Purely Quasi-Dedekind Modules
And
Purely Prime Modules

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Abstract . An R-submodule N of an R-module M is called pure if $IN = N \cap IM$ for every ideal I of R . In this paper we introduce the notion of purely quasi-invertible submodule and a purely quasi-Dedekind module, where an R-submodule N of an R-module M is called purely quasi-invertible if, N is pure and $Hom_R(M/N, M) = 0$. And an R-module M is called purely quasi-Dedekind if, every nonzero pure submodule N of M is quasi-invertible ; that is $Hom_R(M/N, M) = 0$. Beside these, we also introduce the notion of purely prime module, where an R-module M is called purely prime module if $ann_R M = ann_R N$ for all nonzero pure submodule N of M .We gave many properties related with this concepts. And we studied the relationships between these concepts and several other types of modules. In this paper R is a commutative ring with unity and M is a unitary R-module .

Key Words : Purely quasi-invertible Submodules; Pure Submodules; Purely quasi-Dedekind Modules; Purely prime Modules .

0. Introduction

Let R be a ring and M be a unital R-module. If N is a submodule of M, we write $N \leq M$ and if N is an essential submodule of M then we write $N \leq_e M$, also if N is a direct summand of M then we write $N \leq^{\oplus} M$. Recall an R-submodule N of an R-module M is called pure if $IN = N \cap IM$ for every ideal I of R [5], [10], and N is called quasi-invertible if, $Hom_R(M/N, M) = 0$ [14] . And an R-module M is called quasi-Dedekind if, each nonzero submodule of M is quasi-invertible [14] . And an R-module M is called prime module if $ann_R M = ann_R N$ for all nonzero submodule N of M [8] . Ghawi Th.Y. in [11] introduced the concepts of essentially quasi-invertible submodules and essentially quasi-Dedekind modules as a generalization of quasi-invertible submodules and quasi-Dedekind modules, where a submodule N of an R-module M is called essentially quasi-invertible if $N \leq_e M$ and N is quasi-invertible, and M is called essentially quasi-Dedekind if every essential submodule of M is quasi-invertible .

This paper has been organized on three sections. In section 1, we generalized the concept of quasi-invertible submodule to a purely quasi-invertible submodule, where a submodule N of a module M is called purely quasi-invertible if N is a pure and quasi-invertible submodule. We give some basic properties of this class of submodules.

In section 2, we introduce the concept of a purely quasi-Dedekind module as a generalization to concept a quasi-Dedekind module, where an *R*-module *M* is called purely quasi-Dedekind if, every nonzero pure submodule of *M* is quasi-invertible . We prove that if *M* a purely quasi-Dedekind module with M/K is projective for all pure submodule *K* of *M* then M/N is a purely quasi-Dedekind module, for all $N \leq M$. Also, we show by an example a direct sum of purely quasi-Dedekind modules need not be a purely quasi-Dedekind module (see Ex 2.14) . On the other hand we give a condition under which the direct sum of purely quasi-Dedekind modules is a gain purely quasi-Dedekind (see Prop 2.15) .

Finally, in section 3, we introduce and study the concept purely prime module as a generalization of prime module, where an *R*-module *M* is called a purely prime module if $ann_R M = ann_R N$ for all nonzero pure submodule *N* of *M* . We see that every prime module is a purely prime module, but the converse is not true. Also we give some equivalent formulas and results of this concept .

1. Purely Quasi-Invertible Submodules

Firstly, we recall that an *R*-submodule *N* of an R-module *M* is pure if, $IN = N \cap IM$ for every ideal *I* of *R* [5], [10] . Mijbass A.S. in [14] introduced the following concept, an *R*-submodule *N* of an *R*-module *M* is called quasi-invertible if, $Hom_R(M/N, M) = 0$. And an ideal *J* of a ring *R* is called quasi-invertible if *J* is a quasi-invertible *R*-submodule. In this section we introduce and study a generalization of the concept a quasi-invertible submodule namely " purely quasi-invertible " .

Definition 1.1. An *R*-submodule *N* of an *R*-module *M* is called purely quasi-invertible if *N* is pure and $Hom_R(M/N, M) = 0$. And an ideal *I* of a ring *R* is called purely quasi-invertible if *I* is a purely quasi-invertible *R*-submodule . It is clear that every purely quasi-invertible submodule is a quasi-invertible submodule . The following example shows that the converse is false .

