. Atshan on a class of univalent functions in [3].
Definition 3. The Hadamard product of the two functions $F(a,b,c)f$ given by (8) and

$$ g(z)=z^p+e^{i\vartheta}\sum_{n=p+1}^{\infty}\left|b_n\right|z^n \tag{9} $$

is defined by

$$ (g*F(a,b,c)(f))(z)=z^p+e^{i\vartheta}\sum_{n=p+1}^{\infty}\Gamma_n\left|a_n\right|\left|b_n\right|z^n, $$

where

$$ \Gamma_n=\frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(n-p)!}. \tag{10} $$

Definition 4. A function f in M^{ϑ} is in the class $M^{\vartheta}(\beta,p,n)$ if satisfies the condition

$$ \left|\frac{z(g*F(a,b,c)(f))\prime(z)-p(g*F(a,b,c)(f))(z)}{z(g*F(a,b,c)(f))\prime(z)-(p+2(p-\beta))(g*F(a,b,c)(f))(z)}\right|<\gamma, \tag{11} $$

2 Main result

Theorem 1 *A function $f\in M^{\vartheta}$ of the form (2) belong to $M^{\vartheta}(\beta,p,n)$ if and only if*

$$ \sum_{n=p+1}^{\infty}[(n-p)(1-\gamma)+2\gamma(p-\beta)]\Gamma_n\left|a_n\right|\left|b_n\right|\le 2\gamma(p-\beta), \tag{12} $$

where Γ_n is defined by (10) and $0\le\beta<p, 0<\gamma\le 1, z\in U$.

Proof For $|z|=1$, we need to show only sufficient condition for $f\in M^{\vartheta}(\beta,p,n)$ consider

$$ \left|e^{i\vartheta}\sum_{n=p+1}^{\infty}(n-p)\Gamma_n\left|a_n\right|\left|b_n\right|z^n\right|-\gamma\left|-2(p-\beta)z^p-e^{i\vartheta}\sum_{n=p+1}^{\infty}(3p-n-2\beta)\Gamma_n\left|a_n\right|\left|b_n\right|z^n\right| $$

$$\leq \sum_{n=p+1}^{\infty}(n-p)\Gamma_n |a_n| |b_n| - \left[2\gamma(p-\beta) - \gamma\sum_{n=p+1}^{\infty}(3p-n-2\beta)\Gamma_n |a_n| |b_n|\right]$$

$$\leq \sum_{n=p+1}^{\infty}[(n-p)(1-\gamma)+2\gamma(p-\beta)]\Gamma_n |a_n| |b_n| - 2\gamma(p-\beta) \leq 0,$$

if (12) holds and by maximum modulus principle which proves $f \in M^{\vartheta}(\beta,p,n)$.
Now , suppose that $f \in M^{\vartheta}(\beta,p,n)$ so that

$$\left|\frac{z(g*F(a,b,c)(f))\prime(z) - p(g*F(a,b,c)(f))(z)}{z(g*F(a,b,c)(f))\prime(z) - (p+2(p-\beta))(g*F(a,b,c)(f))(z)}\right| < \gamma, \qquad (z \in U)$$

then

$$|z(g*F(a,b,c)(f))\prime(z) - p(g*F(a,b,c)(f))(z)|$$

$$< \gamma\,|z(g*F(a,b,c)(f))\prime(z) - (p+2(p-\beta))(g*F(a,b,c)(f))(z)|\,,$$

we get

$$\left|e^{i\vartheta}\sum_{n=p+1}^{\infty}(n-p)\Gamma_n |a_n| |b_n| z^n\right| < \gamma\left|-2(p-\beta)z^p - e^{i\vartheta}\sum_{n=p+1}^{\infty}(3p-n-2\beta)\Gamma_n |a_n| |b_n| z^n\right|$$

thus

$$\sum_{n=p+1}^{\infty}[(n-p)(1-\gamma)+2\gamma(p-\beta)]\Gamma_n |a_n| |b_n| \leq 2\gamma(p-\beta),$$

and the proof is complete. The result is sharp for the function

$$f(z) = z^p + \frac{2\gamma(p-\beta)}{[(n-p)(1-\gamma)+2\gamma(p-\beta)]\Gamma_n \,||b_n|}z^n, \qquad n \geq p+1. \tag{13}$$

□

Corollary 2 *Let $f \in M^{\vartheta}(\beta,p,n)$. Then*

$$|a_n| \leq \frac{2\gamma(p-\beta)}{[(n-p)(1-\gamma)+2\gamma(p-\beta)]\Gamma_n \,||b_n|}, \qquad n \geq p+1. \tag{14}$$

The equality in (14) is attained for the function f given by (13).

