

On a New Subclass of Harmonic Meromorphic Functions

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Abstract: In the present paper, we introduce a new class of harmonic meromorphic functions $WR_{\bar{H}}(a, b; \lambda, \gamma; v, \mathbb{Q})$, we obtain some geometric properties, like, coefficient inequality, extreme point, convolution and convex combination for this class

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1. Introduction

Let a continuous function $f = u + iv$ be a complex valued harmonic function in a complex domain \mathbb{C} , if both u and v are real harmonic in \mathbb{C} . In any simply connected domain $D \subset \mathbb{C}$, we can write $f = h + \bar{g}$, where h and g are analytic in D . We call h the analytic part and g the co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see Clunie and Sheil-Small (see Clunie and Sheil-Small [5]). Denote by $R_H(1)$ the class of functions $f = h + \bar{g}$ that are harmonic, meromorphic orientation preserving and univalent in the $\bar{U} = \{z \in \mathbb{C}: |z| > 1\}$. So $f = h + \bar{g} \in R_H(1)$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. For $f = h + \bar{g} \in R_H(1)$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^{-n}, g(z) = \sum_{n=1}^{\infty} b_n z^{-n}, |b_1| < 1. \quad (1)$$

We denote by the class $WR_H(a, b; \lambda, \gamma, v, \mathbb{Q})$ the class of all functions of the form (1) that satisfies the condition:

$$Re \left\{ v + \frac{a}{b} \left[\frac{(1 + \mathbb{Q})(zf'(z) + \lambda z^2 f''(z))}{(1 - \lambda)f(z) + \lambda f'(z)} - \mathbb{Q} \right] - v \right\} \geq \gamma. \quad (2)$$

where $|a|, |b| \leq 1, \frac{1}{2} \leq \lambda < 1, 0 \leq \gamma < v < 1, 0 < \mathbb{Q} < 1$.

Also Let $WR_{\bar{H}}(a, b; \lambda, \gamma; v, \mathbb{Q})$ be subclass of $WR_H(a, b; \lambda, \gamma; v, \mathbb{Q})$ consisting of $f = h + \bar{g}$ in which h and g are of the form

$$h(z) = z - \sum_{n=1}^{\infty} a_n z^{-n}; g(z) = - \sum_{n=1}^{\infty} b_n z^{-n}, a_n \geq 0, b_n \geq 0. \quad (3)$$

We obtain sufficient coefficient condition for harmonic meromorphic functions $f = h + \bar{g}$ to be in the class $WR_H(a, b; \lambda, \gamma; v, \mathbb{Q})$. We also show that this coefficient condition is also necessary for $f \in WR_{\bar{H}}(a, b; \lambda, \gamma; v, \mathbb{Q})$. We also obtain distortion bounds, extreme points, convolution condition and convex combination for functions in $WR_{\bar{H}}(a, b; \lambda, \gamma; v, \mathbb{Q})$.

2. Main Result

We begin with a sufficient condition for function in $WR_H(a, b; \lambda, \gamma; v, \mathbb{Q})$.

Theorem (1): Let $f = h + \bar{g}$ with h and g are given by (1) if

$$\sum_{n=2}^{\infty} \left((1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b| \right) |a_n| + \sum_{n=1}^{\infty} \left((1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b| \right) |b_n| \leq |a|(1 - v) + |b|(v - \gamma), \quad (4)$$

where $|a|, |b| \leq 1, \frac{1}{2} \leq \lambda < 1, 0 \leq \gamma < v < 1, 0 < \mathbb{Q} < 1$.

Then f is harmonic Univalent in \bar{U} and $f \in WR_H(a, b; \lambda, \gamma; v, \mathbb{Q})$.

