

On a Subclass of Spiral-Like Functions by Applying Fractional Calculus

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Abstract: In this paper, we study an application of the fractional calculus techniques for the subclass of Spiral-Like functions $\mathcal{R}_\delta(\beta, \alpha, \gamma)$. Distortion theorems for the fractional derivative and fractional integration are obtained. Also we get some geometric properties, like, extreme points, radii of starlikeness, convexity and close-to-convexity, closure theorems and partial sum.

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1. Introduction

Let \mathcal{R} denote the class of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0, n \in N = \{1, 2, 3, \dots\}), \quad (1)$$

which are analytic and univalent in the unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$. For β real, $|\beta| < \frac{\pi}{2}$, a function f in the form (1) is said to be in $\mathcal{R}(\beta)$, the class of β -spiral-Like function if

$$\operatorname{Re} \left(e^{i\beta} \frac{zf'(z)}{f(z)} \right) > 0, z \in U. \quad (2)$$

For $\beta = 0$, $\mathcal{R}(0) \equiv \mathcal{R}$ is the well-known class of functions starlike with respect to the origin, for $\beta \neq 0$, it is known that $\mathcal{R}(\beta)$ is not contained in \mathcal{R} . In fact the class $\mathcal{R}(\beta)$ was introduced and shown to be a subfamily of \mathcal{R} by Spaček [7]. Later, Zomorski [9] obtained sharp coefficient bounds for the class. Recently Several authors studied Spiral-Like function for different classes, like, Atshan [1].

Definition (1)[8]: The fractional integral of order δ ($\delta > 0$), is defined by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(t)}{(z-t)^{1-\delta}} dt, \quad (3)$$

where f is an analytic function in a simply-connected region of the z -plane containing the origin, and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $\log(z-t)$ to be real, when $\operatorname{Re}(z-t) > 0$.

Definition (2)[8]: The fractional derivative of order δ ($0 \leq \delta < 1$) is defined by

$$D_z^\delta f(z) = \frac{1}{\Gamma(1-\delta)} \int_0^z \frac{f(t)}{(z-t)^\delta} dt, \quad (4)$$

where $f(z)$ is as in Definition (1) and the multiplicity of $(z-t)^{-\delta}$ is removed like Definition (1).

Definition (3) [8]: [Under the condition of Definition 2] the fractional derivative of order $n + \delta$, ($n = 0, 1, 2, \dots$) is defined by

$$= \left| - \sum_{n=2}^{\infty} n(n-1)a_n z^n \right| - \left| 2\alpha(1-\gamma)e^{-i\beta} \cos\beta z - \sum_{n=2}^{\infty} n[(n-1)(1+2\alpha) + 2\alpha(1-\gamma)e^{-i\beta} \cos\beta] a_n z^n \right|$$

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^\delta f(z).$$

From Definition (1) and Definition (2) by applying a simple calculation, we get

$$D_z^\delta f(z) = \frac{1}{\Gamma(2+\delta)} z^{\delta+1} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} a_n z^{n+\delta}, \quad (5)$$

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(2-\delta)} z^{\delta-1} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} a_n z^{n+\delta}, \quad (6)$$

Definition (4): We introduce a new subclass of $\mathcal{R}(\beta)$ as functions in the form (1) that satisfy the inequality:

$$\left| \frac{\frac{zf''(z)}{f'(z)}}{2\alpha \left[\frac{zf''(z)}{f'(z)} + (1-\gamma)e^{-i\beta} \cos\beta \right] + \frac{zf''(z)}{f'(z)}} \right| < 1, \quad \text{for } z \in U,$$

where $0 \leq \gamma < 1, 0 < \beta \leq \frac{\pi}{2}, |\beta| < \frac{\pi}{2}$. We denote for our class by $\mathcal{R}_\delta(\beta, \alpha, \gamma)$.

2. Main Results

In the following theorem, we obtain the coefficient inequality for the class $\mathcal{R}_\delta(\beta, \alpha, \gamma)$.

Theorem (1): Let $f(z) \in \mathcal{R}(\beta)$. Then $f(z)$ is in the class $\mathcal{R}_\delta(\beta, \alpha, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|] a_n \leq \alpha(1-\gamma)|e^{i\beta} \cos\beta|, \quad (7)$$

where $0 \leq \gamma < 1, 0 < \beta \leq \frac{\pi}{2}, |\beta| < \frac{\pi}{2}$.

