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On a New Subclass of Univalent Functions with Negative Coefficients Defined by Ruscheweyh Derivative

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Abstract: ***In this paper, we have discussed a subclass$S(\theta,\alpha,\beta,\gamma,\lambda)$ of analytic and univalent function with negative coefficients defined by Ruscheweyh derivative in unit disk $U=\{z\in\mathbb{C}:|z|<1\}$.We obtain basic properties like coefficient inequality, distortion theorem, extreme points and radii of starlikeness, convexity and close-to-convexity, Hadamard product and convolution operator.***

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1. Introduction

Let A denote the class of function given by

$$f(z)=z+\sum_{i=2}^{\infty}a_iz^i, \qquad (1.1)$$

which are analytic and univalent in open unit disk $U=\{z\in\mathbb{C}:\ |z|<1\}$.
Let S be a subclass of A consisting of functions of the form:

$$f(z)=z-\sum_{i=2}^{\infty}a_iz^i, \qquad (a_i\geq 0). \qquad (1.2)$$

We denote by $S^*(\alpha),K(\alpha)$consisting of all functions which are respectively starlike and convex of order α in U with $0\leq\alpha<1$,thus

$$S^*(\alpha)=\left\{f\in S: Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha:0\leq\alpha<1,z\in U\right\}$$

$$k(\alpha)=\left\{f\in S: Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha:0\leq\alpha<1,z\in U\right\}.$$

The Ruscheweyh derivative[4], [5] of $f\in S$denoted by $D^\lambda f(z)$of order λ is defined by

$$D^\lambda f(z)=z-\sum_{i=2}^{\infty}a_iB_i(\lambda)z^i,$$

where

$$B_i(\lambda)=\frac{(\lambda+1)(\lambda+2)\dots(\lambda+i-1)}{(i-1)!}, \qquad \lambda>-1,z\in U.$$

Definition (1):A function $f\in S$is said to be in the class $S(\theta,\alpha,\beta,\gamma,\lambda)$if the following inequality is satisfied:

$$\left|\frac{\frac{z\left(D^\lambda f(z)\right)''}{\left(D^\lambda f(z)\right)'}+1}{2(1-\alpha)\left(\frac{z\left(D^\lambda f(z)\right)''}{\left(D^\lambda f(z)\right)'}-\theta\right)-\gamma\left(\frac{z\left(D^\lambda f(z)\right)''}{\left(D^\lambda f(z)\right)'}+1\right)}\right|<\beta, \quad (1.3)$$

for $|z|<1, 0<\beta\leq 1, 0\leq\alpha<1, 0<\theta\leq 1$ and $\frac{1}{2}\leq\gamma\leq 1$.

2. Coefficient Estimates

Theorem(1): Let the function f be defined by (1.2). Then $f\in S(\theta,\alpha,\beta,\gamma,\lambda)$if and only if

$$\sum_{i=2}^{\infty}i[(1+\beta\gamma)i-2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)\,a_i \leq 1+2\beta\theta(1-\alpha)+\beta\gamma, \qquad (2.1)$$

where $0<\beta\leq 1, 0\leq\alpha<1, 0<\theta\leq 1,\frac{1}{2}\leq\gamma\leq 1$.The result (2.1) is sharp for the function

$$f(z)=z-\frac{1+2\beta\theta(1-\alpha)+\beta\gamma}{i[(1+\beta\gamma)i-2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}z^i, i\geq 2.$$