Example 1.2. Let *R* be an integral domain and let $\overline{R} = R[x, y]$ be the polynomial ring of two independent variables *x* and *y* , then $\overline{R}$ is also an integral domain . Let *I = (x ,y)* is the ideal of $\overline{R}$ generated by *x* and *y* , so by [14, Ex 1.3(1), P.6] *I* is quasi-invertible. But *I* is not pure of $\overline{R}$, thus *I* is not purely quasi-invertible; To see this: Let $R = Z$, $\overline{R} = Z[x, y]$, let $I = (x ,y) = \{xf_1 + yf_2 : f_1, f_2 \in \overline{R}\}$, thus by [14, Ex 1.3(1), P.6] *I* is quasi-invertible . Now, Let $J = \{f \in \overline{R} : f(x, y) = a \quad, a \in 2Z\}$ then $JI = \{axf_1 + ayf_2 : f_1, f_2 \in \overline{R}\} \neq \{0\} = I \cap J\overline{R}$;that is *I* not pure, hence *I* is not purely quasi-invertible .

Remarks and Examples 1.3.

1) In any nonzero module *M*. *0* is not purely quasi-invertible, but *M* is a purely quasi-invertible submodule .

2) If *N* is a proper direct summand of an *R*-module *M* then *N* is pure by [21], but not quasi-invertible, since there exists $0 \neq K \leq M$ such that $M = K \oplus N$ and $Hom_R(M/N, M) = Hom_R(K \oplus N/N, K \oplus N) = Hom_R(K, K \oplus N) \neq 0$.

Recall that an *R*-module *M* is called semisimple if, every submodule of *M* is a direct summand of *M* [12, P.189] .

3) If M is a semisimple module, then M is the only purely quasi-invertible submodule of M ; since every proper submodule of M is direct summand; that is pure not quasi-invertible (see Rem.and.Ex 1.3(2)) .

4) Let $M = Z_4$ as Z-module , $N = (\bar{2})$ is not a purely quasi-invertible submodule of Z_4 as Z-module . In fact N is not quasi-invertible , since $Hom_Z(Z_4/(\bar{2}), Z_4) \cong Z_2 \neq 0$. Also, N is not pure, since $\bar{2} = \bar{2}.\bar{1} \in (\bar{2}) \cap 2(Z_2)$ but $\bar{2} \notin 2(\bar{2})$.

5) If N is a purely quasi-invertible R-submodule of an R-module M, then $ann_R M = ann_R N$.
Proof. Follows by [14, Prop 1.4, P.7] . □

However, the converse of (Rem.and.Ex 1.3(5)) is not true as the following example shows: Consider Z-module $Z \oplus Z_4$, let $N = 2Z \oplus Z_4 \leq Z \oplus Z_4$, then $ann_Z(Z \oplus Z_4) = ann_Z(2Z \oplus Z_2) = 0$ but $N = 2Z \oplus Z_4$ is not purely quasi-invertible of $Z \oplus Z_4$ as Z-module. In fact N is not pure, since $(2, \bar{2}) = 2(1, \bar{1}) \in (2Z \oplus Z_4) \cap 2(Z \oplus Z_4)$ but $(2, \bar{2}) \notin 2(2Z \oplus Z_4)$.

6) Let I be an ideal of a ring R . If I is purely quasi-invertible then $ann_R(I) = 0$.
Proof. Obvious . □

The converse of (Rem.and.Ex 1.3(6)) is not true in general, consider the following example: Let $R = Z$, let $I = 2Z$ then $ann_Z(I) = ann_Z(2Z) = 0$, but I is not pure of Z, since $J = 4Z$ be an ideal of Z and $JI = (4Z)(2Z) = 8Z \neq 4Z = (2Z) \cap (4Z) = I \cap JZ$, so it is not purely quasi-invertible ideal of Z.

7) If $M = M_1 \oplus M_2$ is an R-module and let K be a purely quasi-invertible in M_i for some $i = 1,2$, then it is not necessarily that K is a purely quasi-invertible submodule of M ;
For example: In the Z-module $Z \oplus Z_2$, $K = Z_2$ is a purely quasi-invertible submodule of Z_2 as Z-module , but $Z_2 \cong (0) \oplus Z_2$ which is not a purely quasi-invertible submodule of $Z \oplus Z_2$ as Z-module, since $Hom_Z(Z \oplus Z_2/(0) \oplus Z_2, Z \oplus Z_2) = Hom_Z(Z, Z \oplus Z_2) \neq 0$; that is $(0) \oplus Z_2$ not quasi-invertible .

Remark 1.4. We do not whether the intersection of purely quasi-invertible submodules is purely quasi-invertible.

Recall that an R-module M has the pure intersection property (briefly *PIP*) if, the intersection of any two pure submodules is again pure [3, def 2.1, P.33] .

Now we can introduce the following result .

Proposition 1.5. Let M be an R-module has *PIP*. If N_1, N_2 are purely quasi-invertible submodules of M then $N_1 \cap N_2$ is also .