Proof: To show that $f \in WR_H(a, b; \lambda, \gamma; v, \mathbb{Q})$, we notice according to (2), we must have

$$Re \left\{ \frac{A(z)}{B(z)} \right\} > \gamma \text{ where}$$

$$A(z) = vb((1 - \lambda)f(z) + \lambda zf'(z)) + a(1 + \mathbb{Q})(zf'(z) + \lambda z^2 f''(z)) - a(\mathbb{Q} + v)(1 - \lambda)f(z) + \lambda zf'(z)$$

$$B(z) = b((1 - \lambda)f(z) + \lambda zf'(z))$$

Using the fact that $Re(w) \geq \gamma$ if and only if $|1 - \gamma + w| \geq |1 + \gamma - w|$.

For $0 \leq \gamma < 1$ it is enough to show that

$$|A(z) + (1 - \gamma)B(z)| - |A(z) - (1 + \gamma)B(z)| \geq 0$$

Differentiating h and g and Subalstituting in the above inequality we obtain

$$|A(z) + (1 - \gamma)B(z)| - |A(z) - (1 + \gamma)B(z)|$$

$$\begin{aligned}
 &= \left| (v(b-a) + a)z + \sum_{n=2}^{\infty} (vb(1-\lambda(n+1)) + a(v+\varpi+2n)(\lambda(n+1)-1))a_n z^{-n} \right. \\
 &\quad \left. + \sum_{n=1}^{\infty} (vb(1-\lambda(n+1)) + a(v+\varpi+2n)(\lambda(n+1)-1))b_n (\bar{z})^{-n} + (1-\gamma) \left(bz + \sum_{n=2}^{\infty} b(1-\lambda(n+1))a_n z^{-n} + \sum_{n=1}^{\infty} b(1-\lambda(n+1))b_n (\bar{z})^{-n} \right) \right| \\
 &= \left| (v(b-a) + a)z + \sum_{n=2}^{\infty} (vb(1-\lambda(n+1)) + a(v+\varpi+2n)(\lambda(n+1)-1))a_n z^{-n} \right. \\
 &\quad \left. + \sum_{n=1}^{\infty} (vb(1-\lambda(n+1)) + a(v+\varpi+2n)(\lambda(n+1)-1))b_n (\bar{z})^{-n} - (1+\gamma) \left(bz + \sum_{n=2}^{\infty} b(1-\lambda(n+1))a_n z^{-n} + \sum_{n=1}^{\infty} b(1-\lambda(n+1))b_n (\bar{z})^{-n} \right) \right| \\
 &\geq 2(|a|(1-v) + |b|(v-\gamma))|z| - \sum_{n=2}^{\infty} (2(1-\lambda(n+1))(v+\varpi+2n)|a| + 2(\lambda(n+1)-1)(v-\gamma)|b|)|a_n||z^{-n}| \\
 &\quad - \sum_{n=1}^{\infty} (2(1-\lambda(n+1))(v+m+2n)|a| + 2(\lambda(n+1)-1)(v-\gamma)|b|)|b_n|.
 \end{aligned}$$

The harmonic Univalent function .

$$f(z) = z + \sum_{n=2}^{\infty} \frac{x_n}{[(1-\lambda(n+1))(v+\varpi+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|]} (z)^{-n} + \sum_{n=1}^{\infty} \frac{y_n}{[(1-\lambda(n+1))(v+\varpi+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|]} (\bar{z})^{-n} \quad (5)$$

Where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = |a|(1-v) + |b|(v-\gamma)$.

Show that the Coefficient bound given by (4) is sharp

The function of form (5) are in the class $WR_H(a, b; \lambda, \gamma; v, \varpi)$ because

$$\begin{aligned}
 &\sum_{n=2}^{\infty} [(1-\lambda(n+1))(v+\varpi+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|] \frac{|x_n|}{[(1-\lambda(n+1))(v+\varpi+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|]} \\
 &\quad + \sum_{n=1}^{\infty} [(1-\lambda(n+1))(v+\varpi+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|] \frac{|y_n|}{[(1-\lambda(n+1))(v+\varpi+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|]} \\
 &= \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = |a|(1-v) + |b|(v-\gamma) .
 \end{aligned}$$

The restriction placed in Theorem (1) on moduli of the Coefficients of $f = h + \bar{g}$ enables us to Conclude for arbitrary rotation of the Coefficients of f that the resulting functions would still be harmonic Univalent and $f \in WR_H(a, b; \lambda, \gamma; v, \varpi)$.