The result (7) is sharp for the function $f(z)$ given by:

$$f(z) = z - \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]} z^n. \quad (8)$$

Proof: Let (7) holds true and $|z| = 1$, we have

$$|zf''(z)| - |2\alpha[zf''(z) + (1-\gamma)e^{-i\beta} \cos\beta f'(z)] + zf''(z)|$$

$$\begin{aligned} &\leq \sum_{n=2}^{\infty} n(n-1)a_n - 2\alpha(1-\gamma)|e^{-i\beta} \cos\beta| + \sum_{n=2}^{\infty} n[(n-1)(1+2\alpha) + 2\alpha(1-\gamma)e^{-i\beta} \cos\beta]a_n \\ &= \sum_{n=2}^{\infty} n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]a_n - \alpha(1-\gamma)|e^{i\beta} \cos\beta| \leq 0, \end{aligned}$$

by hypothesis. Thus by Maximum modules theorem $f \in \mathcal{R}_\delta(\beta, \alpha, \gamma)$.
 Conversely, assume that

$$\begin{aligned} &\left| \frac{\frac{zf''(z)}{f'(z)}}{2\alpha \left[\frac{zf''(z)}{f'(z)} + (1-\gamma)e^{-i\beta} \cos\beta f'(z) \right] + \frac{zf''(z)}{f'(z)}}} \right| \\ &= \left| \frac{zf''(z)}{2\alpha [zf''(z) + (1-\gamma)e^{-i\beta} \cos\beta f'(z)] + zf''(z)} \right| \\ &= \left| \frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{2\alpha(1-\gamma)e^{-i\beta} \cos\beta - \sum_{n=2}^{\infty} n[(n-1)(1+2\alpha) + 2\alpha(1-\gamma)e^{-i\beta} \cos\beta]a_n z^{n-1}} \right| < 1. \end{aligned}$$

Since $Re(z) \leq |z|$ for all z , we have

$$Re \left(\frac{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{2\alpha(1-\gamma)e^{-i\beta} \cos\beta - \sum_{n=2}^{\infty} n[(n-1)(1+2\alpha) + 2\alpha(1-\gamma)e^{-i\beta} \cos\beta]a_n z^{n-1}} \right) < 1, \quad (9)$$

we can choose value of z on the real axis so that $f(z)$ is real. Let $z \rightarrow 1^-$, through real values, so we write (9) as

$$\begin{aligned} \sum_{n=2}^{\infty} n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]a_n & \Gamma(2+\delta)z^{-\delta} D_z^{-\delta} f(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)} a_n z^n \\ & \leq \alpha(1-\gamma)|e^{i\beta} \cos\beta|. \blacksquare \\ \text{Corollary (1): Let } f(z) \in \mathcal{R}_\delta(\beta, \alpha, \gamma). \text{ Then} & = z - \sum_{n=2}^{\infty} \theta(n) a_n z^n. \quad (14) \end{aligned}$$

$a_n \leq \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]}$, $n \geq 2$. where

Theorem (2): Let $f(z) \in \mathcal{R}_\delta(\beta, \alpha, \gamma)$. Then

$$\begin{aligned} &\leq \frac{1}{\Gamma(2+\delta)} |z|^{\delta+1} \left[1 + \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{(2+\delta)[1 + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]} |z| \right], \quad (10) \end{aligned}$$

and

$$\begin{aligned} &\geq \frac{1}{\Gamma(2+\delta)} |z|^{\delta+1} \left[1 - \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{(2+\delta)[1 + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]} |z| \right]. \quad (11) \end{aligned}$$

The inequalities in (10) and (11) are attained for the function $f(z)$ given by:

$$\begin{aligned} f(z) &= z - \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{2[1 + \alpha(1 + (1-\gamma))|e^{i\beta} \cos\beta|]} z^2. \quad (12) \end{aligned}$$

Proof: by using Theorem (1), we have

$$\begin{aligned} &\sum_{n=2}^{\infty} a_n \leq \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{2[1 + \alpha(1 + (1-\gamma))|e^{i\beta} \cos\beta|]}, \quad (13) \end{aligned}$$

by Definition (3), we have

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(2+\delta)} z^{1+\delta} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\delta)} a_n z^{n+\delta},$$

and

$$\begin{aligned} |D_z^{-\delta} f(z)| & \text{ We know that } \theta(n) \text{ is a decreasing function of } n \text{ and} \\ & 0 < \theta(n) < \theta(2) = \frac{2}{2+\delta}. \end{aligned}$$

