

# On a New Subclass of Meromorphic Univalent Functions Defined by Integral Operator

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**Abstract:** In this paper we introduce a new subclass of meromorphic univalent functions in the punctured unit disk with integral operator. We obtain coefficient inequality, Closure Theorem, convex combination, distortion bound, Partial sums and neighborhood property.

**Keywords:** Meromorphic univalent functions, integral operator, Closure Theorem, convex combination, distortion bound, Partial sums and neighborhood property

## 1. Introduction

Let  $\mathcal{A}_1^*$  be the class of functions  $f$  of the form:

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k,$$

which are analytic and meromorphic univalent in the punctured unit disk  $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$ .

Let  $\Sigma$  be the subclass of  $\mathcal{A}_1^*$ , consisting of functions of the form:

$$f(z) = z^{-1} - \sum_{k=1}^{\infty} a_k z^k, \quad (a_k \geq 0, k \in \mathbb{N} = \{1, 2, \dots\}), \quad (1.1)$$

which are analytic and meromorphic univalent in  $U^*$ .

A function  $f \in \Sigma$  is said to be meromorphic starlike function of order  $\rho$  ( $0 \leq \rho < 1$ ) if  $-Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \rho$ , ( $z \in U = U^* \cup \{0\}$ ,  $0 \leq \rho < 1$ ). (1.2)

The class of all such functions is defined by  $\Sigma^*(\rho)$ .

A function  $f \in \Sigma$  is said to be meromorphic convex function of order  $\rho$  ( $0 \leq \rho < 1$ ) if

$$-Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \rho, \quad (z \in U = U^* \cup \{0\}, 0 \leq \rho < 1). \quad (1.3)$$

**Definition (1.1)[6]:** Analogous to the operators defined by Jung, Kim, and Srivastava [6] on the normalized analytic functions, by [1] define the following integral operator

$$P_\beta^\alpha: \Sigma \rightarrow \Sigma$$

$$P_\beta^\alpha f(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{1}{z^{\beta+1}} \int_0^z t^\beta \left(\log \frac{z}{t}\right)^{\alpha-1} f(t) dt, \quad (\alpha, \beta > 0; z \in U), \quad (1.4)$$

where  $\Gamma(\alpha)$  is the familiar Gamma function.

Using the integral representation of the Gamma and Beta for  $f(z) \in \Sigma$ , given by (1.1) we have

$$P_\beta^\alpha f(z) = \frac{1}{z} - \sum_{n=1}^{\infty} \left( \frac{\beta}{n + \beta + 1} \right)^\alpha a_n z^n, \quad (\alpha > 0, \beta > 0) \quad (1.5)$$

**Lemma (1.1)[2]:** Let  $\alpha \geq 0$  and  $w = -(u+iv)$  is complex number, Then  $Re(w) > \alpha$  if and only if  $|w - (1 + \alpha)| \leq |w + (1 - \alpha)|$ .

**Lemma (1.2) [4]:** If  $w = u + iv$  is a complex number and  $\delta, k$  are real numbers, then  $-Re(w) \geq k|w + 1| + \delta$  if and only if  $-Re(w(1 + ke^{i\theta})) + ke^{i\theta} \geq \delta, \quad -\pi \leq \theta \leq \pi$ .

**Definition (1.2):** A function  $f \in \Sigma$  is said to be in the class  $A(\delta, \lambda, k, \alpha, \beta)$  if

$$-Re \left\{ \frac{z \left( \frac{P_\beta^\alpha f(z)}{P_\beta^\alpha f(z)} \right)' + z^2 \left( \frac{P_\beta^\alpha f(z)}{P_\beta^\alpha f(z)} \right)'' + \frac{\lambda}{3} z^3 \left( \frac{P_\beta^\alpha f(z)}{P_\beta^\alpha f(z)} \right)'''}{P_\beta^\alpha f(z)} + \delta \right\} > k \left| \frac{z \left( \frac{P_\beta^\alpha f(z)}{P_\beta^\alpha f(z)} \right)' + z^2 \left( \frac{P_\beta^\alpha f(z)}{P_\beta^\alpha f(z)} \right)''}{P_\beta^\alpha f(z)} + \frac{\lambda}{3} z^3 \left( \frac{P_\beta^\alpha f(z)}{P_\beta^\alpha f(z)} \right)'''}{P_\beta^\alpha f(z)} + 1 \right|, \quad (0 \leq \delta < 1, k \geq 0, 0 \leq \lambda < \frac{1}{3}; \alpha, \beta > 0) \quad (1.6)$$

The first theorem gives a necessary and sufficient condition for a function  $f$  to be in the class  $A(\delta, \lambda, k, \alpha, \beta)$ .