In the following Theorem ,it is shown that the condition (4) is also necessary for function in $WR_H(a, b; \lambda, \gamma; v, \varpi)$.

Theorem (2): Let $f = h + \bar{g}$ with h and g are given by (3) . A necessary and sufficient condition for f to be in $WR_H(a, b; \lambda, \gamma; v, \varpi)$ is that

$$\begin{aligned}
 &\sum_{n=2}^{\infty} ((1-\lambda(n+1))(v+\varpi+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|) |a_n| \\
 &\quad + \sum_{n=1}^{\infty} ((1-\lambda(n+1))(v+\varpi+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|) |b_n| \\
 &\leq |a|(1-v) + |b|(v-\gamma) . \quad (6)
 \end{aligned}$$

Where $|a|, |b| \leq 1, \frac{1}{2} \leq \lambda < 1, 0 \leq \gamma < v < 1, 0 < \varpi < 1$.

Proof: In view of Theorem 1 .We need only show that $f \notin WR_{\bar{H}}(a, b; \lambda, \gamma; v, \mathbb{Q})$ does not hold .

$f \notin WR_{\bar{H}}(a, b; \lambda, \gamma; v, \mathbb{Q})$ if the coefficient inequality (6)

We note that if $f \in WR_{\bar{H}}(a, b; \lambda, \gamma; v, \mathbb{Q})$ we must have

$$Re \left\{ v + \frac{a}{b} \left[\left(\frac{(1 + \mathbb{Q})(zf'(z) + \lambda z^2 f''(z))}{(1 - \lambda)f(z) + \lambda f'(z)} - \mathbb{Q} \right) - v \right] \right\} > \gamma .$$

This is equivalent to

$$Re \left\{ \frac{vb((1 - \lambda)f(z) + \lambda f'(z)) + a(1 + \mathbb{Q})(zf'(z) + \lambda z^2 f''(z)) - a(\mathbb{Q} + V)((1 - \lambda)f(z) + \lambda f'(z))}{b[(1 - \lambda)f(z) + \lambda f'(z)]} \right\}$$

$$= Re \left\{ \frac{a(1-v)+b(v-\gamma)z^{-\sum_{n=2}^{\infty}((1-\lambda(n+1))(V+\mathbb{Q}+2n)a+(\lambda(n+1)-1)(v-\gamma)b)} a_n z^{-n}}{bz + \sum_{n=2}^{\infty} b(1-\lambda(n+1))a_n z^{-n} - \sum_{n=1}^{\infty} b(1-\lambda(n+1))b_n z^{-n}} - \frac{\sum_{n=2}^{\infty}((1-\lambda(n+1))(v+\mathbb{Q}+2n)a+(\lambda(n+1)-1)(v-\gamma)b) b_n (\bar{z})^{-n}}{bz + \sum_{n=2}^{\infty} b(1-\lambda(n+1))a_n z^{-n} - \sum_{n=1}^{\infty} b(1-\lambda(n+1))b_n z^{-n}} \right\} \geq 0.$$

This inequality must hold for all $z \in \bar{U}$ and for, $\frac{1}{2} \leq \lambda < 1, 0 \leq \gamma < v < 1, 0 < \mathbb{Q} < 1$ and for any $0 < |a| < 1, 0 < |b| < 1$.