Proof: Assume that the inequality (2.1) holds true and let $|z|=1$, then,we have

$$\left|z\left(D^\lambda f(z)\right)''+\left(D^\lambda f(z)\right)'\right| -\beta\left|2(1-\alpha)\left(z\left(D^\lambda f(z)\right)'' -\theta\left(D^\lambda f(z)\right)'\right) -\gamma\left(z\left(D^\lambda f(z)\right)''+\left(D^\lambda f(z)\right)'\right)\right|$$

$$\left|1 - i^2 \sum_{i=2}^{\infty} a_i z^{i-1}\right| - \beta \left|(-2(1-\alpha)i(i-1-\theta) + \gamma i^2) \sum_{i=2}^{\infty} a_i z^{i-1} - 2\theta(1-\alpha) - \gamma\right|$$

$$\leq \sum_{i=2}^{\infty} i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)\, a_i - 1 - 2\beta\theta(1-\alpha) - \beta\gamma \leq 0,$$

by hypothesis. Hence, by maximum modulus principle, $f \in S(\theta,\alpha,\beta,\gamma,\lambda)$.

Conversely, suppose that f defined by (1.2) is in the class $S(\theta,\alpha,\beta,\gamma,\lambda)$. Hence

$$\left|\frac{z\left(D^\lambda f(z)\right)'' + \left(D^\lambda f(z)\right)'}{2(1-\alpha)\left(z\left(D^\lambda f(z)\right)'' - \theta\left(D^\lambda f(z)\right)'\right) - \gamma\left(z\left(D^\lambda f(z)\right)'' + \left(D^\lambda f(z)\right)'\right)}\right| = \left|\frac{1 - i^2 \sum_{i=2}^{\infty} a_i z^{i-1}}{(-2(1-\alpha)i(i-1-\theta) + \gamma i^2) \sum_{i=2}^{\infty} a_i z^{i-1} - 2\theta(1-\alpha) - \gamma}\right| < \beta.$$

Since $Re(z) < |z|$ for all z, we have

$$Re\left\{\frac{1 - i^2 \sum_{i=2}^{\infty} a_i z^{i-1}}{(-2(1-\alpha)i(i-1-\theta) + \gamma i^2) \sum_{i=2}^{\infty} a_i z^{i-1} - 2\theta(1-\alpha) - \gamma}\right\} < \beta. \quad (2.2)$$

We can choose the value of z on the real axis. Let $z \to 1^-$ through real values, we obtain the inequality (2.1).

Finally, sharpness follows if, we take

$$f(z) = z - \frac{1 + 2\beta\theta(1-\alpha) + \beta\gamma}{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)} z^i, i \geq 2. \quad (2.3)$$

Corollary (1): Let $f \in S(\theta,\alpha,\beta,\gamma,\lambda)$. Then

$$a_n \leq \frac{1 + 2\beta\theta(1-\alpha) + \beta\gamma}{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}, i = 2,3,\ldots . \quad (2.4)$$

3. Growth and Distortion Theorems

In the following theorems, we obtain the growth and distortion theorems for function $f \in S(\theta,\alpha,\beta,\gamma,\lambda)$.

Theorem (2): Let the function $f(z)$ defined by(1.2) be in the class $S(\theta,\alpha,\beta,\gamma,\lambda)$. Then

$$r - \frac{1 + 2\beta\theta(1-\alpha) + \beta\gamma}{4[(1+\beta\gamma) - \beta(1-\alpha)(1-\theta)](\lambda+1)} r^2 \leq |f(z)| \leq r + \frac{1 + 2\beta\theta(1-\alpha) + \beta\gamma}{4[(1+\beta\gamma) - \beta(1-\alpha)(1-\theta)](\lambda+1)} r^2, (|z| = r < 1). (3.1)$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{1 + 2\beta\theta(1-\alpha) + \beta\gamma}{4[(1+\beta\gamma) - \beta(1-\alpha)(1-\theta)](\lambda+1)} z^2.$$

Proof: Let $f(z) \in S(\theta,\alpha,\beta,\gamma,\lambda)$. Then by Theorem (1), we have

$$\sum_{i=2}^{\infty} a_i \leq \frac{1 + 2\beta\theta(1-\alpha) + \beta\gamma}{4[(1+\beta\gamma) - \beta(1-\alpha)(1-\theta)](\lambda+1)}.$$