Next , we begin by presenting the following quasi- Hadamard products results for the class $M^{\vartheta}(\beta,p,n)$ and the same results have been studied by M. K. Aouf in [2] on p-valent functions.

3 Quasi Hadamard products

Theorem 3 *Let the functions f_k defined by*

$$f_k(z) = z^p + e^{i\vartheta}\sum_{n=p+1}^{\infty}|a_{n,k}|\,z^n \tag{15}$$

be in the class $M^{\vartheta}(\beta_k, p, n)$*. Then we have* $(f_1 * \cdots * f_t) \in M^{\vartheta}(\sigma, p, n)$*, where*

$$\sigma = p - \frac{(1-\gamma)\prod_{k=1}^{t} 2\gamma(p-\beta_k)}{2\gamma\left(\prod_{k=1}^{t}[(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_{p+1}\,|b_{p+1}| - \prod_{k=1}^{t} 2\gamma(p-\beta_k)\right)} \tag{16}$$

the result is sharp for the functions f_k *given by*

$$f_k(z) = z^p + \frac{2\gamma(p-\beta_k)}{[(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_{p+1}\,|b_{p+1}|} z^{p+1}. \tag{17}$$

Proof We prove Theorem 3 by using mathematical induction on t . For $t = 2$,(12) gives

$$\sum_{n=p+1}^{\infty} \frac{[(n-p)(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_n\,|b_n|}{2\gamma(p-\beta_k)} |a_{n,k}| \le 1, \qquad (k=1,2) \tag{18}$$

By Cauchy Schwarz inequality , we have

$$\sum_{n=p+1}^{\infty} \sqrt{\prod_{k=1}^{2} \frac{[(n-p)(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_n\,|b_n|}{2\gamma(p-\beta_k)}} \sqrt{|a_{n,1}|\,|a_{n,2}|} \le 1. \tag{19}$$

To prove the case when $t = 2$, we need to find the largest σ such that

$$\sum_{n=p+1}^{\infty} \frac{[(n-p)(1-\gamma)+2\gamma(p-\sigma)]\Gamma_n\,|b_n|}{2\gamma(p-\sigma)} |a_{n,1}|\,|a_{n,2}| \le 1, \tag{20}$$

Thus , it suffices to show that

$$\frac{[(n-p)(1-\gamma)+2\gamma(p-\sigma)]\Gamma_n\,|b_n|}{2\gamma(p-\sigma)} |a_{n,1}|\,|a_{n,2}|$$

$$\le \frac{\sqrt{\prod_{k=1}^{2}[(n-p)(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_n\,|b_n|}}{\sqrt{\prod_{k=1}^{2}\gamma(p-\beta_k)}} \sqrt{|a_{n,1}|\,|a_{n,2}|}$$

or equivalently to

$$\sqrt{|a_{n,1}|\,|a_{n,2}|} \le \frac{2\gamma(p-\sigma)\sqrt{\prod_{k=1}^{2}[(n-p)(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_n\,|b_n|}}{[(n-p)(1-\gamma)+2\gamma(p-\sigma)]\Gamma_n\,|b_n|\sqrt{\prod_{k=1}^{2}\gamma(p-\beta_k)}}.$$

By noting that

$$\sqrt{|a_{n,1}|\,|a_{n,2}|} \le \frac{\sqrt{\prod_{k=1}^{2}\gamma(p-\beta_k)}}{\sqrt{\prod_{k=1}^{2}[(n-p)(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_n\,|b_n|}},$$

consequently , we need only to prove that

$$\frac{\prod_{k=1}^{2}\gamma(p-\beta_k)}{\prod_{k=1}^{2}[(n-p)(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_n\,|b_n|} \le \frac{2\gamma(p-\sigma)}{[(n-p)(1-\gamma)+2\gamma(p-\sigma)]}$$

which is equivalent to

$$\sigma \leq p - \frac{[(n-p)(1-\gamma)]\prod_{k=1}^{2} 2\gamma(p-\beta_k)}{2\gamma\left(\prod_{k=1}^{2}[(n-p)(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_n\,|b_n| - \prod_{k=1}^{2} 2\gamma(p-\beta_k)\right)}$$