Using (13) and (14), we have

$$|\Gamma(2+\delta)z^{-\delta} D_z^{-\delta} f(z)| \leq |z| + \theta(2)|z|^2 \sum_{n=2}^{\infty} a_n$$

$$\begin{aligned} |D_z^{-\delta} f(z)| & \leq |z| + \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{(2+\delta)[1 + \alpha(1 + (1-\gamma))|e^{i\beta} \cos\beta|]} |z|^2, \end{aligned}$$

which gives (10), we also have

$$|\Gamma(2+\delta)z^{-\delta} D_z^{-\delta} f(z)| \geq |z| - \theta(2)|z|^2 \sum_{n=2}^{\infty} a_n$$

$$\begin{aligned} f(z) & \geq |z| - \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{(2+\delta)[1 + \alpha(1 + (1-\gamma))|e^{i\beta} \cos\beta|]} |z|^2, \end{aligned}$$

which gives (11). \blacksquare

Theorem(3): Let $f(z) \in \mathcal{R}_\delta(\beta, \alpha, \gamma)$. Then

$$\begin{aligned} |D_z^\delta f(z)| & \leq \frac{1}{\Gamma(2-\delta)} |z|^{1-\delta} \left[1 + \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{(2-\delta)[1 + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]} |z| \right], \quad (15) \end{aligned}$$

and

$$\begin{aligned} & \geq \frac{1}{\Gamma(2-\delta)} |z|^{1-\delta} \left[1 - \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{(2-\delta)[1+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} |z| \right]. \quad (16) \end{aligned}$$

The inequalities in (15) and (16) are attained for the function $f(z)$ given by (12).

Proof: From Definition (3), we have

$$D_z^\delta f(z) = \frac{1}{\Gamma(2-\delta)} z^{\delta+1} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} a_n z^{n-\delta}$$

and

$$\begin{aligned} \Gamma(2-\delta) z^\delta D_z^\delta f(z) &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} a_n z^n \\ &= z - \sum_{n=2}^{\infty} \theta(n) a_n z^n \end{aligned}$$

where $\Phi(n) = \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}$ for $n \geq 2$, $\Phi(n)$ is a decreasing of n , then

$$\Phi(n) \leq \Phi(2) = \frac{2}{2-\delta}.$$

Also by using Theorem (1), we have

$$\sum_{n=2}^{\infty} a_n \leq \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{2[1+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]},$$

thus

$$\begin{aligned} |\Gamma(2-\delta) z^\delta D_z^\delta f(z)| &\leq |z| - \theta(2) |z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| - \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{(2-\delta)[1+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} |z|^2. \end{aligned}$$

Then

$$\begin{aligned} |D_z^\delta f(z)| &\leq \frac{1}{\Gamma(2-\delta)} |z|^{1-\delta} \left[1 + \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{(2-\delta)[1+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} |z| \right], \end{aligned}$$

and by the same way, we obtain

$$\begin{aligned} & \geq \frac{1}{\Gamma(2-\delta)} |z|^{1-\delta} \left[1 - \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{(2-\delta)[1+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} |z| \right]. \quad \blacksquare \end{aligned}$$

Corollary (2): For every $f(z) \in \mathcal{R}_\delta(\beta, \alpha, \gamma)$, we have

$$\begin{aligned} \frac{|z|^2}{2} \left[1 - \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{3[1+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} |z| \right] &\leq \left| \int_0^z f(t) dt \right| \leq \frac{|z|^2}{2} \left[1 + \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{3[1+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} |z| \right], \quad (17) \end{aligned}$$

and

$$\begin{aligned} |z| \left[1 - \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{2[1+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} |z| \right] &\leq |f(z)| \\ &\leq |z| \left[1 + \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{2[1+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} |z| \right] \quad (18) \end{aligned}$$

Proof: i) By Definition (1) and Theorem (2) for $\delta = 1$, we have

$$D_z^{-1} f(z) = \int_0^z f(t) dt,$$

the result is true.

ii) By Definition (2) and Theorem (2) for $\delta = 0$, we have

$$D_z^0 f(z) = \frac{d}{dz} \int_0^z f(t) dt = f(z),$$

the result is true. ■

Corollary (3): $D_z^{-\delta} f(z)$ and $D_z^\delta f(z)$ are included in the disk with center at origin and radii

$$\begin{aligned} & \frac{1}{\Gamma(2+\delta)} \left[1 - \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{(2+\delta)[1+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} \right], \frac{1}{\Gamma(2-\delta)} \left[1 - \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{(2-\delta)[1+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} \right]. \end{aligned}$$

In the following theorem, we obtain the extreme points of the class $\mathcal{R}_\delta(\beta, \alpha, \gamma)$.