Letting $z = r > 1$ and a, b are real and positive so that $|a| = a, |b| = b$, we have

$$= Re \left\{ \frac{a(1 - v) + b(v - \gamma) - \sum_{n=2}^{\infty} ((1 - \lambda(n + 1))(V + \mathbb{Q} + 2n)a + (\lambda(n + 1) - 1)(v - \gamma)b) a_n r^{-n-1}}{b + \sum_{n=2}^{\infty} b(1 - \lambda(n + 1))a_n r^{-n-1} - \sum_{n=1}^{\infty} b(1 - \lambda(n + 1))b_n r^{-n-1}} - \frac{\sum_{n=2}^{\infty} ((1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)a + (\lambda(n + 1) - 1)(v - \gamma)b) b_n (\bar{r})^{-n-1}}{b + \sum_{n=2}^{\infty} b(1 - \lambda(n + 1))a_n r^{-n-1} - \sum_{n=1}^{\infty} b(1 - \lambda(n + 1))b_n r^{-n-1}} \right\} = \frac{A(r)}{B(r)} \geq 0$$

If the condition (6) does not hold , then $A(r)$ is negative for r sufficiently close to 1 . Thus there exists a $z_0 = r > 1$ for which the quotient $\frac{A(r)}{B(r)}$ is negative .This contradicts that $\frac{A(r)}{B(r)} \geq 0$ and so that the proof is complete.

Extreme points for $WR_{\bar{H}}(a, b; \lambda, \gamma; v, \mathbb{Q})$ are given in following Theorem .

Theorem (3): Let the function f be given by (3) . Then $f \in WR_{\bar{H}}(a, b; \lambda, \gamma; v, \mathbb{Q})$ if and only if

$$f(z) = \sum_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z)) , z \in \bar{U} \quad (7)$$

where $h_1(z) = z$,

$$h_n(z) = z - \frac{|a|(1 - v) + |b|(v - \gamma)}{(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|} z^{-n} \quad (n = 2, 3, \dots)$$

and

$$g_n(z) = z - \frac{|a|(1 - v) + |b|(V - \gamma)}{(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|} (\bar{z})^{-n} \quad (n = 1, 2, \dots)$$

$\sum_{n=1}^{\infty} (x_n + y_n) = 1, (x_n \geq 0, y_n \geq 0)$.

In particular , the extreme points of $WR_{\bar{H}}(a, b; \lambda, \gamma; v, \mathbb{Q})$ are h_n and g_n .

Proof: Assume that f can be expressed by (7) . Then , we have

$$f(z) = \sum_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z))$$

$$= \sum_{n=1}^{\infty} (x_n + y_n)z - \sum_{n=2}^{\infty} \frac{|a|(1 - v) + |b|(v - \gamma)}{(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|} x_n z^{-n}$$

$$- \sum_{n=1}^{\infty} \frac{|a|(1 - v) + |b|(v - \gamma)}{(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|} y_n (\bar{z})^{-n}$$

$$= z - \sum_{n=2}^{\infty} \frac{|a|(1 - v) + |b|(v - \gamma)}{(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|} x_n z^{-n}$$

$$- \sum_{n=1}^{\infty} \frac{|a|(1 - v) + |b|(v - \gamma)}{(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|} y_n (\bar{z})^{-n}$$

Therefore

$$\sum_{n=2}^{\infty} (1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a|$$

$$+ (\lambda(n + 1) - 1)(v - \gamma)|b| \frac{|a|(1 - v) + |b|(v - \gamma)}{(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|} x_n$$

$$+ \sum_{n=1}^{\infty} (1 - \lambda(n+1))(V + m + 2n)|a|$$

$$+ (\lambda(n+1) - 1)(V - \gamma)|b| \frac{|a|(1-v) + |b|(v-\gamma)}{(1 - \lambda(n+1))(v + \square + 2n)|a| + (\lambda(n+1) - 1)(v - \gamma)|b|} y_n$$

$$= (|a|(1-v) + |b|(v-\gamma)) \left(\sum_{n=1}^{\infty} (x_n + y_n) - x_1 \right)$$

$$= (|a|(1-v) + |b|(v-\gamma))(1 - x_1) \leq (|a|(1-v) + |b|(v-\gamma))$$

So $f \in WR_{\bar{H}}(a, b; \lambda, \gamma; v, \square)$

Conversely, Let $f \in WR_{\bar{H}}(a, b; \lambda, \gamma; v, \square)$, by putting

$$x_n = \frac{(1 - \lambda(n+1))(v + \square + 2n)|a| + (\lambda(n+1) - 1)(v - \gamma)|b|}{|a|(1-v) + |b|(v-\gamma)} a_n, (n = 2, 3, \dots)$$

and

$$y_n = \frac{(1 - \lambda(n+1))(v + \square + 2n)|a| + (\lambda(n+1) - 1)(v - \gamma)|b|}{|a|(1-v) + |b|(v-\gamma)} b_n, (n = 1, 2, \dots).$$