Since

$$B(n) = p - \frac{[(n-p)(1-\gamma)]\prod_{k=1}^{2} 2\gamma(p-\beta_k)}{2\gamma\left(\prod_{k=1}^{2}[(n-p)(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_n\,|b_n| - \prod_{k=1}^{2} 2\gamma(p-\beta_k)\right)}$$

is an increasing function of $n \quad (n \geq p+1)$, then

$$\sigma \leq B(p+1) = p - \frac{(1-\gamma)\prod_{k=1}^{2} 2\gamma(p-\beta_k)}{2\gamma\left(\prod_{k=1}^{2}[(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_{p+1}\,|b_{p+1}| - \prod_{k=1}^{2} 2\gamma(p-\beta_k)\right)}$$

Therefore , the result is true for $t = 2$.
Suppose that the result is true for any positive integer $t = s$. Then we have $(f_1 * \cdots * f_s * f_{s+1}) \in M^{\vartheta}(\mu, p, n)$, where

$$\mu = p - \frac{2\gamma(p-\sigma)(1-\gamma)2\gamma(p-\beta_{s+1})}{2\gamma\left([(1-\gamma)+2\gamma(p-\sigma)2\gamma(p-\beta_{s+1})]\Gamma_{p+1}\,|b_{p+1}| - 2\gamma(p-\sigma)2\gamma(p-\beta_{s+1})\right)}$$

and σ is given by (16). After simple calculation , we have

$$\mu = p - \frac{(1-\gamma)\prod_{k=1}^{s+1} 2\gamma(p-\beta_k)}{2\gamma\left(\prod_{k=1}^{s+1}[(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_{p+1}\,|b_{p+1}| - \prod_{k=1}^{s+1} 2\gamma(p-\beta_k)\right)}. \tag{21}$$

This shows that the result is true for $t = s+1$. Therefore , by mathematical induction , the result is true for any positive integer $t \quad (t \geq 2)$.
Taking the functions f_k given by (17) we have

$$(f_1 * \cdots * f_t)(z) = z^p + \prod_{k=1}^{t} \frac{2\gamma(p-\beta_k)}{[(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_{p+1}\,|b_{p+1}|} z^{p+1} = z^p + G_{p+1}z^{p+1}, \tag{22}$$

which shows that

$$\sum_{n=p+1}^{\infty} \frac{[(n-p)(1-\gamma)+2\gamma(p-\sigma)]\Gamma_n\,|b_n|}{2\gamma(p-\sigma)} G_n$$

$$= \frac{[(n-p)(1-\gamma)+2\gamma(p-\sigma)]\Gamma_n\,|b_n|}{2\gamma(p-\sigma)} \prod_{k=1}^{t} \frac{2\gamma(p-\beta_k)}{[(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_{p+1}\,|b_{p+1}|} = 1.$$

Consequently , the result is sharp for functions f_k given by (17).

□

Letting $\beta_k = \beta$ in Theorem 3, we obtain the following corollary :

Corollary 4 *Let the functions f_k defined by (15) be in the class $M^{\vartheta}(\beta, p, n)$. Then we have$(f_1 * \cdots * f_t) \in M^{\vartheta}(\sigma, p, n)$, where*

$$\sigma = p - \frac{(1-\gamma)[2\gamma(p-\beta)]^t}{2\gamma([(1-\gamma)+2\gamma(p-\beta)]^t\Gamma_{p+1}\,|b_{p+1}| - [2\gamma(p-\beta)]^t)}. \tag{23}$$

The result is sharp for the functions f_k given by

$$f_k(z) = z^p + \frac{2(p-\beta)}{[(1-\gamma)+2\gamma(p-\beta)]\Gamma_{p+1}\,|b_{p+1}|} z^{p+1}. \tag{24}$$

Theorem 5 *Let the functions f_k defined by (15) be in the class $M^{\vartheta}(\beta_k, p, n)$. Then the function*

$$F(z) = z^p + e^{i\vartheta} \sum_{n=p+1}^{\infty} (\sum_{k=1}^{t} |a_{n,k}|^m) z^n, \qquad (m > 1) \tag{25}$$

belong to the class $M^{\vartheta}(\sigma_t, p, n)$, where

$$\sigma_t = p - \frac{t(1-\gamma)[2\gamma(p-\beta)]^m}{2\gamma([(1-\gamma)+2\gamma(p-\beta)]^m \Gamma_{p+1} |b_{p+1}| - t[2\gamma(p-\beta)]^m)}, \qquad (\beta = \min_{1 \le k \le t} \{\beta_k\}) \tag{26}$$

and $2\gamma[(1-\gamma)+2\gamma(p-\beta)]^m \Gamma_{p+1} |b_{p+1}| \ge t[2\gamma(p-\beta)]^m (2\gamma p + 1 - \gamma)$.
The result is sharp for the functions $f_k, (k = 1, \ldots, t)$ given by (17).