## 2. Coefficient Bounds

**Theorem (2.1):** Let  $f \in \Sigma$ . Then  $f \in A(\delta, \lambda, k, \alpha, \beta)$  if and only if

$$\sum_{n=1}^{\infty} \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right] \left( \frac{\beta}{n + \beta + 1} \right)^\alpha a_n \leq 2k(1-\lambda) - 2\lambda + 1 + \delta. \quad (2.1)$$

**Proof :** Let  $f \in A(\delta, \lambda, k, \alpha, \beta)$  and  $|z| = 1$ , Then by Lemma (1.2) we have

$$-Re \left\{ \frac{z(P_\beta^\alpha f(z))'}{P_\beta^\alpha f(z)} + \frac{z^2(P_\beta^\alpha f(z))''}{P_\beta^\alpha f(z)} + \frac{\frac{\lambda}{3}z^3(P_\beta^\alpha f(z))'''}{P_\beta^\alpha f(z)}(1+ke^{i\theta}) + ke^{i\theta} \right\} \geq \delta. \quad (2.2)$$

Let

$$A(z) = - \left( z(P_\beta^\alpha f(z))' + z^2(P_\beta^\alpha f(z))'' + \frac{\lambda}{3}z^3(P_\beta^\alpha f(z))''' \right) (1+ke^{i\theta}) - P_\beta^\alpha f(z)ke^{i\theta}$$

and

$$B(z) = P_\beta^\alpha f(z).$$

The equation (2.2) is equivalent to

$$Re \left\{ \frac{A(z)}{B(z)} \right\} \geq \delta.$$

In view of Lemma (1.1), we only need to prove that

$$|A(z) + (1-\delta)B(z)| - |A(z) - (1+\delta)B(z)| \geq 0.$$

Therefore

$$\begin{aligned} |A(z) + (1-\delta)B(z)| &= \\ &= \left| -(2ke^{i\theta}(1-\lambda) - 2\lambda + \delta)z^{-1} \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \left[ n \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) (1 + ke^{i\theta}) + ke^{i\theta} - (1 - \delta) \right] \left( \frac{\beta}{n + \beta + 1} \right)^\alpha a_n z^n \right| \\ &\geq (2k(1-\lambda) - 2\lambda + \delta)|z^{-1}| - \\ &\quad \sum_{n=1}^{\infty} \left[ n \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) (1+k) + k - (1 - \delta) \right] \left( \frac{\beta}{n + \beta + 1} \right)^\alpha a_n z^n, \end{aligned}$$

and

$$|A(z) - (1+\delta)B(z)| =$$

$$\begin{aligned} &= \left| (-2 + 2\lambda) - \delta + 2ke^{i\theta}(-1 + \lambda)z^{-1} \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \left[ n \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) (1 + ke^{i\theta}) + ke^{i\theta} + (1 + \delta) \right] \left( \frac{\beta}{n + \beta + 1} \right)^\alpha a_n z^n \right| \\ &\leq (-2 + 2\lambda) - \delta + 2k(-1 + \lambda)|z^{-1}| \\ &\quad + \sum_{n=1}^{\infty} \left[ n \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) (1+k) + k + (1 + \delta) \right] \left( \frac{\beta}{n + \beta + 1} \right)^\alpha a_n |z^n|, \end{aligned}$$

then

$$\begin{aligned} &|A(z) + (1-\delta)B(z)| - |A(z) - (1+\delta)B(z)| \\ &\geq (2k(1-\lambda) - 2\lambda + \delta)|z^{-1}| \\ &\quad - \sum_{n=1}^{\infty} \left[ n \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) (1+k) - k - (1-\delta) \right] \left( \frac{\beta}{n + \beta + 1} \right)^\alpha a_n z^n \\ &\quad + 2 - 2\lambda + \delta + 2k(1-\lambda)|z^{-1}| \\ &\quad - \sum_{n=1}^{\infty} \left[ n \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) (1+k) + k + \right. \\ &\quad \left. (1+\delta)\beta n + \beta + 1 \alpha a_n z^n \right], \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\sum_{n=1}^{\infty} \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right] \left( \frac{\beta}{n + \beta + 1} \right)^\alpha a_n \\ &\leq 2k(1-\lambda) - 2\lambda + 1 + \delta. \end{aligned}$$

Conversely, suppose that (2.1) hold true, then, we must show

$$-Re \left\{ \frac{z(P_\beta^\alpha f(z))'}{P_\beta^\alpha f(z)} + \frac{z^2(P_\beta^\alpha f(z))''}{P_\beta^\alpha f(z)} + \frac{\frac{\lambda}{3}z^3(P_\beta^\alpha f(z))'''}{P_\beta^\alpha f(z)}(1+ke^{i\theta}) + ke^{i\theta} \right\} \geq \delta.$$