Hence

$$|f(z)| \leq |z| + \sum_{i=2}^{\infty} a_i |z|^i = r + r^2 \sum_{i=2}^{\infty} a_i \leq r + \frac{1 + 2\beta\theta(1-\alpha) + \beta\gamma}{4[(1+\beta\gamma) - \beta(1-\alpha)(1-\theta)](\lambda+1)} r^2. \quad (3.2)$$

Similarly, we obtain

$$|f(z)| \geq |z| - \sum_{i=2}^{\infty} a_i |z|^i = r - r^2 \sum_{i=2}^{\infty} a_i \geq r - \frac{1 + 2\beta\theta(1-\alpha) + \beta\gamma}{4[(1+\beta\gamma) - \beta(1-\alpha)(1-\theta)](\lambda+1)} r^2. \quad (3.3)$$

From bounds (3.2) and (3.3), we get (3.1).

Theorem (3): Let the function $f(z)$ defined by (1.2) be in the class $S(\theta,\alpha,\beta,\gamma,\lambda)$. Then

$$1 - \frac{1 + 2\beta\theta(1-\alpha) + \beta\gamma}{2[(1+\beta\gamma) - \beta(1-\alpha)(1-\theta)](\lambda+1)} r \leq |f'(z)| \leq 1 + \frac{1 + 2\beta\theta(1-\alpha) + \beta\gamma}{2[(1+\beta\gamma) - \beta(1-\alpha)(1-\theta)](\lambda+1)} r. \quad (3.4)$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{1 + 2\beta\theta(1-\alpha) + \beta\gamma}{4[(1+\beta\gamma) - \beta(1-\alpha)(1-\theta)](\lambda+1)} z^2.$$

Proof: Let $f(z) \in S(\theta,\alpha,\beta,\gamma,\lambda)$. Then by Theorem (1), we have

$$\sum_{i=2}^{\infty} a_i \leq \frac{1 + 2\beta\theta(1-\alpha) + \beta\gamma}{2[(1+\beta\gamma) - \beta(1-\alpha)(1-\theta)](\lambda+1)}.$$

Hence

$$|f'(z)| \leq |1| + \sum_{i=2}^{\infty} i a_i |z|^{i-1} = 1 + r \sum_{i=2}^{\infty} a_i \leq 1 + \frac{1 + 2\beta\theta(1-\alpha) + \beta\gamma}{2[(1+\beta\gamma) - \beta(1-\alpha)(1-\theta)](\lambda+1)} r. \quad (3.5)$$

Similarly, we obtain

$$|f'(z)| \geq |1| - \sum_{i=2}^{\infty} i a_i |z|^{i-1} = 1 - r \sum_{i=2}^{\infty} a_i \geq 1 - \frac{1 + 2\beta\theta(1-\alpha) + \beta\gamma}{2[(1+\beta\gamma) - \beta(1-\alpha)(1-\theta)](\lambda+1)} r. \quad (3.6)$$

From bounds (3.5) and (3.6), we get (3.4).

4. Radii of Starlikeness, Convexity and Close-to-convexity

In the following theorems, we obtain the radii of starlikeness and convexity and close-to-convexity for the class $S(\theta,\alpha,\beta,\gamma,\lambda)$.

Theorem (4): Let $f \in S(\theta,\alpha,\beta,\gamma,\lambda)$. Then f is starlike in the disk $|z| < R_1$, of order $\alpha, 0 \leq \alpha < 1$, where

$$R_1 = \inf_i \left[\frac{(1-\alpha)i[(1+\beta\gamma)i-2\beta(1-\alpha)(i-1-\theta)]B_i(z)}{(i-\alpha)(1+2\beta\theta(1-\alpha)+\beta\gamma)}\right]^{\frac{1}{i-1}}, i \geq 2. \quad (4.1)$$

Proof: A function f is starlike of order $\alpha, 0 \leq \alpha < 1$, if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha.$$

We must show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\alpha, for\ |z| < R_1.$$