Proof. Since M has *PIP* then $N_1 \cap N_2$ is pure in M. But it is easy to see that $Hom(M/N_1 \cap N_2, M) \subseteq Hom(M/N_1, M) + Hom(M/N_2, M)$. Hence $Hom(M/N_1 \cap N_2, M) = 0$ and so that $N_1 \cap N_2$ is a purely quasi- invertible submodule of M . □

Recall that an R-module M is called multiplication if, for each submodule N of M, $N=IM$ for some ideal I of R. Equivalently, M is multiplication if, for each submodule N of M, $N=[N:M].M$, where $[N:M]=\{r\in R: rM\subseteq N\}$ [19] .

Corollary 1.6. Let M be a multiplication R-module. If N_1,N_2 are purely quasi-invertible submodules of M then $N_1\cap N_2$ is also .

Proof. Follows by [3, Prop 2.3, p.33] and (Prop 1.5) . □

However, the following results (1.5) , (1.6) gives necessary conditions for make (Rem 1.4) is true .

Remark 1.7. Let M be an R-module and let N be a purely quasi-invertible submodule of M. If $K\leq M$ such that $K\cong N$ then it is not necessarily that K is a purely quasi-invertible submodule of M. We can give the following example show that .

Example 1.8. Let $M = Z$ as Z-module, let $N = Z$ be a submodule of M, then N is a purely quasi-invertible submodule of M, but $K=2Z\cong Z=N$ is not a purely quasi-quasi-invertible submodule of M . In the fact $K = 2Z$ is not pure in M .

Remark 1.9. Let M_1, M_2 be R-modules and let $f: M_1\longrightarrow M_2$ be R-homomorphism . If N is a purely quasi-invertible submodule of M_1 then not necessary that the image of N is a purely quasi-invertible submodule of M_2 . For example : Consider Z-modules Z_4, Z_6. Let $f: Z_6\longrightarrow Z_4$ be Z-homomorphism define by $f(\overline{x})=2\overline{x}$ for all $\overline{x}\in Z_6$. Let $N = Z_6$, it is well known that N is a purely quasi-invertible submodule of Z_6 as Z-module, but $f(N)=f(Z_6)=\{\overline{0},\overline{2}\}=(\overline{2})$ is not purely quasi-invertible submodule of Z_4 as Z-module (see Rem.and.Ex 1.3(4)) .

Recall that a nonzero R-module M is called a rational extension of the R-submodule N of M if, for all $m_1,m_2\in M, m_2\neq 0$, there exists an element $r\in R$ such that $rm_1\in N$ and $rm_2\neq 0$ [20] . And recall that an R-module M is regular if for all $a\in M$ and for all $r\in R$, there exists $x\in R$ such that $rxra=ra$. Equivalently, every submodule of M is pure [7] .

Proposition 1.10. Let M be a module over regular ring R and let $N\leq M$. If M is a rational extension of N then N is a purely quasi-invertible submodule of M .

Proof. Since M is a rational extension of N then by [14, Prop 3.3, P.14] N is a quasi-invertible submodule of M . On the other hand, since R is a regular ring then M is a regular R-module ; that is every submodule of M is pure, thus N is a purely quasi-invertible submodule of M . □

Recall that an R-submodule N of an R-module M is called small (briefly $N\ll M$) if, for all $K\leq M$ with $N+K = M$ implies $K = M$ [12, P.106] . And recall that an R-submodule N of R-module M is called SQI-submodule if, for each $f\in Hom_R(M/N,M)$ then $f(\frac{M}{N})$ is a small in M [17, p.44] .

Remark 1.11. It is clear that every quasi-invertible submodule is SQI-submodule, hence every purely quasi-invertible submodule is SQI-submodule. But the converse is not true in general, the following example shows .

Example 1.12. Let $M = Z_4$ as Z-module and let $N = (\bar{2}) \leq M$. Then N is SQI-submodule of Z_4, but it is known that N is not a purely quasi-invertible submodule of Z_4 (See Rem.and.Ex 1.3(4)).

We end this section by the following theorem .

Theorem 1.13. Let M be a faithful multiplication over integral domain R . If N is a pure submodule of M then $[N : M]$ is a purely quasi-invertible ideal of R .