Up on choose the value of z on the positive real axis, where  $0 \leq z=r < 1$ , the above inequality reduces to

$$Re \left\{ \frac{\left( ((1-2\lambda)(1+ke^{i\theta}) + ke^{i\theta} + \delta)z^{-1} - \sum_{n=1}^{\infty} \left( n \left( n + \frac{\lambda}{3}(n-1)(n-2) (1+k) + k + \delta \right) \left( \frac{\beta}{n + \beta + 1} \right)^\alpha \right) \right)}{z^{-1} - \sum_{n=1}^{\infty} \left( \frac{\beta}{n + \beta + 1} \right)^\alpha a_n z^n} \right\} \geq 0.$$

Since  $Re\{-e^{i\theta}\} \geq -|e^{i\theta}| = -1$ , the above inequality reduce to

$$\left\{ \frac{\left( ((1-2\lambda)(1+k) + k + \delta)z^{-1} - \sum_{n=1}^{\infty} \left( n \left( n + \frac{\lambda}{3}(n-1)(n-2) (1+k) + k + \delta \right) \left( \frac{\beta}{n + \beta + 1} \right)^\alpha \right) \right)}{z^{-1} - \sum_{n=1}^{\infty} \left( \frac{\beta}{n + \beta + 1} \right)^\alpha a_n z^n} \right\} \geq 0$$

Letting  $r \rightarrow 1^-$  and by the mean value Theorem, we have desired inequality (2.1).

**Corollary (2.1):** Let  $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ . Then

$$a_n \leq \frac{2k(1-\lambda) - 2\lambda + 1 + \delta}{\left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right] \left( \frac{\beta}{n + \beta + 1} \right)^\alpha}, \quad (n \geq 1)$$

### 3. Convex Linear Combination

In the following theorem, we prove the class  $\mathcal{A}(\delta, \lambda, k, \alpha, \beta)$  is closed under convex linear combination.

**Theorem (2):** The class  $\mathcal{A}(\delta, \lambda, k, \alpha, \beta)$  is a closed under convex linear combination.

**Proof:** Let  $f_1$  and  $f_2$  be the arbitrary elements of  $\mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ . Then for every  $t$  ( $0 \leq t \leq 1$ ), we show that  $(1-t)f_1 + tf_2 \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ . Thus, we have

$$(1-t)f_1 + tf_2 = z^{-1} + \sum_{n=1}^{\infty} [(1-t)a_n + tb_n]z^n.$$

Hence,

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k \right. \\ & \quad \left. + \delta \left( \frac{\beta}{n+\beta+1} \right)^\alpha [(1-t)a_n + tb_n] \right] \\ &= (1-t) \sum_{n=1}^{\infty} \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k \right. \\ & \quad \left. + \delta \left( \frac{\beta}{n+\beta+1} \right)^\alpha a_n \right] \\ & \quad + t \sum_{n=1}^{\infty} \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k \right. \\ & \quad \left. + \delta \left( \frac{\beta}{n+\beta+1} \right)^\alpha b_n \right] \\ &\leq (1-t)2k(1-\lambda) - 2\lambda + 1 + \delta + t2k(1-\lambda) - 2\lambda \\ & \quad + 1 + \delta = 2k(1-\lambda) - 2\lambda + 1 + \delta. \end{aligned}$$

This completes the proof.

### 4. Closure Theorem

We shall prove the closure theorem of the functions in the class  $\mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ .

**Theorem (4.3):** Let the functions  $f_k$  defined by

$$f_k(z) = z^{-1} + \sum_{n=1}^{\infty} a_{n,k}, \quad (a_{n,k} \geq 0, n \in \mathbb{N}, k = 1, 2, \dots, l)$$

be in the class  $\mathcal{A}(\delta, \lambda, k, \alpha, \beta)$  for every  $k = 1, 2, 3, \dots, l$ , then the function  $h$  defined by

$$h(z) = z^{-1} + \sum_{n=1}^{\infty} e_n z^n, \quad (e_n \geq 0, n \in \mathbb{N})$$

also belong to the class  $\mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ , where

$$e_n = \frac{1}{l} \sum_{k=1}^l a_{n,k}$$

**Proof:** Since  $f_k \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$  it follows from Theorem (1) that

$$\sum_{n=1}^{\infty} \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \beta n + \beta + 1 \alpha a_n, k \leq 2k1 - \lambda - 2\lambda + 1 + \delta. \right]$$

For every  $k = 1, 2, 3, \dots, l$ . Hence

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k \right. \\ & \quad \left. + \delta \left( \frac{\beta}{n+\beta+1} \right)^\alpha e_n \right] \\ &= \sum_{n=1}^{\infty} \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k \right. \\ & \quad \left. + \delta \left( \frac{\beta}{n+\beta+1} \right)^\alpha \left( \frac{1}{l} \sum_{k=1}^l a_{n,k} \right) \right] \\ &= \frac{1}{l} \sum_{k=1}^l \left( \sum_{n=1}^{\infty} \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k \right. \right. \\ & \quad \left. \left. + \delta \left( \frac{\beta}{n+\beta+1} \right)^\alpha \right] \right) \\ &\leq \frac{1}{l} \sum_{k=1}^l 2k(1-\lambda) - 2\lambda + 1 + \delta = 2k(1-\lambda) - 2\lambda + 1 + \delta \end{aligned}$$

Then  $h \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ .

