Principally Pseudo-Injective Modules

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Abstract

The concepts of pseudo-injective modules and principally quasi-injective modules are generalized in this paper to principally pseudo-injective modules . Many characterizations and properties of principally pseudo-injective modules are obtained. Relationships between principally pseudo-injective modules and other classes of modules are given for example we proved that for each integer $n \geq 2$, then M^n is principally pseudo-injective R-module if and only if M is principally quasi-injective R-module. New characterizations of semi-simple Artinian ring in terms of principally pseudo-injective modules are introduced. Endomorphisms ring of principally pseudo-injective modules are studied .

§0:- Introduction

Throughout this paper, R will denote an associative, commutative ring with identity, and all R-modules are unitary (left) R-modules. Given two R-modules M and N. M is called pseudo-N-injective if for any R-submodule A of N and every R-monomorphism from A into M can be extended to an R-homomorphism from N into M [16] . An R-module M is called pseudo-injective if M is pseudo-M-injective[19]. An R-module M is called principally N-injective if for any cyclic R-submodule A of N and every R-homomorphism from A into M can be extended to an R-homomorphism from N into M. An R-module M is called principally quasi-injective (or semi-fully stable[2]) if M is principally M-injective[14]. An R-module M is called p-injective if M is principally R-injective[13]. An R-module M is called pointwise injective if for each R-monomorphism $f:A \to B$ (where A and B are two R-modules), each R-homomorphism $g:A \to M$ and for each $a \in A$, there exists an R-homomorphism $h_a:B \to M$ (h_a may depend on a) such that $(h_a \circ f)(a)=g(a)$ [8].An R-module M is pointwise injective if and only if M is principally N-injective for every R-module N [8].An R-module M is called pointwise ker-injective if for each R-monomorphism $f:A\to B$ (where A and B are R-modules), each R-homomorphism $g:A\to M$ and for each $a\in A$, there exist an R-monomorphism $\alpha:M\to M$ and R-homomorphism $\beta_a:B\to M$ (β_a may depend on a) such that $(\beta_a\circ f)(a)=(\alpha\circ g)(a)$ [12]. An R-monomorphism $f:N\to M$ is called p-split if for each $a\in N$, there exists an R-homomorphism $g_a:M\to N$ (g_a may depend on a) such that $(g_a\circ f)(a)=a$ [8] . An R-monomorphism $f:N\to M$ is called pointwise ker-split if for each $a\in N$, there exist an R-monomorphism $\alpha:N\to N$ and an R-homomorphism $g_a:M\to N$ (g_a may depend on a) such that $(g_a\circ f)(a)=\alpha(a)$ [12]. Recall that an R-module M is fully stable (fully p-stable) if for each R-submodule N of M and each R-homomorphism (resp. R-monomorphism) $f:N\to M$, then $f(N)\subseteq N$ [1].A ring R is called Von Neumann regular(in short, regular) if for each $a\in R$, there exsits $b\in R$ such that a=aba .For an R-module M , J(M),E(M) and $S=End_R(M)$ will respectively stand for the Jacobson radical of M ,the injective envelope of M and the endomorphism ring of M. $Hom_R(N,M)$ denoted to the set of all R-homomorphism from R-module N into R-module M . For a submodule N of an R-module M and $a\in M$, $[N:a]_R=\{r\in R \mid ra\in N\}$. For an R-module M and $a\in M$, then $ann_R(a)$ denoted to the set $[(0):a]_R$. A submodule N of an R-module M is called essential and denoted by $N\subset^e M$. if every non zero submodule of M has non zero intersection with N. An R-module M is called uniform if every non zero R-submodule of M is essential .

§1:- Principally pseudo-N-injectivity

In this section we introduced the concept of principally pseudo-N-injective modules as generalization of both pseudo-N-injective modules and principally N-injective modules.

Definition(1.1):- Let M and N be two R-modules. M is said to be principally pseudo-N-injective (in short, p-pseudo-N-injective) if for any cyclic R-submodule A of N and any R-monomorphism f: $A\to M$ can be extended to an R-homomorphism form N to M . An R-module M is called principally pseudo-injective (in short , p-pseudo-injective) if M is principally pseudo-M-injective . A ring R is called principally pseudo-injective if R is a principally pseudo-injective R-module .