Proof. Assume that N is a pure submodule of M . Since M be a faithful multiplication R-module, so by [4, Coro 1.2, P.65] $[N : M]$ is a pure ideal of R. But R is an integral domain, hence by [14, Ex 1.3(1), P.6] every nonzero ideal of R is quasi-invertible,thus $[N : M]$ is a quasi-invertible ideal of R. Hence $[N : M]$ is a purely quasi-invertible ideal of R . □

2. Purely Quasi-Dedekind Modules

Recall that an R-module M is called quasi-Dedekind if, every nonzero submodule of M is quasi-invertible; that is $Hom_R(M/N, M) = 0$ for all nonzero submodule N of M [14, P.24] . In this section we give generalization of the concept a quasi-Dedekind module namely " purely quasi-Dedekind module ". We list some basic properties of purely quasi-Dedekind modules. Also we give a characterization of this concept. We study the relationships between a purely quasi-Dedekind modules with other related modules .We begin with the following definition :

Definition 2.1. An R-module M is said to be purely quasi-Dedekind if, every proper nonzero pure submodule of M is quasi-invertible. And a ring R is called purely quasi-Dedekind if R is a purely quasi-Dedekind R-module .

It is clear that every quasi-Dedekind R-module is a purely quasi-Dedekind R-module . But the converse may note be, as the following example shows :

Example 2.2. Consider Z-module Z_4, it is clear that Z_4 is purely quasi-Dedekind , since Z_4 as Z-module has no proper pure submodule. But it is not quasi-Dedekind , since $(\bar{2}) \leq Z_4$ and $Hom_Z(Z_4/(\bar{2}), Z_4) \cong Z_2 \neq 0$.

Remarks and Examples 2.3.

1) Every simple R-module is a purely quasi-Dedekind R-module .

2) Every nonzero semisimple and (not simple) module is not a purely quasi-Dedekind module. In particular Z_6 as Z-module is semisimple and (not simple) but it is not purely quasi-Dedekind .

3) Every integral domain R is a quasi-Dedekind R-module [14, Ex 1.4(1), P.24] , so it is a purely quasi-Dedekind R-module. But the converse need not be in general; For example: Let $M = Z_4$ as Z_4-module, then Z_4 is purely quasi-Dedekind, but Z_4 is not an integral domain .

4) Z as Z-module is purely quasi-Dedekind . *0* , Z are the only pure submodules of Z .

5) Let M be a regular R-module . Then M is purely quasi-Dedekind if and only if M is quasi-Dedekind .
   Proof. Clear . □

6) Let M be a module over regular ring R. Then M is purely quasi-Dedekind if and only if M is quasi-Dedekind .
   Proof. Follows by (Rem.and.Ex 2.3(5)) and since every module over a regular ring is regular . □

7) If M is a purely quasi-Dedekind R-module then $ann_R N = ann_R M$ for all nonzero pure submodule N of M .
   Proof. Follows by (Rem.and.Ex 1.3(5)) . □

Proposition 2.4. Let M be an R-module with $\overline{R} = R/J$, where J is an ideal of R such that $J \subseteq ann_R M$. M is a purely quasi-Dedekind R-module if and only if M is a purely quasi-Dedekind $\overline{R}$ -module .

Proof. We have by [12, P.51] $Hom_R(M/N, M) = Hom_{\overline{R}}(M/N, M)$ for all submodule N of M . Thus the result is obtained . □

Proposition 2.5. Let M be a uniform R-module with $ann_R M$ is a maximal ideal of R, then M is a purely quasi-Dedekind R-module .

Proof. Follows by [11, Coro 1.2.10 and (Rem.and.Ex 1.2.2(5))] . □

Theorem 2.6. Let M be an R-module. If M is purely quasi-Dedekind then for all $f \in End_R(M)$ and $Kerf$ is a pure submodule of M implies $f = 0$.

Proof. Let $f \in End_R(M)$ and $Kerf$ is a pure submodule of M . Suppose that $f \neq 0$, define $g : M/Kerf \longrightarrow M$ by $g(m + Kerf) = f(m)$ for all $m \in M$. It is easy to see that g is Well-defined and $g \neq 0$ (since $f \neq 0$) . Hence $Hom_R(M/Kerf, M) \neq 0$ which is a Contradiction. □

Proposition 2.7. Let M be an R-module such that for all pure submodule N of M, and for all $K \leq M$ such that $N \leq K \leq M$ implies K is pure in M . If for all $f \in End_R(M)$, $Kerf$ is a pure submodule of M implies $f = 0$, then M is a purely quasi-Dedekind R-module .

Proof. Suppose that there exists $0 \neq N \leq M$, N is pure such that $Hom_R(M/N, M) \neq 0$; that is there exists R-homomorphism $f : M/N \longrightarrow M$ and $f \neq 0$. Now, consider the following diagram : $M \xrightarrow{\pi} M/N \xrightarrow{f} M$, where π is the canonical projection map. Let $\phi = f o \pi$, so $\phi \in End_R(M)$, but $N \subseteq Ker\phi$ and N is a nonzero pure submodule of M, thus $Ker\phi$ is a nonzero pure submodule of M (by hypothesis) . On the other hand $\phi(M) = f(M/N) \neq 0$ which is a contradiction . □

We will need the following lemma for the proof next proposition .