We define $x_1 = 1 - \sum_{n=2}^{\infty} x_n - \sum_{n=1}^{\infty} y_n$.

Then, note that $0 \leq x_n \leq 1 (n = 2, 3, \dots), 0 \leq y_n \leq 1 (n = 1, 2, \dots)$.

Hence,

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n (\bar{z})^{-n}$$

$$= z - \sum_{n=2}^{\infty} \frac{|a|(1-v) + |b|(v-\gamma)}{(1 - \lambda(n+1))(v + \square + 2n)|a| + (\lambda(n+1) - 1)(v - \gamma)|b|} x_n z^{-n}$$

$$- \sum_{n=1}^{\infty} \frac{|a|(1-v) + |b|(v-\gamma)}{(1 - \lambda(n+1))(v + \square + 2n)|a| + (\lambda(n+1) - 1)(v - \gamma)|b|} y_n (\bar{z})^{-n}$$

$$= z - \sum_{n=2}^{\infty} (z - h_n(z)) x_n - \sum_{n=1}^{\infty} (z - g_n(z)) y_n$$

$$= \left(1 - \sum_{n=2}^{\infty} x_n - \sum_{n=1}^{\infty} y_n \right) z + \sum_{n=2}^{\infty} x_n h_n(z) + \sum_{n=1}^{\infty} y_n g_n(z)$$

$$= x_1 h_1(z) + \sum_{n=2}^{\infty} x_n h_n(z) + \sum_{n=1}^{\infty} y_n g_n(z)$$

$$= \sum_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z))$$

That is the required representation

Now, we show $WR_{\bar{H}}(a, b; \lambda, \gamma; v, \square)$ is closed under Convex Combination of this members.

Theorem (4): The class $WR_{\bar{H}}(a, b; \lambda, \gamma; v, \square)$ is closed under Convex Combination.

Proof: For $j = 1, 2, 3, \dots$, let $f_j \in WR_{\bar{H}}(a, b; \lambda, \gamma; v, \square)$, where f_j is given

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^{-n} - \sum_{n=1}^{\infty} b_{n,j} (\bar{z})^{-n}.$$

Then by (4), we have

For $\sum_{j=1}^{\infty} t_j = 1, 0 \leq t_j \leq 1$, the Convex Combination of f_j may be written as

$$\sum_{j=1}^{\infty} t_j f_j(z) = z + \sum_{n=2}^{\infty} \left(\sum_{j=1}^{\infty} t_j a_{n,j} \right) z^{-n}$$

$$- \sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} t_j b_{n,j} \right) (\bar{z})^{-n}.$$

Then, by (4), we have

$$\sum_{n=2}^{\infty} \left((1 - \lambda(n+1))(v + \square + 2n)|a| \right.$$

$$\left. + (\lambda(n+1) - 1)(v - \gamma)|b| \right) \left(\sum_{j=1}^{\infty} t_j a_{n,j} \right)$$

$$+ \sum_{n=1}^{\infty} \left((1 - \lambda(n+1))(v + \square + 2n)|a| \right.$$

$$\left. + (\lambda(n+1) - 1)(v - \gamma)|b| \right) \left(\sum_{j=1}^{\infty} t_j b_{n,j} \right)$$