We have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \left|\frac{-\sum_{i=2}^{\infty}(i-1)a_i z^{i-1}}{1-\sum_{i=2}^{\infty} a_i z^{i-1}}\right| \leq \frac{\sum_{i=2}^{\infty}(i-1)a_i|z|^{i-1}}{1-\sum_{i=2}^{\infty} a_i|z|^{i-1}}.$$

The last expression above is bounded by $(1-\alpha)$ if

$$\sum_{i=2}^{\infty}\frac{i-\alpha}{1-\alpha}a_i|z|^{i-1} \leq 1. \quad (4.2)$$

Hence, by Theorem (1), (4.2) will be true if

$$\frac{i-\alpha}{1-\alpha}|z|^{i-1} \leq \frac{i[(1+\beta\gamma)i-2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{1+2\beta\theta(1-\alpha)+\beta\gamma},$$

or equivalently

$$|z| \leq \left[\frac{(1-\alpha)i[(1+\beta\gamma)i-2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(i-\alpha)(1+2\beta\theta(1-\alpha)+\beta\gamma)}\right]^{\frac{1}{i-1}}, i \geq 2 \quad (4.3)$$

the theorem follows easily from (4.3).

Theorem (5): Let $f \in S(\theta,\alpha,\beta,\gamma,\lambda)$ Then f is convex in the disk $|z| < R_2$, of order $\alpha, 0 \leq \alpha < 1$, where

$$R_2 = \inf_i \left[\frac{(1-\alpha)i[(1+\beta\gamma)i-2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{i(i-\alpha)(1+2\beta\theta(1-\alpha)+\beta\gamma)}\right]^{\frac{1}{i-1}}, i \geq 2. \quad (4.4)$$

Proof: A function f is convex of order $\alpha, 0 \leq \alpha < 1$, if

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha.$$

Thus is enough to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\alpha, for\ |z| < R_2.$$

We have

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{-\sum_{i=2}^{\infty} i(i-1)a_i z^{i-1}}{1-\sum_{i=2}^{\infty} i a_i z^{i-1}}\right| \leq \frac{\sum_{i=2}^{\infty} i(i-1)a_i|z|^{i-1}}{1-\sum_{i=2}^{\infty} a_i|z|^{i-1}}.$$

The last expression above is bounded by $(1-\alpha)$ if

$$\sum_{i=2}^{\infty}\frac{i(i-\alpha)}{1-\alpha}a_i|z|^{i-1} \leq 1. \quad (4.5)$$

Hence, by Theorem (1), (4.5) will be true if

$$\frac{i(i-\alpha)}{1-\alpha}|z|^{i-1} \leq \frac{i[(1+\beta\gamma)i-2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{1+2\beta\theta(1-\alpha)+\beta\gamma},$$

or equivalently

$$|z| \leq \left[\frac{(1-\alpha)i[(1+\beta\gamma)i-2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{i(i-\alpha)(1+2\beta\theta(1-\alpha)+\beta\gamma)}\right]^{\frac{1}{i-1}}, i \geq 2 \quad (4.6)$$

the theorem follows easily from (4.6).

Theorem (6): Let $f \in S(\theta,\alpha,\beta,\gamma,\lambda)$ Then f is close-to-convex in the disk $|z| < R_3$, of order $\alpha, 0 \leq \alpha < 1$, where

$$R_3 = \inf_i \left[\frac{(1-\alpha)i[(1+\beta\gamma)i-2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{i(1+2\beta\theta(1-\alpha)+\beta\gamma)}\right]^{\frac{1}{i-1}}. \quad (4.7)$$

Proof: A function f is close-to-convex function of order $\alpha, 0 \leq \alpha < 1$, if

$$Re\{f'(z)\} > \alpha.$$

Thus it is enough to show that

$$|f'(z) - 1| \leq 1-\alpha, for\ |z| < R_3.$$

We have

$$|f'(z) - 1| = \left|\sum_{i=2}^{\infty} i a_i z^{i-1}\right| \leq \sum_{i=2}^{\infty} i a_i |z|^{i-1}.$$