Lemma 2.8. Let M_1 , M_2 be R-modules and let $f : M_1 \longrightarrow M_2$ be R-epimorphism . If N is a pure submodule of M_2 then $f^{-1}(N)$ is a pure submodule of M_1 .

Proof. Assume that I is an ideal of R , then $I f^{-1}(N) = f^{-1}(IN) = f^{-1}(N \cap IM_2) =$ $f^{-1}(N) \cap f^{-1}(IM_2) = f^{-1}(N) \cap I f^{-1}(M_2) = f^{-1}(N) \cap I.M_1$, since f is epimorphism . Thus $f^{-1}(N)$ is a pure submodule of M_1 . □

Now, we can introduce the following proposition .

Proposition 2.9. Let M_1 , M_2 be R-modules such that M_1 is isomorphic to M_2 . Then M_1 is purely quasi-Dedekind if and only if M_2 is purely quasi-Dedekind .

Proof. Suppose that M_1 is a purely quasi-Dedekind R-module. Since $M_1 \cong M_2$, so there exists $f : M_1 \longrightarrow M_2$ be R-isomorphism. Let N be a nonzero pure submodule of M_2, thus by above lemma $f^{-1}(N)$ is a nonzero pure submodule of M_1, so $Hom_R(M_1/f^{-1}(N), M_1) = 0$.
But $Hom_R(M_2/N, M_2) \cong (Hom_R(M_1/f^{-1}(N), M_1)$, since $M_1 \cong M_2$. Thus $Hom_R(M_2/N, M_2) = 0$ for all nonzero pure submodule N of M_2 . Therefore M_2 is purely quasi-Dedekind .
The proof of the converse is similarly . □

Remark 2.10. Let M be a purely quasi-Dedekind R-module and $N \leq M$ then not necessary that M/N is a purely quasi-Dedekind R-module, as the following example shows .

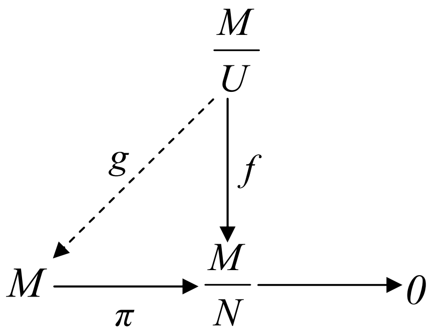
Example 2.11. It is know that Z as Z-module is purely quasi-Dedekind, let $N = 6Z \leq Z$. But $Z/6Z \cong Z_6$ is not a purely quasi-Dedekind as Z-module (see Rem.and.Ex 2.3(2)) .

Now, we shall give a necessary condition under which the (Rem 2.10) is true .

Proposition 2.12. Let M be a purely quasi-Dedekind R-module with $\frac{M}{K}$ is projective for all pure submodule K of M, then $\frac{M}{N}$ is a purely quasi-Dedekind R-module for all $N \leq M$.

Proof. Let $N \leq M$. If $N = 0$, then nothing to prove . Now, let $0 \neq N \leq M$. Suppose that $\frac{U}{N}$ is a pure submodule of $\frac{M}{N}$, then by (Lemma 2.8) $\pi^{-1}(\frac{U}{N})$ is a pure submodule of M , where π is the canonical projection map, so U is a pure submodule of M, hence $\frac{M}{U}$ is projective by hypothesis. Assume that $\frac{M}{N}$ is not purely quasi-Dedekind , thus there exists a nonzero R-homomorphism

$f:\frac{M/N}{U/N}\longrightarrow\frac{M}{N}$. But $Hom_R(\frac{M/N}{U/N},\frac{M}{N})\cong Hom_R(\frac{M}{U},\frac{M}{N})$, so there exists *R*-homomorphism $g:\frac{M}{U}\longrightarrow M$ such that $\pi o g=f$.



$g\neq 0$ **(** since $f\neq 0$) , thus $Hom_R(\frac{M}{U},M)\neq 0$, *U* is pure . Hence *M* is not a purely quasi-Dedekind *R*-module which is a contradiction . Therefore $\frac{M}{N}$ must to be a purely quasi-Dedekind *R*-module . □

Remark 2.13. Let *M* be an *R*-module and $N\leq M$. If M/N is a purely quasi-Dedekind *R*-module then not necessary that *M* is a quasi-Dedekind *R*-module; For example: Consider *Z*-module Z_6, $N=(\bar{2})\leq Z_6$. Then $Z_6/(\bar{2})\cong Z_2$ is a purely quasi-Dedekind as *Z*-module, but Z_6 is not a purely quasi-Dedekind as *Z*-module (see Rem.and.Ex 2.3 (1), (2)) .