Proof By virtue of (12) , we have

$$\sum_{n=p+1}^{\infty} \frac{[(n-p)(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_n |b_n|}{2\gamma(p-\beta_k)} |a_{n,k}| \le 1.$$

By the Cauchy Schwarz inequality , we have

$$\sum_{n=p+1}^{\infty} \left(\frac{[(n-p)(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_n |b_n|}{2\gamma(p-\beta_k)} \right)^m |a_{n,k}|^m$$

$$\le \left(\sum_{n=p+1}^{\infty} \frac{[(n-p)(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_n |b_n|}{2\gamma(p-\beta_k)} |a_{n,k}| \right)^m \le 1. \tag{27}$$

It follows from (27) that

$$\sum_{n=p+1}^{\infty} \left(\frac{1}{t} \sum_{k=1}^{t} \left(\frac{[(n-p)(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_n |b_n|}{2\gamma(p-\beta_k)} \right)^m |a_{n,k}|^m \right) \le 1.$$

By setting $\beta = \min_{1 \le k \le t} \{\beta_k\}$, suppose also that

$$\sigma_t \le p - \frac{t(n-p)(1-\gamma)[2\gamma(p-\beta)]^m}{2\gamma([(n-p)(1-\gamma)+2\gamma(p-\beta)]^m \Gamma_n |b_n| - t[2\gamma(p-\beta)]^m)}.$$

By virtue of (12) ,we have

$$\sum_{n=p+1}^{\infty} \left(\frac{[(n-p)(1-\gamma)+2\gamma(p-\sigma_m)]\Gamma_n |b_n|}{2\gamma(p-\sigma_m)} \left(\sum_{k=1}^{t} |a_{n,k}|^m \right) \right)$$

$$\le \sum_{n=p+1}^{\infty} \left(\frac{1}{t} \left(\frac{[(n-p)(1-\gamma)+2\gamma(p-\beta)]\Gamma_n |b_n|}{2\gamma(p-\beta)} \right)^m \left(\sum_{k=1}^{t} |a_{n,k}|^m \right) \right)$$

$$\le \sum_{n=p+1}^{\infty} \left(\frac{1}{t} \sum_{k=1}^{t} \left(\frac{[(n-p)(1-\gamma)+2\gamma(p-\beta_k)]\Gamma_n |b_n|}{2\gamma(p-\beta_k)} \right)^m |a_{n,k}|^m \right) \le 1.$$

Now , let

$$A(n) = p - \frac{t(n-p)(1-\gamma)[2\gamma(p-\beta)]^m}{2\gamma([(n-p)(1-\gamma)+2\gamma(p-\beta)]^m \Gamma_n |b_n| - t[2\gamma(p-\beta)]^m)}, \quad n \ge p+1.$$

Since $A(n)$ is an increasing function of $n \quad (n \geq p+1)$, then we have

$$\sigma_t \leq A(p+1) = p - \frac{t(1-\gamma)[2\gamma(p-\beta)]^m}{2\gamma([(1-\gamma)+2\gamma(p-\beta)]^m \Gamma_{p+1} |b_{p+1}| - t[2\gamma(p-\beta)]^m)}.$$

and by noting that $2\gamma[(1-\gamma)+2\gamma(p-\beta)]^m \Gamma_{p+1} |b_{p+1}| \geq t[2\gamma(p-\beta)]^m(2\gamma p+1-\gamma)$, we can see that $0 \leq \sigma_t < p$. The result is sharp for the functions $f_k(k=1,\ldots,t)$ given by (17).

□

Putting $m=2$ and $\beta_k = \beta(k=1,\ldots,t)$ in Theorem 5 , we obtain the following corollary:

Corollary 6 *Let the functions $f_k(k=1,\ldots,t)$ defined by (15) be in the class $M^{\vartheta}(\beta,p,n)$. Then the function*

$$G(z) = z^p + e^{i\vartheta} \sum_{n=p+1}^{\infty} \left(\sum_{k=1}^{t} |a_{n,k}|^2 \right) z^n, \tag{28}$$

belong to the class $M^{\vartheta}(\sigma_t,p,n)$, where

$$\sigma_t = p - \frac{t(1-\gamma)[2\gamma(p-\beta)]^2}{2\gamma([(1-\gamma)+2\gamma(p-\beta)]^2 \Gamma_{p+1} |b_{p+1}| - t[2\gamma(p-\beta)]^2)}, \tag{29}$$

and $2\gamma[(1-\gamma)+2\gamma(p-\beta)]^2 \Gamma_{p+1} |b_{P+1}| \geq t[2\gamma(p-\beta)]^2(2\gamma p+1-\gamma)$. The result is sharp for the functions $f_k(k=1,\ldots,t)$ given by (24).