Theorem (4):

Let $f_1(z) = z, f_n(z) = z + \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} z^n, (n \geq 2, (19))$ where $n \in \mathbb{N}, 0 \leq \gamma < 1, 0 < \beta \leq \pi/2$. Then the function $f \in \mathcal{R}_\delta(\beta, \alpha, \gamma)$ if and only if it can be expressed in the form:

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \quad (20)$$

where $\mu_n \geq 0, \sum_{n=1}^{\infty} \mu_n = 1$ or $1 = \mu_1 + \sum_{n=2}^{\infty} \mu_n$.

Proof: Let $f(z)$ can be expressed as in (20). Then

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \mu_n f_n(z) = \mu_1 f_1(z) + \sum_{n=2}^{\infty} \mu_n f_n(z) \\ &= \mu_1 z + \sum_{n=2}^{\infty} \mu_n \left(z + \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} z^n \right) \\ &= z \left(\mu_1 + \sum_{n=2}^{\infty} \mu_n \right) + \sum_{n=2}^{\infty} \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} \mu_n z^n \\ &= z + \sum_{n=2}^{\infty} \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} \mu_n z^n \\ &= z - \sum_{n=2}^{\infty} h_n z^n, \end{aligned}$$

where

$$h_n = \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} \mu_n.$$

Thus

$$\begin{aligned} \sum_{n=2}^{\infty} h_n \frac{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]}{\alpha(1-\gamma)|e^{i\beta} \cos\beta|} &= \sum_{n=2}^{\infty} \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]} \mu_n \\ &\times \frac{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta} \cos\beta|]}{\alpha(1-\gamma)|e^{i\beta} \cos\beta|} \end{aligned}$$

$$= \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1$$

< 1.

Therefore, we have $f \in \mathcal{R}_\delta(\beta, \alpha, \gamma)$.

Conversely, suppose that $f \in \mathcal{R}_\delta(\beta, \alpha, \gamma)$. Then by (7), we have

$$a_n \leq \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]}, \quad (n \geq 2)$$

we may set,

$$\mu_n = \frac{n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]}{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}, \quad (n \geq 2)$$

and

$$\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n.$$

Then

$$\begin{aligned} f(z) &= z + \sum_{n=2}^{\infty} a_n z^n, \\ &= z + \sum_{n=2}^{\infty} \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]} \mu_n z^n \\ &= z + \sum_{n=2}^{\infty} \mu_n (z - f_n(z)) = \left(1 - \sum_{n=2}^{\infty} \mu_n\right) z + \sum_{n=2}^{\infty} \mu_n f_n(z) \\ &= z\mu_1 + \sum_{n=2}^{\infty} \mu_n f_n(z) = \sum_{n=1}^{\infty} \mu_n f_n(z). \quad \blacksquare \end{aligned}$$

In the following theorems, we obtain radii of starlikeness, convexity and close-to-convexity of the class $\mathcal{R}_\delta(\beta, \alpha, \gamma)$.

Theorem (5): If $f \in \mathcal{R}_\delta(\beta, \alpha, \gamma)$, then f is starlike of order δ ($0 \leq \delta < 1$) in the disk $|z| < r_1(\beta, \alpha, \gamma, \delta)$, where

$$r_1(\beta, \alpha, \gamma, \delta) = \left\{ \frac{(1-\delta)n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]}{(n-\delta)\alpha(1-\gamma)|e^{i\beta} \cos\beta|} \right\}^{\frac{1}{n-1}} \geq 2. \quad (21)$$

The result is sharp with extremal function f given by

$$f(z) = z + \frac{\alpha(1-\gamma)|e^{i\beta} \cos\beta|}{n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]} z^n. \quad (22)$$

Proof: It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \delta, \quad (0 \leq \delta < 1)$$

for $|z| < r_1(\beta, \alpha, \gamma, \delta)$. We have

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - 1 \right| &= \left| \frac{zf'(z) - f(z)}{f(z)} \right| = \left| \frac{\sum_{n=2}^{\infty} a_n(n-1)z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} a_n(n-1)|z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n|z|^{n-1}}. \end{aligned}$$

The last expression above is bounded by $(1-\delta)$ if

$$\sum_{n=2}^{\infty} \frac{(n-\delta)a_n|z|^{n-1}}{(1-\delta)} \leq 1. \quad (23)$$