The following example shows the direct sum of purely quasi-Dedekind modules is not necessary that a purely quasi-Dedekind module .

Example 2.14. Each of Z_2, Z_3 as *Z*-module is purely quasi-Dedekind (see Rem.and.Ex 2.3(1)), but $Z_2\oplus Z_3\cong Z_6$ is not a purely quasi-Dedekind as *Z*-module .

Now, we gives a condition under which the direct sum of purely quasi-Dedekind modules is also purely quasi-Dedekind in the next proposition .

Proposition 2.15. Let *M* and *N* be a purely quasi-Dedekind *R*-modules with $ann_R M+ann_R N=R$ then $M\oplus N$ is a purely quasi-Dedekind *R*-module .

Proof. Assume that *K* is a pure submodule of $M\oplus N$. And since $ann_R M+ann_R N=R$ then by same way of the proof of [1, Prop 4.2, Ch.1] $K=K_1\oplus K_2$, where $K_1\leq M$ and $K_2\leq N$ .But $K_1\leq^{\oplus}K$ and $K_2\leq^{\oplus}K$ then by [21] K_1 , K_2 are pure in *K*, but *K* is pure in $M\oplus N$ by hypothesis, then K_1 is pure in *M* and K_2 is pure in *N*; to show this : Assume that there exists be an ideal *I* of *R* such that $IK_1\neq K_1\cap IM$ and ($IK_2\neq K_2\cap IN$ or $IK_2=K_2\cap IN$) then $IK=I(K_1\oplus K_2)=IK_1\oplus IK_2\neq(K_1\cap IM)\oplus(K_2\cap IN)=(K_1\oplus K_2)\cap I(M\oplus N)=K\cap I(M\oplus N)$ which is a contradiction . So $Hom_R(M/K_1,M)=0$ and $Hom_R(N/K_2,N)=0$, since *M* and *N* is purely quasi-Dedekind . On the other hand we have $Hom_R(M\oplus N/K,M\oplus N)=$ $Hom_R(M\oplus N/K_1\oplus K_2,M\oplus N)\subseteq Hom_R(M/K_1,M)\cap Hom_R(N/K_2,N)=0$. Hence $M\oplus N$ is a purely quasi-Dedekind *R*-module . □

Recall that an R-module M is scalar if, for all $f \in End_R(M)$ then there exists $r \in R$ such that $f(x) = rx$ for all $x \in M$ [18, P.8] .

In the following proposition we shall study the endomorphism ring of purely quasi-Dedekind module .

Proposition 2.16. Let M be a scalar R-module with $ann_R M$ is a prime ideal of R, then $End_R(M)$ is a purely quasi-Dedekind ring .

Proof. Since M be a scalar R-module, then by [15, Lemma 6.2, P.80] $End_R(M) \cong R/ann_R M$, But $ann_R M$ is a prime, so $End_R(M)$ is an integral domain. Hence by (Rem.and.Ex 2.3(3)) $End_R(M)$ is a purely quasi-Dedekind ring . □

Corollary 2.17. If M is a scalar and prime R-module, then $End_R(M)$ is a purely quasi-Dedekind ring .

Proof. It is clearly, since M is prime implies $ann_R M$ is a prime ideal, so the result is obtained by (Prop 2.16) . □

Proposition 2.18. Let M be a scalar faithful R-module . $End_R(M)$ is a purely quasi-Dedekind ring if and only if R is a purely quasi-Dedekind ring .

Proof. Suppose that M is a scalar R-module, so $End_R(M) \cong R/ann_R M$ by [15,Lemma 6.2, P.80] , but M is a faithful, thus $R/ann_R M \cong R$, so $End_R(M) \cong R$. Hence we have on the result . □

Corollary 2.19. Let M be a finitely generated multiplication faithful R-module . $End_R(M)$ is a purely quasi-Dedekind ring if and only if R is a purely quasi-Dedekind ring .

Proof. Since M is a finitely generated multiplication R-module, then by [16, The.3.2] M is scalar R-module; that is M is a scalar faithful R-module, thus by (Prop 2.18) the result is obtained . □

Recall that an R-module M is called quasi-prime if $ann_R N$ is a prime ideal of R for each $0 \neq N \leq M$ [2, def 1.2.1] .

Proposition 2.20. Let M be a quasi-injective scalar and quasi-prime R-module then $End_R(N)$ is a purely quasi-Dedekind ring for all $0 \neq N \leq M$.