Thus

$$|f'(z) - 1| \leq 1-\alpha\ if \sum_{i=2}^{\infty}\frac{i a_i|z|^{i-1}}{1-\alpha} \leq 1. \quad (4.8)$$

Hence, by Theorem (1), (4.8) will be true if

$$\frac{i|z|^{i-1}}{1-\alpha} \leq \frac{i[(1+\beta\gamma)i-2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{1+2\beta\theta(1-\alpha)+\beta\gamma}.$$

Or equivalently

$$|z| \leq \left[\frac{(1-\alpha)i[(1+\beta\gamma)i-2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{i(1+2\beta\theta(1-\alpha)+\beta\gamma)}\right]^{\frac{1}{i-1}}, i \geq 2, \quad (4.9)$$

the theorem follows easily from (4.9).

5. Extreme Points

In the following theorem, we obtain extreme points for the class $S(\theta,\alpha,\beta,\gamma,\lambda)$.

Theorem (7): Let $f_1(z) = z$ and

$$f_i(z) = z - \frac{(1+2\beta\theta(1-\alpha)+\beta\gamma)}{i[(1+\beta\gamma)i-2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}z^i, for\ i = 2,3,\ldots .$$

Then $f \in S(\theta,\alpha,\beta,\gamma,\lambda)$ if and only if it can be expressed in the form

$$f(z) = \sum_{i=1}^{\infty}\mu_i f_i(z),$$

where

$$\left(\mu_i \geq 0 \; and \sum_{i=1}^{\infty} \mu_i = 1 \; or \; 1 = \mu_1 + \sum_{i=2}^{\infty} \mu_i\right).$$

Proof: Let

$$f(z) = \sum_{i=1}^{\infty} \mu_i f_i(z)$$

$$= z - \sum_{i=2}^{\infty} \frac{(1+2\beta\theta(1-\alpha)+\beta\gamma)}{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)} \mu_i z^i,$$

then

$$\sum_{i=2}^{\infty} \frac{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\gamma)} \mu_n \times \frac{(1+2\beta\theta(1-\alpha)+\beta\gamma)}{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}$$

$$= \sum_{i=2}^{\infty} \mu_i = 1 - \mu_1 \leq 1.$$

Using Theorem (1), we easily get $f \in S(\theta,\alpha,\beta,\gamma,\lambda)$.

Conversely, let $f \in S(\theta,\alpha,\beta,\gamma,\lambda)$is of the form (1.2). Then

$$a_n \leq \frac{(1+2\beta\theta(1-\alpha)+\beta\gamma)}{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}, i \geq 2.$$

Setting

$$\mu_i = \frac{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\gamma)} a_i, for\ i \geq 2$$

and

$$\mu_1 = 1 - \sum_{i=2}^{\infty} \mu_i.$$

Then

$$f(z) = z - \sum_{i=2}^{\infty} a_i z^i$$

$$= z - \sum_{i=2}^{\infty} \frac{(1+2\beta\theta(1-\alpha)+\beta\gamma)}{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)} \mu_i z^i$$

$$= \mu_1 z - \sum_{i=2}^{\infty} \mu_i f_i(z).$$

Thus

$$f(z) = \sum_{i=1}^{\infty} \mu_i f_i(z) = \mu_1 f_1(z) + \sum_{i=2}^{\infty} \mu_i f_i(z).$$