### 5. Partial Sums and Neighborhood Property

We introduce the partial sums and the same property has been found for other class in [8]

**Theorem (5.6):** Let  $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$  be given by (1.1) and define the partial sums  $S_1(z)$  and  $S_k(z)$  as follows  $S_1(z) = z^{-1}$  and

$$S_n(z) = z^{-1} - \sum_{n=1}^{k-1} a_n z^n, \quad (k \in \mathbb{N} \setminus \{1\}). \quad (5.1)$$

Also suppose that  $\sum_{n=1}^{\infty} d_n a_n \leq 1$

$$d_n = n1 + kn + \lambda 3n - 1n - 2 + k + \delta \beta n + \beta + 1 \alpha 2k1 - \lambda - 2\lambda + 1 + \delta. \quad (5.2)$$

Then, we have

$$Re \left\{ \frac{f(z)}{S_k(z)} \right\} > 1 - \frac{1}{d_k} \quad (z \in U, k \in \mathbb{N}) \quad (5.3)$$

and

$$Re \left\{ \frac{S_k(z)}{f(z)} \right\} > \frac{d_k}{1 + d_k} \quad (z \in U, k \in \mathbb{N}). \quad (5.4)$$

Each of the bounds in (5.3) and (5.4) is the best possible for  $n \in \mathbb{N}$ .

**Proof:** For the coefficients  $d_n$  given by (5.2), it is not difficult to verify that

$$d_{n+1} > d_n > 1 \quad (n \in \mathbb{N}).$$

Therefore, by using hypothesis (5.2), we have

$$\sum_{n=1}^{k-1} a_n + d_k \sum_{n=k}^{\infty} a_n \leq \sum_{n=1}^{\infty} d_n a_n \leq 1. \quad (5.5)$$

By setting

$$g_1(z) = d_k \left[ \frac{f(z)}{S_k(z)} - \left(1 - \frac{1}{C_k}\right) \right]$$

$$= 1 - \frac{d_k \sum_{n=1}^{\infty} a_n z^{n+1}}{1 - \sum_{n=1}^{\infty} a_n z^{n+1}}, \quad (5.6)$$

and applying (5.5), we find that

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{d_k \sum_{n=1}^{\infty} a_n}{2 - 2 \sum_{n=1}^{k-1} a_n - d_k \sum_{k=n}^{\infty} a_n}$$

$$\leq 1 \quad (z \in U), \quad (5.7)$$

which readily yields the assertion (5.2). If we take

$$f(z) = z^{-1} - \frac{z^k}{C_k}, \quad (5.8)$$

then

$$\frac{f(z)}{S_k(z)} = 1 - \frac{z^k}{C_k} \rightarrow 1 - \frac{1}{C_k} \quad (z \rightarrow 1^-),$$

which shows that the bound in (5.3) is the best possible for  $n \in \mathbb{N}$ .

Similarly, if we put

$$g_2(z) = (1 + d_k) \left[ \frac{S_k(z)}{f(z)} - \frac{d_k}{1 + C_k} \right]$$

$$= 1 - \frac{(1 + d_k) \sum_{n=k}^{\infty} a_n z^{n+1}}{1 - \sum_{n=1}^{\infty} a_n z^{k+1}}, \quad (5.9)$$

and make use of (5.5), we have

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + d_k) \sum_{n=k}^{\infty} a_n}{2 - 2 \sum_{n=1}^{k-1} a_n + (1 + d_k) \sum_{n=k}^{\infty} a_n}, \quad (5.10)$$

which leads us to the assertion (5.4). The bound in (5.4) is sharp for each  $k \in \mathbb{N}$  with function given by (5.8). The proof of the theorem is complete.

**Theorem (5.1):** If  $g \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$  and

$$\eta = 1 - \frac{\gamma((1+k)(k+\delta+1)) \left(\frac{\beta}{n+\beta+1}\right)^\alpha}{((1+k)(k+\delta+1)) \left(\frac{\beta}{n+\beta+1}\right)^\alpha - (2k(1-\lambda) - 2\lambda + (1+\delta))}, \quad (5.13)$$

then  $N_{n,\gamma}(g) \subset \mathcal{A}^\eta(\delta, \lambda, k, \alpha, \beta)$ .