Proof. Assume that $0 \neq N \leq M$. Since M is a quasi-injective scalar R-module, then by [18, Prop 1.1.16] N is a scalar R-module, thus $End_R(N) \cong R/ann_R N$ by [15, Lemma 6.2, P.80]. But M is a quasi-prime R-module , so $ann_R N$ is a prime ideal of R; that is $End_R(N) \cong R/ann_R N$ is an integral domain . Hence by (Rem.and.Ex 2.3(3)) $End_R(N)$ is a purely quasi-Dedekind ring . □

We end this section by the following two corollaries .

Corollary 2.21. If M is an injective scalar and quasi-prime R-module then $End_R(N)$ is a purely quasi-Dedekind ring for all $0 \neq N \leq M$.

Proof. Obvious . □

Corollary 2.22. Let M be a quasi-injective scalar R-module and let $0 \neq N \leq M$ be a faithful R-module. Then $End_R(N)$ is a purely quasi-Dedekind ring if and only if R is a purely quasi-Dedekind ring .

Proof. Follows by [18 , Prop 1.1.16] and (Prop 2.18) . □

3. Purely Prime Modules

Recall that an R-module M is called prime if, $ann_R M = ann_R N$ for all nonzero submodule N of M [8] . In this section we see that if M is purely quasi-Dedekind then $ann_R M = ann_R N$ for all nonzero pure submodule N of M (Prop 3.2). This leads us to introduce many of important statement to this concept with other concepts in this section . We start this section with the following definition :

Definition 3.1. An R-module M is said to be purely prime if, $ann_R M = ann_R N$ for all nonzero pure submodule N of M .

It is clear that every prime module is a purely prime module, but the converse need not be in general; for example : Z_4 as Z-module is purely prime . In fact Z_4 has no proper nonzero pure submodule as Z-module, but it is not prime as Z-module, since $(\bar{2}) \leq Z_4$, $ann_Z(\bar{2}) = 2Z \neq 4Z = ann_Z(Z_4)$.

Proposition 3.2. Every purely quasi-Dedekind module is a purely prime module .

Proof. Follows by (Rem.and.Ex 2.3(7)) . □

Proposition 3.3. Let M be an R-module. Then M is a purely prime R-module if and only if M is a purely prime $\bar{R}$ -module, where $\bar{R} = R/ann_R M$.

Proof. $\Rightarrow$) Suppose that N is a nonzero pure $\bar{R}$ -submodule of M . It is easy to see that N is a nonzero pure R-submodule of M . Let I be an ideal of $\bar{R}$, so it is also ideal of R, thus $IN = N \cap IM$ hence N is a pure R-submodule of M, so that $ann_R M = ann_R N$. Now, it is clear that $ann_{\bar{R}} M \subseteq ann_{\bar{R}} N$, beside let $r + ann_{\bar{R}} M \in ann_{\bar{R}} N$ then $rN = 0$; that is $r \in ann_R N = ann_R M$, hence $r + ann_{\bar{R}} M \in ann_{\bar{R}} M$, therefore $ann_R M = ann_R N$.

$\Leftarrow$) The proof is similarly . □

Proposition 3.4. Let M be a uniform regular R-module. Then the following statements are equivalent :

1) M is a prime R-module .
2) M is a purely prime R-module .
3) M is a purely quasi-Dedekind R-module .
4) M is a quasi-Dedekind R-module .

Proof.

$(1) \Leftrightarrow (2)$: Clear .

$(3) \Rightarrow (2)$: Follows by (Prop 3.2) .

$(2) \Leftarrow (3)$: Suppose that M is purely prime, and since M is regular , so M is prime; that is M is prime uniform, thus by [14, The 3.11, P.37] M is quasi-Dedekind and hence M is purely quasi-Dedekind .

$(3) \Leftrightarrow (4)$: Follows by (Rem.and.Ex 2.3(5)) . □

Corollary 3.5. Let M be a multiplication uniform regular R-module. Then $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6) \Rightarrow (7)$

1) M is a prime R-module .
2) M is a purely prime R-module .
3) M is a purely quasi-Dedekind R-module .
4) M is a quasi-Dedekind R-module .
5) $End_R(M)$ is an integral domain .
6) $End_R(M)$ is a quasi-Dedekind ring .
7) $End_R(M)$ is a purely quasi-Dedekind ring .

Proof.

$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$: Follows by (Prop 3.4) .

$(4) \Leftrightarrow (5)$: Follows by [11, Prop 2.1.27] .

$(5) \Leftrightarrow (6)$: Follows by [11, Rem.and.Ex 1.1.2(7)]

$(6) \Rightarrow (7)$: Clear . □

Recall that an R-module M is monoform if for each $N \le M$ and for each $f \in Hom_R(N, M)$, $f \ne 0$ implies $Kerf = 0$ [22] .