4 Integral means

An analytic function g is said to be subordinate to an analytic function $f(writteng \prec f)$ if $g(z) = f(w(z)), z \in U$, for some analytic function w with $|w(z)| \leq |z|$.in 1925, Littlewood [5] proved the following subordination result which will be required in our present investigation.

Lemma 7 *(see ([5]) If f and g are analytic in U with $g \prec f$, then*

$$\int_0^{2\Pi} \left| g(re^{i\vartheta}) \right|^{\delta} d\vartheta \leq \int_0^{2\Pi} \left| f(re^{i\vartheta}) \right|^{\delta} d\vartheta, \tag{30}$$

where $\delta > 0, z = re^{i\vartheta}$ and $0 < r < 1$.

Applying Theorem 1 and Lemma 7 ,we prove the following.

Theorem 8 *Let $\delta > 0$. If $f \in M^{\vartheta}(\beta,p,n)$ and*

$$f_{p+1}(z) = z^p + \frac{2\gamma(p-\beta)}{[(1-\gamma)+2\gamma(p-\beta)]\Gamma_{p+1} |b_{p+1}|} z^{p+1},$$

then for $z = re^{i\vartheta}$ and $0 < r < 1$

$$\int_0^{2\Pi} \left| f(re^{i\vartheta}) \right|^{\delta} d\vartheta \leq \int_0^{2\Pi} \left| f_{p+1}(re^{i\vartheta}) \right|^{\delta} d\vartheta. \tag{31}$$

Proof Let

$$f(z) = z^p + e^{i\vartheta} \sum_{n=p+1}^{\infty} |a_n| z^n, \qquad n \geq p+1$$

$$f_{p+1}(z) = z^p + \frac{2\gamma(p-\beta)}{[(1-\gamma)+2\gamma(p-\beta)]\Gamma_{p+1} |b_{p+1}|} z^{p+1}, \tag{32}$$

then ,we must show that

$$\int_0^{2\pi}\left|1+e^{i\vartheta}\sum_{n=p+1}^{\infty}|a_n|\,z^{n-p}\right|^{\delta}d\vartheta\le\int_0^{2\pi}\left|1+\frac{2\gamma(p-\beta)}{[(1-\gamma)+2\gamma(p-\beta)]\Gamma_{p+1}\,|b_{p+1}|}z\right|^{\delta}d\vartheta. \qquad (33)$$

By Lemma 7, it suffices to show that

$$1+e^{i\vartheta}\sum_{n=p+1}^{\infty}|a_n|\,z^{n-p}\prec 1+\frac{2\gamma(p-\beta)}{[(1-\gamma)+2\gamma(p-\beta)]\Gamma_{p+1}\,|b_{p+1}|}z. \qquad (34)$$

Set

$$1+e^{i\vartheta}\sum_{n=p+1}^{\infty}|a_n|\,z^{n-p}=1+\frac{2\gamma(p-\beta)}{[(1-\gamma)+2\gamma(p-\beta)]\Gamma_{p+1}\,|b_{p+1}|}z. \qquad (35)$$

From (35) and (12) , we obtain

$$|w(z)|=\left|\frac{[(1-\gamma)+2\gamma(p-\beta)]\Gamma_{p+1}\,|b_{p+1}|}{2\gamma(p-\beta)}\right|\left|e^{i\vartheta}\sum_{n=p+1}^{\infty}|a_n|\,z^{n-p}\right| \qquad (36)$$

$$|z|\sum_{n=p+1}^{\infty}\frac{[(n-p)(1-\gamma)+2\gamma(p-\beta)]\Gamma_n\,|b_n|}{2\gamma(p-\beta)}|a_n|\le|z|\,.$$

□

The proof for the first derivative is similar.