**Proof:** Let  $f \in N_{n,\gamma}(g)$ . Then we find from (5.11) that

$$\sum_{k=1}^{\infty} k |a_k - b_k| \leq \delta,$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, \quad (k \in \mathbb{N}). \quad (5.14)$$

Since  $g \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ , then by using Theorem (1), we get

$$\sum_{n=1}^{\infty} b_n \leq \frac{2k(1-\lambda) - 2\lambda + (1+\delta)}{(1+k)(1+k+\delta) \left(\frac{\beta}{n+\beta+1}\right)^\alpha}, \quad (5.15)$$

so that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \frac{\sum_{n=1}^{\infty} |a_n - b_n|}{1 - \sum_{n=1}^{\infty} b_n}$$

$$\leq \frac{\gamma((1+k)(k+\delta+1)) \left(\frac{\beta}{n+\beta+1}\right)^\alpha}{((1+k)(k+\delta+1)) \left(\frac{\beta}{n+\beta+1}\right)^\alpha - (2k(1-\lambda) - 2\lambda + (1+\delta))}.$$

$$= 1 - \eta$$

Hence, by Definition (5.1),  $f \in \mathcal{A}^\eta(\delta, \lambda, k, \alpha, \beta)$  for  $\eta$  given by (5.13).

This completes the proof.

Now, following the earlier works on neighborhoods of analytic functions by Goodman [5] and Ruscheweyh [9] investigated this concept for the element of several famous subclasses of analytic function and Altintas and Owa [1] considered for a certain family of analytic functions with negative coefficients, also Liu and Srivastava [8] and Atshan [3] extended this concept for a certain subclass of meromorphically univalent and multivalent functions.

We begin by introducing here the  $(n, \gamma)$ -neighborhood of a function  $f \in \Sigma$  of the form (1.1) by means of the definition below:

$$N_{n,\gamma}(f) = \left\{ g \in \Sigma : g(z) = z^{-1} - \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n |a_n - b_n| \leq \gamma, 0 \leq \gamma < 1 \right\}. \quad (5.11)$$

Particular for the identity function  $e(z) = z^{-1}$ , we have

$$N_\delta(e) = \left\{ g \in \Sigma : g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n |b_n| \leq \gamma, 0 \leq \gamma < 1 \right\}. \quad (5.12)$$

**Definition (5.1):** A function  $f \in \Sigma$  is said to be in the class  $\mathcal{A}^\eta(\delta, \lambda, k, \alpha, \beta)$  if there exists a function  $g \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ , such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \eta \quad (z \in U, 0 \leq \eta < 1).$$

## 6. The Radii of Starlikeness and Convexity

In the following theorems, we obtain the radii of starlikeness and convexity for  $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ .

**Theorem (6.1):** If  $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ , then  $f$  is univalent meromorphic starlike of order  $\varphi$  ( $0 \leq \varphi < 1$ ) in the disk  $|z| < r_1$ , where

$$r_1 = \inf_k \left\{ \frac{(1-\varphi) \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right] \left( \frac{\beta}{n+\beta+1} \right)^\alpha}{(n-\varphi+2)2k(1-\lambda) - 2\lambda + 1 + \delta} \right\}^{\frac{1}{n+1}}.$$

The result is sharp for the function  $f$  given

$$f(z) = z^{-1} - \frac{2k(1-\lambda) - 2\lambda + 1 + \delta}{\left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right] \left( \frac{\beta}{n+\beta+1} \right)^\alpha} \quad (6.1)$$

**Proof:** It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq 1 - \varphi \quad \text{for } |z| < r_1. \quad (6.2)$$

But

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| = \left| \frac{zf'(z) + f(z)}{f(z)} \right| \leq \frac{\sum_{n=1}^{\infty} (n+1)a_n |z|^{n+1}}{1 - \sum_{n=1}^{\infty} a_n |z|^{n+1}}.$$

Thus, (6.2) will be satisfied if

$$\frac{\sum_{n=1}^{\infty} (n+1)a_n |z|^{n+1}}{1 - \sum_{n=1}^{\infty} a_n |z|^{n+1}} \leq 1 - \varphi,$$

or if

$$\sum_{n=1}^{\infty} \frac{(n-\varphi+2)}{1-\varphi} a_n |z|^{n+1} \leq 1. \quad (6.3)$$

Since  $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ , we have

$$\sum_{n=1}^{\infty} \frac{\left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right] \left( \frac{\beta}{n+\beta+1} \right)^\alpha}{2k(1-\lambda) - 2\lambda + 1 + \delta} a_n \leq 1.$$

Hence, (6.3) will be true if

$$\frac{(n-\varphi+2)}{1-\varphi} |z|^{k+1} \leq \frac{\left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right] \left( \frac{\beta}{n+\beta+1} \right)^\alpha}{2k(1-\lambda) - 2\lambda + 1 + \delta},$$

or equivalently

$$|z| \leq \left\{ \frac{(1-\varphi) \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right] \left( \frac{\beta}{n+\beta+1} \right)^\alpha}{(n-\varphi+2)2k(1-\lambda) - 2\lambda + 1 + \delta} \right\}^{\frac{1}{n+1}} \quad (n \geq 1),$$

which follows the result.