6. Hadamard Product

Theorem (8): Let $f \; and \; g \in S(\theta,\alpha,\beta,\gamma,\lambda)$.Then$f * g \in S(\theta,\alpha,\beta,\delta,\lambda)$for

$$f(z) = z - \sum_{i=2}^{\infty} a_i z^i, \quad g(z) = z - \sum_{i=2}^{\infty} b_i z^i,$$

where

$$\delta \leq \left((1+2\beta\theta(1-\alpha)+\beta\gamma)^2[i-2\beta(1-\alpha)(i-1-\theta)] - i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]^2 B_i(\lambda)[1+2\beta\theta(1-\alpha)]\right) \Big/ \left(\beta i([(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]^2 B_i(\lambda) - (1+2\beta\theta(1-\alpha)+\beta\gamma)^2)\right).$$

Proof: Since $f \; and \; g \in S(\theta,\alpha,\beta,\gamma,\lambda)$.Then we have

$$\sum_{i=2}^{\infty} \frac{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\gamma)} a_i \leq 1$$

and

$$\sum_{i=2}^{\infty} \frac{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\gamma)} b_i \leq 1.$$

We have to find the largest δ such that

$$\sum_{i=2}^{\infty} \frac{i[(1+\beta\delta)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\delta)} a_i b_i \leq 1.$$

By Cauchy-Schwarz inequality, we have

$$\sum_{i=2}^{\infty} \frac{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\gamma)} \sqrt{a_i b_i} \leq 1. \qquad (6.1)$$

We want only to show that

$$\frac{i[(1+\beta\delta)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\delta)} a_i b_i \leq \frac{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\gamma)} \sqrt{a_i b_i},$$

This inequality is equivalent to

$$\sqrt{a_i b_i} \leq \frac{[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)](1+2\beta\theta(1-\alpha)+\beta\delta)}{[(1+\beta\delta)i - 2\beta(1-\alpha)(i-1-\theta)](1+2\beta\theta(1-\alpha)+\beta\gamma)}. \quad (6.2)$$

From (6.1), we get

$$\sqrt{a_i b_i} \leq \frac{(1+2\beta\theta(1-\alpha)+\beta\gamma)}{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}. \qquad (6.3)$$

Therefore, in view of (6.2) and (6.3) it is enough to show that

$$\frac{(1+2\beta\theta(1-\alpha)+\beta\gamma)}{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)} \leq \frac{[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)](1+2\beta\theta(1-\alpha)+\beta\delta)}{[(1+\beta\delta)i - 2\beta(1-\alpha)(i-1-\theta)](1+2\beta\theta(1-\alpha)+\beta\gamma)},$$

and

$$\delta \leq \left((1+2\beta\theta(1-\alpha)+\beta\gamma)^2[i-2\beta(1-\alpha)(i-1-\theta)] - i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]^2 B_i(\lambda)[1+2\beta\theta(1-\alpha)]\right) \Big/ \left(\beta i([(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]^2 B_i(\lambda) - (1+2\beta\theta(1-\alpha)+\beta\gamma)^2)\right).$$

This complete the proof.

Theorem (9): Let the function $f \; and \; g \in S(\theta,\alpha,\beta,\gamma,\lambda)$.Then

$$T(z) = z - \sum_{i=2}^{\infty} (a_i^2 + b_i^2) z^i$$

belong to the class $S(\theta,\alpha,\beta,\delta,\lambda)$,where

$$\delta \geq \big(2[i - 2\beta(1-\alpha)(i-1-\theta)][1+2\beta\theta(1-\alpha)+\beta\gamma]^2 - i[1 + 2\beta\theta(1-\alpha)][(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]^2 B_i(\lambda)\big) / (\beta i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]^2 B_i(\lambda) - 2\beta i[1+2\beta\theta(1-\alpha)+\beta\gamma]^2).$$