$$\begin{aligned}
 &= \sum_{j=1}^{\infty} t_j \left\{ \sum_{n=2}^{\infty} \left((1-\lambda(n+1))(v+\square+2n)|a| \right. \right. \\
 &\quad \left. \left. + (\lambda(n+1)-1)(v-\gamma)|b| \right) a_{n,j} \right. \\
 &\quad \left. + \sum_{n=1}^{\infty} \left((1-\lambda(n+1))(v+\square+2n)|a| \right. \right. \\
 &\quad \left. \left. + (\lambda(n+1)-1)(v-\gamma)|b| \right) b_{n,j} \right\} \\
 &\leq \sum_{j=1}^{\infty} t_j [|a|(1-v) + |b|(v-\gamma)] \\
 &= |a|(1-v) + |b|(v-\gamma),
 \end{aligned}$$

therefore

$$\sum_{j=1}^{\infty} t_j f_j(z) \in WR_H(a, b; \lambda, \gamma; v, \square),$$

This complete this proof .

The distortion bound for functions in $WR_H(a, b; \lambda, \gamma; v, \square)$ are given by following theorem

Theorem (5): Let $f \in WR_H(a, b; \lambda, \gamma; v, \square)$. Then for $|z| = r < 1$, we have

$$|f(z)| \geq (1-b_1)r^{-1} - \frac{[|a|(1-v) + |b|(v-\gamma)](1-b_1)}{(1-3\lambda)(4+v+\square) + (3\lambda-1)(v-\gamma)} r^{-2} \quad (9)$$

and

$$\begin{aligned}
 |f(z)| &\leq (1-b_1)r^{-1} \\
 &+ \frac{[|a|(1-v) + |b|(v-\gamma)](1-b_1)}{(1-3\lambda)(4+v+\square) + (3\lambda-1)(v-\gamma)} r^{-2} \quad (10)
 \end{aligned}$$

Proof: Assume that $f \in WR_H(a, b; \lambda, \gamma; v, \square)$. Then by (4) , we get

$$\begin{aligned}
 |f(z)| &= \left| z + \sum_{n=2}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n (\bar{z})^{-n} \right| \\
 &\geq (1-b_1)r^{-1} - \sum_{n=2}^{\infty} (a_n + b_n) r^{-n} \\
 &\geq (1-b)r^{-1} - \sum_{n=2}^{\infty} (a_n + b_n) r^{-2} \\
 &= (1-b_1)r^{-1} \\
 &- \frac{1}{(1-3\lambda)(4+v+\square) + (3\lambda-1)(v-\gamma)} \sum_{n=2}^{\infty} (1-3\lambda)(4 \\
 &+ v + \square) + (3\lambda-1)(v-\gamma)(a_n + b_n) \\
 &\geq (1-b_1)r^{-1} \\
 &- \frac{1}{(1-3\lambda)(4+v+\square) + (3\lambda-1)(v-\gamma)} \sum_{n=2}^{\infty} (1 \\
 &- \lambda(n+1)(v+\square+2n) + (\lambda(n+1)(v-\gamma)(a_n + b_n) \\
 &\geq (1-b_1)r^{-1} \\
 &- \frac{1}{(1-3\lambda)(4+v+\square) + (3\lambda-1)(v-\gamma)} [|a|(1-v) \\
 &+ |b|(v-\gamma)] - [|a|(1-v) + |b|(v-\gamma)] b_1] r^{-2} \\
 &= (1-b_1)r^{-1} \\
 &- \frac{[|a|(1-v) + |b|(v-\gamma)](1-b_1)}{(1-3\lambda)(4+v+\square) + (3\lambda-1)(v-\gamma)} r^{-2}
 \end{aligned}$$

Relation (10) can be proved by using the similar statement . So, the proof is Complete .

Theorem (6): The class $WR_H(a, b; \lambda, \gamma; v, \square)$ is Convex set .

Proof: Let the function $f_j (j = 1, 2)$ be in class $WR_H(a, b; \lambda, \gamma; v, \square)$.