**Theorem (6.2):** If  $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ , then  $f$  is univalent meromorphic convex of order  $\varphi$  ( $0 \leq \varphi < 1$ ) in the disk  $|z| < r_2$ , where

$$r_2 = \inf_k \left\{ \frac{(1-\varphi) \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right] \left( \frac{\beta}{n+\beta+1} \right)^\alpha}{(k-\varphi+1)2k(1-\lambda) - 2\lambda + 1 + \delta} \right\}^{\frac{1}{k+1}}.$$

The result is sharp for the function  $f$  given by (3.9).

**Proof:** It is sufficient to show that

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right| \leq 1 - \varphi \quad \text{for } |z| < r_2. \quad (6.4)$$

But

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right| = \left| \frac{zf''(z) + 2f'(z)}{f'(z)} \right| \leq \frac{\sum_{n=1}^{\infty} n^2 a_n |z|^{n+1}}{1 - \sum_{n=1}^{\infty} n a_n |z|^{n+1}}.$$

Thus, (6.4) will be satisfied if

$$\frac{\sum_{n=1}^{\infty} n^2 a_n |z|^{n+1}}{1 - \sum_{n=1}^{\infty} n a_n |z|^{n+1}} \leq 1 - \varphi,$$

**Volume 4 Issue 3, March 2015**

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or if

$$\sum_{n=1}^{\infty} \frac{n(n-\varphi+1)}{1-\varphi} a_n |z|^{n+1} \leq 1. \quad (6.5)$$

Since  $f \in \Sigma(\alpha, \gamma, \lambda)$ , we have

$$\sum_{n=1}^{\infty} \frac{\left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right] \left( \frac{\beta}{n+\beta+1} \right)^\alpha}{2k(1-\lambda) - 2\lambda + 1 + \delta} a_n \leq 1.$$

Hence, (6.5) will be true if

$$\frac{k(k-\varphi+1)}{1-\varphi} |z|^{k+1} \leq \frac{\left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right] \left( \frac{\beta}{n+\beta+1} \right)^\alpha}{2k(1-\lambda) - 2\lambda + 1 + \delta},$$

or equivalently

$$|z| \leq \left\{ \frac{(1-\varphi) \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right] \left( \frac{\beta}{n+\beta+1} \right)^\alpha}{(k-\varphi+1) 2k(1-\lambda) - 2\lambda + 1 + \delta} \right\}^{\frac{1}{k+1}} \quad (n \geq 1),$$

which follows the result.

## 7. Integral Transformation

In the following theorems, we obtain integral transformations in the class  $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ .

**Theorem (7.1):** Let the function  $f$  given by (1.1) be in the class  $\mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ . Then the integral operator

$$F_c(f(z)) = \frac{c}{z^{c+1}} \int_0^z t^c f(t) dt, \quad (0 < t \leq 1, 0 < c < \infty)$$

is in the class  $\mathcal{A}(\sigma, \lambda, k, \alpha, \beta)$ , where

$$\sigma \leq \frac{(c+n+1)(2k(1-\lambda) - 2\lambda + 1)y - cw \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) (1+k) + k}{(w - (c+n+1))y}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} - \frac{2k(1-\lambda) - 2\lambda + \delta + 1}{(2k + \delta + 1) \left( \frac{\beta}{2+\beta} \right)^\alpha} z$$

**Proof :** Let

$$f(z) = \frac{1}{z} - \sum_{n=1}^{\infty} a_n z^n$$

in the class  $\mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ . Then

$$\begin{aligned} F_c(f(z)) &= \frac{c}{z^{c+1}} \int_0^z t^c f(t) dt \\ &= \frac{c}{z^{c+1}} \int_0^z \left[ t^{c-1} - \sum_{n=1}^{\infty} a_n t^{n+1} \right] dt \\ &= \frac{1}{z} - \sum_{n=1}^{\infty} \left( \frac{c}{c+n+1} \right) a_n \end{aligned} \quad (7.2)$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \sigma \right] \left( \frac{\beta}{n+\beta+1} \right)^\alpha}{(c+n+1)(2k(1-\lambda) - 2\lambda + \sigma + 1)} a_n \leq 1. \quad (7.3)$$