Examples and remarks(1.2):-

(1) All principally quasi-injective modules (also,pseudo-injective modules) are trivial examples of p-pseudo-injective modules.

(2) The concept of p-pseudo-injective modules is a proper generalization of both pseudo-injective modules and principally quasi-injective modules ; for examples :-

i-) Let $R=Z_2[x,y]/(x^2,y^2)$ be the polynomial ring in two indeterminates x,y over Z_2 modulo the ideal (x^2,y^2). Since R is a principally quasi-injective ring [1] thus by (1) above we have R is p-pseudo-injective. Assume that R is a self pseudo-injective ring. Since R is a Noetherian ring, thus by [5] R is a self-injective ring and this contradiction since R is not self-injective ring [4] . Therefore R is p-pseudo-injective ring is not self pseudo-injective.

ii-) Let R be an algebra over Z_2 having basis $\{e_1,e_2,e_3,n_1,n_2,n_3,n_4\}$ with the following multiplication table :-

	e_1	e_2	e_3	n_1	n_2	n_3	n_4
e_1	e_1	0	0	n_1	n_2	0	0
e_2	0	e_2	0	0	0	0	0
e_3	0	0	e_3	0	0	n_3	n_4
n_1	0	n_1	0	0	0	0	0
n_2	0	0	n_2	0	0	0	0
n_3	n_3	0	0	0	0	0	0
n_4	0	n_4	0	0	0	0	0

Let $M=Re_2$, then by [9] we have that M is pseudo-injective R-module is not quasi-injective R-module. By (1) above we have M is p-pseudo-injective R-module. Since every R-submodule of M is cyclic[3] , thus M is not principally quasi-injective R-module. Therefore M is p-pseudo-injective R-module is not principally quasi-injective.

(3) The examples (**i**) and (**ii**) in (**2**) are showed that the concept of p-pseudo-N-injective modules is a proper generalization of both pseudo-N-injective modules and principally N-injective modules, respectively .

(4) Every pointwise injective R-module is p-pseudo-N-injective, for all R-module N and so every pointwise injective R-module is p-pseudo- injective.

(5) Every p-injective R-module is p-pseudo-R-injective.

(6) Isomorphic R-module to p-pseudo-N-injective R-module is p-pseudo-N-injective, for any R-module N.

(7) If N_1 and N_2 are isomorphic R-modules and M is a p-pseudo-N_1-injective R-module , then M is p-pseudo-N_2-injective R-module .

In the following theorem we give many characterizations of p-pseudo-N-injective modules.

Theorem(1.3):- Let M and N be two R-modules and $S=End_R(M)$.Then the following statements are equivalent :-

(1) M is p-pseudo-N-injective.

(2) For each $m\in M$, $n\in N$ such that $ann_R(n)=ann_R(m)$, there exists an R- homomorphism $g:N\rightarrow M$ such that $g(n)=m$.

(3) For each $m\in M$, $n\in N$ such that $ann_R(n)= ann_R(m)$, we have $Sm\subseteq Hom_R(N,M)n$.

(4) For each R-monomorphism $f:A\rightarrow M$ (where A be any R-submodule of N) and each $a\in A$, there exists an R-homomorphism $g:N\rightarrow M$ such that $g(a)=f(a)$.

Proof:- (1)⇒(2) Let M be a p-pseudo-N-injective R-module. Let $m\in M$, $n\in N$ such that $ann_R(n)= ann_R(m)$. Define $f:Rn\rightarrow M$ by $f(rn)=rm$, for all $r\in R$. It is clear that f is a well-defined R-monomorphism. Since M is p-pseudo-N-injective R-module, thus there exists an R-homomorphism $g:N\rightarrow M$ such that $g(x)=f(x)$ for all $x\in Rn$. Therefore $g(n)=f(n)=m$.