Proof: Since f and $g \in S(\theta,\alpha,\beta,\gamma,\lambda)$so by Theorem (1), yields

$$\sum_{i=2}^{\infty} \left[\frac{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\gamma)} a_i\right]^2 \leq 1,$$

and

$$\sum_{i=2}^{\infty} \left[\frac{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\gamma)} b_i\right]^2 \leq 1,$$

we obtain from the last two inequalities

$$\sum_{i=2}^{\infty} \frac{1}{2}\left[\frac{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\gamma)}\right]^2 (a_i^2 + b_i^2) \leq 1, \quad (6.4)$$

but $T(z) \in S(\theta,\alpha,\beta,\delta,\lambda)$if and only if

$$\sum_{i=2}^{\infty} \frac{i[(1+\beta\delta)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\delta)} (a_i^2 + b_i^2) \leq 1, \quad (6.5)$$

where$0 < \delta < 1$, however (6.4) implies (6.5) if

$$\frac{i[(1+\beta\delta)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\delta)} \leq \frac{1}{2}\left[\frac{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\gamma)}\right]^2.$$

Simplifying, we get

$$\delta \geq \big(2[i - 2\beta(1-\alpha)(i-1-\theta)][1+2\beta\theta(1-\alpha)+\beta\gamma]^2 - i[1 + 2\beta\theta(1-\alpha)][(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]^2 B_i(\lambda)\big) / (\beta i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]^2 B_i(\lambda) - 2\beta i[1+2\beta\theta(1-\alpha)+\beta\gamma]^2).$$

This complete the proof.

7. Convolution Operator

Definition (2) [1,3]: The Gaussian hypergeometric function denoted by

$${}_2F_1(a,b;c;z) = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i} \frac{z^i}{i!}, |z| < 1,$$

where$c > b > 0, c > a + b$and

$$(x)_n = \begin{cases} x(x+1)(x+2)\ldots(x+n-1) & for\ n = 1,2,3,\ldots \\ 1 & n = 0 \end{cases}.$$

Definition (3) [2]: For every $f \in S(\theta,\alpha,\beta,\gamma,\lambda)$we defined the convolution operator $W_{a,b,c}(f)(z)$as below:

$$W_{a,b,c}(f)(z) = {}_2F_1(a,b;c;z) * f(z) = z - \sum_{i=2}^{\infty} \frac{(a)_i (b)_i}{(c)_i} \frac{z^i}{i!},$$

where ${}_2F_1(a,b;c;z)$ is Gaussian hypergeometric function (see [1] and [3])introduced in definition (2).

Theorem (10): Let $f$is given by (1.2) be in the class$S(\theta,\alpha,\beta,\gamma,\lambda)$.Then the convolution operator $W_{a,b,c}(f)(z)$ is in the class$S(\theta,\alpha,\beta,\gamma,\lambda)for|z| \leq r(\gamma,\delta)$,where

$$r(\gamma,\delta) = \inf_i \left[\frac{\delta[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]}{\gamma[(1+\beta\delta)i - 2\beta(1-\alpha)(i-1-\theta)]\frac{(a)_i(b)_i}{(c)_i i!}}\right]^{\frac{1}{i-1}}.$$

The result is sharp for the function

$$f_i(z) = z - \frac{(1+2\beta\theta(1-\alpha)+\beta\gamma)}{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)} z^i, i \geq 2.$$

Proof: Since $f \in S(\theta,\alpha,\beta,\gamma,\lambda)$,we have

$$\sum_{i=2}^{\infty} \frac{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\gamma)} a_i \leq 1.$$

It is sufficient to show that

$$\sum_{i=2}^{\infty} \frac{i[(1+\beta\delta)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)\frac{(a)_i(b)_i}{(c)_i i!}}{(1+2\beta\theta(1-\alpha)+\beta\delta)} a_i \leq 1. \quad (7.1)$$

Note that (7.1) is satisfied if

$$\frac{i[(1+\beta\delta)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)\frac{(a)_i(b)_i}{(c)_i i!}}{(1+2\beta\theta(1-\alpha)+\beta\delta)} a_i |z|^{i-1} \leq \frac{i[(1+\beta\gamma)i - 2\beta(1-\alpha)(i-1-\theta)]B_i(\lambda)}{(1+2\beta\theta(1-\alpha)+\beta\gamma)} a_i,$$

Solving for$|z|$we get the result.

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