Since  $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ , we have

$$\sum_{n=1}^{\infty} \frac{\left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right] \left( \frac{\beta}{n+\beta+1} \right)^\alpha}{2k(1-\lambda) - 2\lambda + \delta + 1} a_n \leq 1. \quad (7.4)$$

Note that (7.1) it satisfies if

$$\frac{c \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \sigma \right] \left( \frac{\beta}{n+\beta+1} \right)^\alpha}{(c+n+1)(2k(1-\lambda) - 2\lambda + \sigma + 1)} \leq \frac{\left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right] \left( \frac{\beta}{n+\beta+1} \right)^\alpha}{2k(1-\lambda) - 2\lambda + \delta + 1}.$$

Rewriting the inequality, we have

$$\begin{aligned} & c(2k(1-\lambda) - 2\lambda + \delta + 1) \left( n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \sigma \right) \\ & \leq (c+n+1)(2k(1-\lambda) - 2\lambda + \sigma + 1) \left( n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right). \end{aligned}$$

Solving for  $\sigma$ , we have

$$\sigma \leq \frac{(c+n+1)(2k(1-\lambda) - 2\lambda + 1)y - cw \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) (1+k) + k}{(w - (c+n+1))y} = F(n),$$

where

$$w = 2k(1-\lambda) - 2\lambda + \delta + 1 \text{ and } y = n \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) (1+k) + k + \delta$$

$$F(n) \geq F(1).$$

Using this, the results follows.

**Theorem (7.2)** : If  $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ , then the integral operator

$$F_c(f(z)) = \frac{c}{z^{c+1}} \int_0^z t^c f(t) dt, \quad (0 < t \leq 1, 0 < c < \infty)$$

is in the class  $f \in \mathcal{A} \left( \frac{1+\delta c}{2+c}, \lambda, k, \alpha, \beta \right)$ ,

The result is sharp for

$$f_n(z) = z^{-1} - \frac{2k(1-\lambda) - 2\lambda + 1 \left( 1 + \frac{1+\delta c}{2+c} \right)}{n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \frac{1+\delta c}{2+c} \left( \frac{\beta}{n+\beta+1} \right)^\alpha} z^n.$$

**Proof:** By definition of  $F_c$ , we obtain

$$F_c(f(z)) = \frac{c}{z^{c+1}} \int_0^z t^c f(t) dt = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n z^n.$$

By Theorem (2.1), it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c \left( \frac{\beta}{n+\beta+1} \right)^\alpha \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \frac{1+\delta c}{2+c} \right]}{(c+n+1) \left( 2k(1-\lambda) - 2\lambda + 1 + \frac{1+\delta c}{2+c} \right)} a_n \leq 1. \quad (7.5)$$

Since, if  $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ , then (7.5) satisfied if

$$\frac{c}{(c+n+1) \left( 2k(1-\lambda) - 2\lambda + 1 + \frac{1+\delta c}{2+c} \right)} \leq \frac{1}{2k(1-\lambda) - 2\lambda + 1 + \delta}.$$

Or equivalent, when

$$\phi(n, \lambda, k, \delta, c) = \frac{c(2k(1-\lambda) - 2\lambda + \delta + 1)}{(c+n+1) \left( 2k(1-\lambda) - 2\lambda + \left( 1 + \frac{1+\delta c}{2+c} \right) \right)} \leq 1.$$

Since  $\phi(n, \lambda, k, \delta, c)$  is a decreasing function of  $n$  ( $n \geq 1$ ). Then the proof is complete. The result sharp for the function

$$f_n(z) = z^{-1} - \frac{2k(1-\lambda) - 2\lambda + 1 \left( 1 + \frac{1+\delta c}{2+c} \right)}{n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \frac{1+\delta c}{2+c} \left( \frac{\beta}{n+\beta+1} \right)^\alpha} z^n$$

**Theorem (7.3)** : Let  $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ . Then the function  $F$  defined by

$$F_c(f(z)) = \frac{c}{z^{c+1}} \int_0^z t^c f(t) dt = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n z^n, \quad 0 < c < \infty \quad (7.6)$$

is meromorphically starlike in the disk  $|z| < R_1$ , where

$$R_1 = \inf_n \left\{ \frac{(c+n+1) \left( \frac{\beta}{n+\beta+1} \right)^\alpha \left[ n(1+k) \left( n + \frac{\lambda}{3}(n-1)(n-2) \right) + k + \delta \right]}{c(n+2)(2k(1-\lambda) - 2\lambda + \delta + 1)} \right\}^{\frac{1}{n+1}}. \quad (7.7)$$