Hence, by Theorem (1), (23) will be true if

$$\frac{(n-\delta)}{(1-\delta)}|z|^{n-1} \leq \frac{n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]}{\alpha(1-\gamma)|e^{i\beta} \cos\beta|},$$

or equivalently

$$|z| \leq \left(\frac{n(1-\delta)[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]}{(n-\delta)\alpha(1-\gamma)|e^{i\beta} \cos\beta|} \right)^{\frac{1}{n-1}}.$$

Setting $|z| = r_1(\beta, \alpha, \gamma, \delta)$, we get the desired result. \blacksquare

Theorem (6): If $f \in \mathcal{R}_\delta(\beta, \alpha, \gamma)$, then f is convex of order δ ($0 \leq \delta < 1$) in the disk $|z| < r_2(\beta, \alpha, \gamma, \delta)$, where

$$r_2(\beta, \alpha, \gamma, \delta) = \left\{ \frac{(1-\delta)n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]}{n(n-\delta)\alpha(1-\gamma)|e^{i\beta} \cos\beta|} \right\}^{\frac{1}{n-1}}, n \geq 2. \quad (24)$$

The result is sharp with extremal function f given by (22).

Proof: It is sufficient to show that

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - \delta, \quad (0 \leq \delta < 1)$$

for $|z| < r_2(\beta, \alpha, \gamma, \delta)$. We have

$$\begin{aligned} \left| \frac{zf''(z)}{f'(z)} \right| &= \left| \frac{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} na_n z^{n-1}} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} na_n|z|^{n-1}}. \end{aligned}$$

The last expression above is bounded by $(1-\delta)$ if

$$\sum_{n=2}^{\infty} \frac{n(n-\delta)a_n|z|^{n-1}}{(1-\delta)} \leq 1. \quad (25)$$

Hence, by Theorem (1), (25) will be true if

$$\frac{n(n-\delta)}{(1-\delta)}|z|^{n-1} \leq \frac{n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]}{\alpha(1-\gamma)|e^{i\beta} \cos\beta|},$$

or equivalently,

$$|z| \leq \left(\frac{(1-\delta)n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]}{n(n-\delta)\alpha(1-\gamma)|e^{i\beta} \cos\beta|} \right)^{\frac{1}{n-1}}.$$

Setting $|z| = r_2(\beta, \alpha, \gamma, \delta)$, we get the desired result. \blacksquare

Theorem (7): If $f \in \mathcal{R}_\delta(\beta, \alpha, \gamma)$, then f is close-to-convex of order δ ($0 \leq \delta < 1$) in the disk $|z| < r_3(\beta, \alpha, \gamma, \delta)$, where

$$r_3(\beta, \alpha, \gamma, \delta) = \left\{ \frac{(1-\delta)[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]}{\alpha(1-\gamma)|e^{i\beta} \cos\beta|} \right\}^{\frac{1}{n-1}}, n \geq 2. \quad (26)$$

The result is sharp with extremal function f given by (22)

Proof: It is sufficient to show that

$$|f'(z) - 1| \leq 1 - \delta, \quad (0 \leq \delta < 1)$$

for $|z| < r_3(\beta, \alpha, \gamma, \delta)$. We have

$$|f'(z) - 1| = \left| \sum_{n=2}^{\infty} na_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} na_n|z|^{n-1}.$$

The last expression above is bounded by $(1-\delta)$ if

$$\sum_{n=2}^{\infty} \frac{na_n|z|^{n-1}}{(1-\delta)} \leq 1. \quad (27)$$

Hence, by Theorem (1), (27) will be true if

$$\frac{n}{(1-\delta)}|z|^{n-1} \leq \frac{n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]}{\alpha(1-\gamma)|e^{i\beta} \cos\beta|},$$

or equivalently,

$$|z| \leq \left(\frac{(1-\delta)[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta} \cos\beta|]}{\alpha(1-\gamma)|e^{i\beta} \cos\beta|} \right)^{\frac{1}{n-1}}.$$

Setting $|z| = r_3(\beta, \alpha, \gamma, \delta)$, we get the desired result. \blacksquare

In the following Theorem, we prove the class $\mathcal{R}_\delta(\beta, \alpha, \gamma)$ is closed under linear combination.

Theorem (8): Let the function $f_j(z) \in \mathcal{R}_\delta(\beta, \alpha, \gamma)$ defined by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad (a_{n,j} \geq 0, j = 1, 2, 3, \dots, l).$$

Then the function $h(z)$ defined by

$$h(z) = \sum_{j=1}^l c_j f_j(z),$$

is in the class $\mathcal{R}_\delta(\beta, \alpha, \gamma)$, where $\sum_{j=1}^l c_j = 1, c_j \geq 0$.