Theorem 9 *Let $\delta>0$. If $f\in M^{\vartheta}(\beta,p,n)$ and*

$$f_{p+1}(z)=z^p+\frac{2\gamma(p-\beta)}{[(1-\gamma)+2\gamma(p-\beta)]\Gamma_{p+1}\,|b_{p+1}|}z^{p+1}$$

then for $z=re^{i\vartheta}$ and $0<r<1$.

$$\int_0^{2\Pi}\left|f\prime(re^{i\vartheta})\right|^{\delta}d\vartheta\le\int_0^{2\Pi}\left|f\prime_{p+1}(re^{i\vartheta})\right|^{\delta}d\vartheta. \qquad (37)$$

Proof It is suffices to show that

$$1+e^{i\vartheta}\sum_{n=p+1}^{\infty}\frac{n}{p}|a_n|\,z^{n-p}\prec 1+\frac{2\gamma(p-\beta)}{[(1-\gamma)+2\gamma(p-\beta)]\Gamma_{p+1}\,|b_{p+1}|}\frac{p+1}{p}z. \qquad (38)$$

This follows because

$$|w(z)|=\left|e^{i\vartheta}\sum_{n=p+1}^{\infty}\frac{[(n-p)(1-\gamma)+2\gamma(p-\beta)]\Gamma_n\,|b_n|}{2\gamma(p-\beta)}|a_n|\,z^{n-p}\right|$$

$$\le|z|^{n-p}\sum_{n=p+1}^{\infty}\frac{[(n-p)(1-\gamma)+2\gamma(p-\beta)]\Gamma_n\,|b_n|}{2\gamma(p-\beta)}|a_n|\le|z|^{n-p}\le|z|$$

□

5 Integral means inequality for f and h

We introduce an analytic and multivalent function h defined by:

$$h(z) = z^p + e^{i\vartheta} |b_k| z^k + e^{i\vartheta} |b_{2k-p}| z^{2k-p} + e^{i\vartheta} |b_{3k-2p}| z^{3k-2p}, \quad k \geq p+1 \tag{39}$$

For the above function h , we show.

Theorem 10 *Let $f \in M^{\vartheta}(\beta, p, n)$ and h be given by (39). If f satisfies*

$$\sum_{n=p+1}^{\infty} |a_n| \leq |b_{3k-2p}| - |b_{2k-p}| - |b_k|, \qquad (|b_k| + |b_{2k-p}| < |b_{3k-2p}|)$$

and there exists an analytic function w such that

$$e^{i\vartheta} |b_{3k-2p}| (w(z))^{3(k-p)} + e^{i\vartheta} |b_{2k-p}| (w(z))^{2(k-p)} + e^{i\vartheta} |b_k| (w(z))^{(k-p)} - e^{i\vartheta} \sum_{n=p+1}^{\infty} |a_n| z^{n-p} = 0, \tag{40}$$

then for $\delta > 0$ and $z = re^{i\vartheta}, (0 < r < 1)$

$$\int_0^{2\pi} |f(z)|^{\delta} d\vartheta \leq \int_0^{2\pi} |h(z)|^{\delta} d\vartheta.$$

Proof We have to show that there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1, (z \in U)$ such that $f(z) = h(w(z))$. Note that this function w is defined by (40) since , for $z = 0$

$$e^{i\vartheta}(w(0))^{(k-p)} \left\{ |b_{3k-2p}| (w(0))^{2(k-p)} + \left|b_{(2k-p)}\right| (w(0))^{(k-p)} + |b_k| \right\} = 0,$$

we consider w satisfies $w(0) = 0$.
On the other hand, we have that

$$|b_{3k-2p}| |w(z)|^{3(k-p)} + |b_{2k-p}| |w(z)|^{2(k-p)} + |b_k| |w(z)|^{(k-p)} - \sum_{n=p+1}^{\infty} |a_n| < 0.$$

Putting $t = |w(z)|^{(k-p)}$ $(t0)$, we define the function $H(t)$ by

$$H(t) = |b_{3k-2p}| |t^3 - |b_{2k-p}| t^2 - |b_k| t - \sum_{n=p+1}^{\infty} |a_n| .$$

It follows that $H(0) \leq 0$ and

$$H\prime(t) = 3 |b_{3k-2p}| t^2 - 2 |b_{2k-p}| t - |b_k| .$$

Since the discriminant of $H\prime(t) = 0$ is greater than 0, if $H\prime(1) \geq 0$, then $t < 1$ for $H(t) < 0$. Therefore , we need the following inequality ,

$$H(1) = |b_{3k-2p}| - |b_{2k-p}| - |b_k| - \sum_{n=p+1}^{\infty} |a_n| \geq 0$$

or

$$\sum_{n=p+1}^{\infty} |a_n| \leq |b_{3k-2p}| - |b_{2k-p}| - |b_k| .$$

□

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