Remark 3.6. Every monoform module is a purely quasi-Dedekind module and hence it is a purely prime module .

The converse of above remark is not true in general; for example : Consider Z-module $Z \oplus Z$ then it is known that is purely prime, since it is prime. But $Z \oplus Z$ is not monoform as Z-module.

Proposition 3.7. Let M be a uniform regular ring. Then the following statements are equivalent :

1) R is a monoform ring .
2) R is an integral domain .
3) R is a quasi-Dedekind ring .
4) R is a purely quasi-Dedekind ring .
5) R is a purely prime ring .
6) R is a prime ring .

Proof.
$(1) \Leftrightarrow (2) \Leftrightarrow (3)$: Follows by [11, Coro 2.3.20] .
$(3) \Leftrightarrow (4)$: Clear .
$(4) \Rightarrow (5)$: Clear .
$(5) \Rightarrow (4)$: Assume that R is purely prime , and since R is regular, then R is prime. But R is uniform, so by [14, The 3.11, P.37] R is quasi-Dedekind, hence R is a purely quasi-Dedekind ring .
$(5) \Leftrightarrow (6)$: Clear . □

Proposition 3.8. Let M be an R-module. If M is embedded in each of its nonzero pure submodule then M is a purely prime R-module .

Proof. Suppose that N is a nonzero pure submodule of M . It is known that $ann_R M \subseteq ann_R N$.
On the other hand, let $r \in ann_R N$ then $rN = 0$. But M is embedded in N (by hypothesis), so there exists a monomorphism $f : M \longrightarrow N$, thus $f(rM) = rf(M) \subseteq rN = 0$ implies $rM = 0$ (since f is monomorphism), so $r \in ann_R M$ and $ann_R M = ann_R N$. Hence M is a purely prime R-module . □

Corollary 3.9. Let M be a uniform regular R-module such that M is embedded in each of its nonzero pure submodule then M is a quasi-Dedekind R-module and hence it is a purely quasi-Dedekind R-module .

Proof. Follows by (Prop 3.8) and (Prop 3.4) . □

Recall that an R-module M is said to be weak cancellation if, for any two ideals A , B of R with $AM = BM$ implies that $A + ann_R M = B + ann_R M$. And recall that an R-module M is cancellation if M is weak cancellation and faithful [6] .

Mijbass A.S. in [13, P.62 , P.63] introduce the following two results :

Theorem 3.10. Let M be an R-module and let N be a pure in M with $ann_R N = ann_R M$. If N is a weak cancellation R-module then M is a weak cancellation R-module .

Corollary 3.11. Let M be an R-module and let N be a pure in M with $ann_R N = ann_R M$. If N is a cancellation R-module then M is a cancellation R-module .

We end this section by the following two corollaries .

Corollary 3.12. Let M be a purely prime R-module and let N be a pure in M. If N is a weak cancellation R-module then M is a weak cancellation R-module .

Proof. Follows by (Th 3.10) . □

Corollary 3.13. Let M be a purely prime R-module and let N be a pure in M. If N is a cancellation R-module then M is a cancellation R-module .

Proof. Follows by (Coro 3.11) . □

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المقاسات شبه- ديديكاندية النقية
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المستخلص :
يسمّى المقاس الجزئـي N من المقـاس M علـى الحلقـة R بالمقـاس الجزئـي النقـي أذا كـان $IN = N \cap IM$ لكل مثالي I من الحلقة R . في بحثنا هذا قدّمنا المقاس الجزئي شبه- معكوس النقي، حيث أن المقـاس الجزئـي N مـن المقـاس M يسمى شبه- معكوس نقي أذا كان N مقاس جزئي نقي ومقاس جزئـي شبـه- معكـوس أي $Hom_R(M/N, M) = 0$. يسمّى المقـاس M بأنه مقاس شبه- ديديكاندي نقي أذا كان كل مقاس جزئي غير صفري نقي N من M هو مقاس شبه- معكـوس . مـن جانـب أخـر نحـن أيضاً قدّمنا مفهوم أخر من المقاسات يسمّى المقاس الأولـي النقي، حيث يسمّى المقاس M *علـى الحلقة* R بأنه مقاس أولي نقـي أذا كـان تالف M = تالف N لكل مقاس جزئي غيرصفري نقي N من M . لقد أعطينا العديد من الخواص الأساسية المتعلقة بهذه المفـاهيم . كذلك درسنا العلاقات بين هذه المفاهيم وأنواع عديدة أخرى من المقاسات . في هذا البحث الحلقة R هـي أبدالية بمحايد و M مقاساً أحادياً على R .