**Proof :** We show that

$$\left| \frac{zF'_c(z)}{F_c(z)} + 1 \right| \leq 1 \text{ in } |z| < R_1. \quad (7.8)$$

$R_1$  is given by (7.7). In view of (7.6) we have

$$\begin{aligned} \left| \frac{zF'_c(z) + F_c(z)}{F_c(z)} \right| &= \left| \frac{-\sum_{n=1}^{\infty} \left( \frac{c}{c+n+1} \right) (n+1) a_n z^{n+1}}{1 - \sum_{n=1}^{\infty} \frac{c}{(c+n+1)} a_n z^{n+1}} \right| \\ &\leq \frac{\sum_{n=1}^{\infty} \left( \frac{c(n+1)}{c+n+1} \right) a_n |z|^{n+1}}{1 - \sum_{n=1}^{\infty} \frac{c}{(c+n+1)} a_n |z|^{n+1}}. \end{aligned}$$

Thus (7.8) will be satisfied if

$$\sum_{n=1}^{\infty} \frac{c(n+1)}{c+n+1} a_n |z|^{n+1} \leq 1. \quad (7.9)$$

Hence (7.9) will be true if

$$|z|^{n+1} \leq \left\{ \frac{(c+n+1) \left( \frac{\beta}{n+\beta+1} \right)^\alpha \left[ n(1+k) \left( n + \frac{\lambda}{3} (n-1)(n-2) \right) + k + \delta \right]}{c(n+2)(2k(1-\lambda) - 2\lambda + \delta + 1)} \right\}.$$

Therefore

$$|z| \leq \left\{ \frac{(c+n+1) \left( \frac{\beta}{n+\beta+1} \right)^\alpha \left[ n(1+k) \left( n + \frac{\lambda}{3} (n-1)(n-2) \right) + k + \delta \right]}{c(n+2)(2k(1-\lambda) - 2\lambda + \delta + 1)} \right\}^{\frac{1}{n+1}}.$$

For  $n \geq 1, n \in N$ , The result follows by setting  $|z| = R_1$

**Theorem (7.4)** : Let  $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ . Then the function  $F$  defined by

$$F_c(f(z)) = \frac{c}{z^{c+1}} \int_0^z t^c f(t) dt = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n z^n, \quad 0 < c < \infty \quad (7.10)$$

is convex in the disk  $|z| < R_2$ , where

$$R_2 = \inf_n \left\{ \frac{(c+n+1) \left( \frac{\beta}{n+\beta+1} \right)^\alpha \left[ n(1+k) \left( n + \frac{\lambda}{3} (n-1)(n-2) \right) + k + \delta \right]}{cn^2(2k(1-\lambda) - 2\lambda + \delta + 1)} \right\}^{\frac{1}{n+1}} \quad (7.11)$$

**Proof** : We show that

$$\left| \frac{zF'_c(z)}{F_c(z)} + 2 \right| \leq 1 \text{ in } |z| < R_2, \quad (7.12)$$

$R_2$  is given by (7.7). In view of (7.6) we have

$$\begin{aligned} \left| \frac{zF'_c(z) + 2F_c(z)}{F_c(z)} \right| &= \left| \frac{-\sum_{n=1}^{\infty} \left( \frac{cn^2}{c+n+1} \right) a_n z^{n+1}}{1 - \sum_{n=1}^{\infty} \frac{cn}{(c+n+1)} a_n z^{n+1}} \right| \\ &\leq \frac{\sum_{n=1}^{\infty} \left( \frac{cn^2}{c+n+1} \right) a_n |z|^{n+1}}{1 - \sum_{n=1}^{\infty} \frac{cn}{(c+n+1)} a_n |z|^{n+1}}. \end{aligned}$$

Thus (7.12) will be satisfied if

$$\sum_{n=1}^{\infty} \frac{cn^2}{(c+n+1)} a_n |z|^{n+1} \leq 1. \quad (7.13)$$

Hence (7.13) will be true if

$$|z|^{n+1} \leq \left\{ \frac{(c+n+1) \left( \frac{\beta}{n+\beta+1} \right)^\alpha \left[ n(1+k) \left( n + \frac{\lambda}{3} (n-1)(n-2) \right) + k + \delta \right]}{cn^2(2k(1-\lambda) - 2\lambda + \delta + 1)} \right\}.$$

Therefore



$$|z| \leq \left\{ \frac{(c+n+1) \left( \frac{\beta}{n+\beta+1} \right)^\alpha \left[ n(1+k) \left( n + \frac{\lambda}{3} (n-1)(n-2) \right) + k + \delta \right]}{cn^2(2k(1-\lambda) - 2\lambda + \delta + 1)} \right\}^{\frac{1}{n+1}}$$

For  $n \geq 1, n \in N$ , The result follows by setting  $|z| = R_2$

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