Proof: By definition of $h(z)$, we have

$$h(z) = \left[\sum_{j=1}^l c_j \right] z - \sum_{n=2}^{\infty} \left[\sum_{j=1}^l c_j a_{n,j} \right] z^n. \quad (28)$$

Further, since $f_j(z)$ are in the class $\mathcal{R}_\delta(\beta, \alpha, \gamma)$ for every $j = 1, 2, 3, \dots, l$.

Hence, we can see that

$$\begin{aligned} & \sum_{n=2}^{\infty} (n[(n-1)(1+\alpha) + \alpha(1-\gamma)] |e^{i\beta} \cos\beta|) \left[\sum_{j=1}^l c_j a_{n,j} \right] \\ &= \sum_{j=1}^l c_j \left[\sum_{n=2}^{\infty} n[(n-1)(1+\alpha) + \alpha(1-\gamma)] |e^{i\beta} \cos\beta| a_{n,j} \right] \\ &\leq \alpha(1-\gamma) |e^{i\beta} \cos\beta| \sum_{j=1}^l c_j = \alpha(1-\gamma) |e^{i\beta} \cos\beta|. \blacksquare \end{aligned}$$

Let $f \in \mathcal{R}$ be a function of the form (1). Motivated by Silverman [3] and Silvia [5], see also [4], [6], we define the partial sums f_m defined by

$$f_m(z) = z - \sum_{n=2}^m a_n z^n, \quad (m \in \mathbb{N}). \quad (29)$$

Theorem (9): Let $f \in \mathcal{R}$ be given by (1) and define the partial sums $f_1(z)$ and $f_m(z)$ as follows: $f_1(z) = z$ and

$$f_m(z) = z - \sum_{n=2}^m a_n z^n, \quad (m > 2). \quad (30)$$

Also suppose that

$$\begin{aligned} & \sum_{n=2}^{\infty} d_n a_n \\ &\leq 1, \quad \left(d_n \right. \\ &= \left. \frac{n[(n-1)(1+\alpha) + \alpha(1-\gamma)] |e^{i\beta} \cos\beta|}{\alpha(1-\gamma) |e^{i\beta} \cos\beta|} \right). \end{aligned} \quad (31)$$

Then, we have

$$Re \left\{ \frac{f(z)}{d_m(z)} \right\} > 1 - \frac{1}{d_m}, \quad (32)$$

and

$$Re \left\{ \frac{f_m(z)}{f(z)} \right\} > \frac{d_{m+1}}{1 + d_{m+1}}. \quad (33)$$

Each of the bounds in (32) and (33) is the best possible for $m \in \mathbb{N}$.

Proof: For the coefficients d_n given by (31), it is not difficult to verify that

$d_{n+1} > d_n > 1, n = 2, 3, \dots$. Therefore, we have

$$\sum_{n=2}^m a_n + d_m \sum_{n=m+1}^{\infty} a_n \leq \sum_{n=2}^{\infty} d_n a_n \leq 1. \quad (34)$$

By setting

$$\begin{aligned} g_1(z) &= d_m \left[\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_m} \right) \right] \\ &= 1 \\ &+ \frac{d_m \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 - \sum_{n=2}^m a_n z^{n-1}}, \end{aligned} \quad (35)$$

and applying (34), we find that

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{d_m \sum_{n=m+1}^{\infty} a_n}{2 - 2 \sum_{n=2}^m a_n - d_m \sum_{n=m+1}^{\infty} a_n} \leq 1, \quad (z \in U)$$

which readily yields the left assertion (32). If we take

$$f(z) = z - \frac{z^m}{d_m}, \quad (36)$$

then

$$\frac{f(z)}{f_m(z)} = 1 - \frac{z^m}{d_m} \rightarrow 1 - \frac{1}{d_m} \quad (z \rightarrow 1^-),$$

Similarly, if we take

$$g_2(z) = (1 + d_m) \left[\frac{f_m(z)}{f(z)} - \frac{d_m}{1 + d_m} \right]$$

and making use of (34), we deduce that

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + d_{m+1}) \sum_{n=m+1}^{\infty} a_n}{2 - 2 \sum_{n=2}^m a_n + (1 - d_m) \sum_{n=m+1}^{\infty} a_n} \leq 1, \quad (37)$$

which leads us to the assertion (33). The bound in (33) is sharp for each $m \in \mathbb{N}$ with the function given by (36).

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