It is sufficient to show that the function h defined by $h(z) = (1-t)f_1(z) + tf_2(z)$, $(0 \leq t < 1)$ it is in the class $WR_H(a, b; \lambda, \gamma; v, \square)$, where

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^{-n} - \sum_{n=1}^{\infty} b_{n,j} (\bar{z})^{-n} , (j = 1, 2)$$

Since for $0 \leq t \leq 1$,

$$\begin{aligned}
 h(z) &= z + \sum_{n=2}^{\infty} \left((1-t)a_{n,1} - ta_{n,2} \right) z^{-n} \\
 &- \sum_{n=1}^{\infty} \left((1-t)b_{n,1} - tb_{n,2} \right) (\bar{z})^{-n} .
 \end{aligned}$$

In view of Theorem (2) , we have

$$\begin{aligned}
 &\sum_{n=2}^{\infty} \left((1-\lambda(n+1))(v+\square+2n)|a| \right. \\
 &\quad \left. + (\lambda(n+1)-1)(v-\gamma)|b| \right) \left((1-t)a_{n,1} \right. \\
 &\quad \left. - ta_{n,2} \right) \\
 &+ \sum_{n=1}^{\infty} \left((1-\lambda(n+1))(v+\square+2n)|a| \right. \\
 &\quad \left. + (\lambda(n+1)-1)(v-\gamma)|b| \right) \left((1-t)b_{n,1} \right. \\
 &\quad \left. - tb_{n,2} \right) \\
 &= (1-t) \left(\sum_{n=2}^{\infty} \left((1-\lambda(n+1))(v+\square+2n)|a| \right. \right. \\
 &\quad \left. \left. + (\lambda(n+1)-1)(v-\gamma)|b| \right) a_{n,1} \right. \\
 &\quad \left. + \sum_{n=1}^{\infty} \left((1-\lambda(n+1))(v+\square+2n)|a| \right. \right. \\
 &\quad \left. \left. + (\lambda(n+1)-1)(v-\gamma)|b| \right) b_{n,1} \right) \\
 &+ t \left(\sum_{n=2}^{\infty} \left((1-\lambda(n+1))(v+\square+2n)|a| \right. \right. \\
 &\quad \left. \left. + (\lambda(n+1)-1)(v-\gamma)|b| \right) a_{n,2} \right. \\
 &\quad \left. + \sum_{n=1}^{\infty} \left((1-\lambda(n+1))(v+\square+2n)|a| \right. \right. \\
 &\quad \left. \left. + (\lambda(n+1)-1)(v-\gamma)|b| \right) b_{n,2} \right)
 \end{aligned}$$

$$\leq (1-t) (|a|(1-v) + |b|(v-\gamma) + t (|a|(1-v) + |b|(v-\gamma))) = |a|(1-v) + |b|(v-\gamma)$$

Hence $h \in WR_H(a, b; \lambda, \gamma; v, \square)$

We define the Convolution two harmonic function f and F by

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^{-n} - \sum_{n=1}^{\infty} b_n B_n (\bar{z})^{-n}, \quad (11)$$

$$F(z) = z^p + \sum_{n=2}^{\infty} A_n z^{-n} - \sum_{n=1}^{\infty} B_n (\bar{z})^{-n}.$$

where

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n (\bar{z})^{-n},$$

and

Theorem (7): For $0 \leq \beta \leq \gamma < 1$. Let $f \in WR_{\bar{H}}(a, b; \lambda, \gamma; v, \mathbb{Q})$ and $F \in WR_{\bar{H}}(a, b; \lambda, \gamma, \beta, \mathbb{Q})$. Then $f * F \in WR_{\bar{H}}(a, b; \lambda, \gamma; v, \mathbb{Q}) \subset WR_{\bar{H}}(a, b; \lambda, \gamma; \beta, \mathbb{Q})$

Proof: Let the Convolution $(f * F)$ be of form (10). Then we want to prove that the Coefficient of $(f * F)$ Satisfy the Condition of Theorem (2).

Since $f \in WR_{\bar{H}}(a, b; \lambda, \gamma, v, \mathbb{Q})$ and $F \in WR_{\bar{H}}(a, b; \lambda, \gamma, \beta, \mathbb{Q})$. Then by Theorem (2), we have

$$\sum_{n=2}^{\infty} \frac{(1 - \lambda(n + 1)) + (v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|}{|a|(1 - v) + |b|(v - \gamma)} a_n + \sum_{n=1}^{\infty} \frac{(1 - \lambda(n + 1)) + (v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|}{|a|(1 - v) + |b|(v - \gamma)} b_n \leq 1$$

and

$$\sum_{n=2}^{\infty} \frac{(1 - \lambda(n + 1)) + (v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \beta)|b|}{|a|(1 - v) + |b|(v - \beta)} A_n + \sum_{n=1}^{\infty} \frac{(1 - \lambda(n + 1)) + (v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \beta)|b|}{|a|(1 - v) + |b|(v - \beta)} B_n \leq 1$$

From (4), we obtain the following inequalities

$$A_n \leq \frac{|a|(1 - v) + |b|(v - \beta)}{(1 - \lambda(n + 1)) + (v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \beta)|b|} \quad (n = 2, 3, \dots)$$

$$B_n \leq \frac{|a|(1 - v) + |b|(v - \beta)}{(1 - \lambda(n + 1)) + (v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \beta)|b|} \quad (n = 1, 2, \dots)$$

Therefore,

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|}{|a|(1 - v) + |b|(v - \gamma)} a_n A_n \\ & + \sum_{n=1}^{\infty} \frac{(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|}{|a|(1 - v) + |b|(v - \gamma)} b_n B_n \\ & = \frac{(1 - 2\lambda)(2 + v + \mathbb{Q})|a| + (2\lambda - 1)(v - \gamma)|b|}{|a|(1 - v) + |b|(v - \gamma)} b_1 B_1 \\ & + \sum_{n=2}^{\infty} \frac{(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|}{|a|(1 - v) + |b|(v - \gamma)} a_n A_n \\ & + \sum_{n=1}^{\infty} \frac{(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|}{|a|(1 - v) + |b|(v - \gamma)} b_n B_n \end{aligned}$$

$$\begin{aligned} & < \frac{[(1 - 2\lambda)(2 + v + \mathbb{Q})|a| + (2\lambda - 1)(v - \gamma)|b|][|a|(1 - v) + |b|(v - \beta)]}{[|a|(1 - v) + |b|(v - \gamma)][(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \beta)|b|]} \\ & + \sum_{n=2}^{\infty} \frac{[(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|][|a|(1 - v) + |b|(v - \beta)]}{[|a|(1 - v) + |b|(v - \gamma)][(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \beta)|b|]} a_n \\ & + \sum_{n=2}^{\infty} \frac{[(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|][|a|(1 - v) + |b|(v - \beta)]}{[|a|(1 - v) + |b|(v - \gamma)][(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \beta)|b|]} b_n \\ & = \sum_{n=2}^{\infty} \frac{[(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|][|a|(1 - v) + |b|(v - \beta)]}{[|a|(1 - v) + |b|(v - \gamma)][(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \beta)|b|]} a_n \\ & + \sum_{n=1}^{\infty} \frac{[(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|][|a|(1 - v) + |b|(v - \beta)]}{[|a|(1 - v) + |b|(v - \gamma)][(1 - \lambda(n + 1))(v + \mathbb{Q} + 2n)|a| + (\lambda(n + 1) - 1)(v - \beta)|b|]} b_n \end{aligned}$$

$$\leq \sum_{n=2}^{\infty} \frac{(1 - \lambda(n + 1))(v + \alpha + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|}{|a|(1 - v) + |b|(v - \gamma)} a_n + \sum_{n=1}^{\infty} \frac{(1 - \lambda(n + 1))(v + \alpha + 2n)|a| + (\lambda(n + 1) - 1)(v - \gamma)|b|}{|a|(1 - v) + |b|(v - \gamma)} b_n \leq 1$$

Then $(f * F) \in WR_H(a, b; \lambda, \gamma; v, \alpha) \subset WR_H(a, b; \lambda, \gamma; \beta, \alpha)$.

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