(2)⇒(3) Let $m\in M$, $n\in N$ such that $ann_R(n)=ann_R(m)$.By hypothesis, there exists an R-homomorphism $g:N\rightarrow M$ such that $g(n)=m$. Let $\alpha\in S$, thus $\alpha(m)=\alpha(g(n))=(\alpha\circ g)(n)$. Since $\alpha\circ g\in Hom_R(N,M)$, thus $\alpha(m)\in Hom_R(N,M)n$. Therefore $Sm\subseteq Hom_R(N,M)n$.

(3)⇒(4) Let $f:A\rightarrow M$ be any R-monomorphism where A be any R-submodule of N, and let $a\in A$. Put $m=f(a)$, since $m\in M$ and $ann_R(m)=ann_R(a)$, thus by hypothesis we have $Sm\subseteq Hom_R(N,M)a$. Let $I_M:M\rightarrow M$ be the identity R-homomorphism. Since $I_M\in S$, thus there exists an R-homomorphism $g\in Hom_R(N,M)$ such that $I_M(m)=g(a)$. Thus $g(a)=m=f(a)$.

(4)⇒(1) Let $A=Ra$ be any cyclic R-submodule of N and $f:A\rightarrow M$ be any R-monomorphism. Since $a\in A$, thus by hypothesis there exists an R-homomorphism $g:N\rightarrow M$ such that $g(a)=f(a)$. For each $x\in A$, $x=ra$ for some $r\in R$, we have that $g(x)=g(ra)=rg(a)=rf(a)=f(ra)=f(x)$. Therefore M is p-pseudo-N-injective R- module. □

As an immediate consequence of Theorem(1.3) we have the following corollary in which we get many characterizations of p-pseudo-injective modules.

Corollary(1.4):- The following statements are equivalent for an R-module M :-

(1) M is p-pseudo-injective.

(2) For each $n,m \in M$ such that $ann_R(n)=ann_R(m)$, there exists an R-homomorphism $g:M \to M$ such that $g(n)=m$.

(3) For each $n,m \in M$ such that $ann_R(n)=ann_R(m)$, we have $Sn \subseteq Sm$ where $S= End_R(M)$.

(4) For each R-monomorphism $f:A \to M$ (where A be any R-submodule of M) and each $a \in A$, there exists an R-homomorphism $g:M \to M$ such that $g(a)=f(a)$.

Proposition(1.5):-Let M and N be two R-modules. If M is p-pseudo-N-injective, then every R-monomorphism $\alpha:M \to N$ is p-split .

Proof:-Let $\alpha:M \to N$ be any R-monomorphism and $a \in M$. Define $\beta:\alpha(M) \to M$ by $\beta(\alpha(m))=m$ for all $m \in M$. β is a well-defined R-monomorphism. Since M is p-pseudo-N-injective R-module and $\alpha(a) \in \alpha(M)$, thus by Theorem(1.3) there exists an R-homomorphism $h:N \to M$ such that $h(\alpha(a))=\beta(\alpha(a))$.Put $h_a=h$ and since $\beta(\alpha(a))=a$, thus $(h_a \circ \alpha)(a)=a$. Therefore α is p-split R-homomorphism. □

Corollary(1.6):-If M is p-pseudo-injective R-module , then every R-monomorphism $\alpha:M \to M$ is p-split.

It is easy to prove the following lemma by using [8 , Theorem(1.2.4)] .

Lemma(1.7):- An R-module M is pointwise injective if and only if every R-monomorphism $\alpha:M \to E(M)$ is p-split.

In the following proposition we get a new characterization of pointwise injective modules.

Proposition(1.8):-An R-module M is pointwise injective if and only if M is p-pseudo-E(M)-injective.

Proof:- Let M be a pointwise injective R-module. By remark(1.2(4)), then M is p-pseudo-N-injective for all R-module N. Thus M is p-pseudo-E(M)-injective R-module. Conversely, let M be a p-pseudo-E(M)-injective R-module. By proposition(1.5), every R-monomorphism α:M→E(M) is p-split and hence by lemma(1.7), then M is pointwise injective R-module. □

By proposition(1.8) and [8,Proposition(2.1.1)] we have the following corollary.

Corollary(1.9) :- Let M be a cyclic R-module. Then M is injective if and only if M is p-pseudo-E(M)-injective. In particular, a ring R is self-injective if and only if R is p-pseudo-E(R)-injective R-module.

By proposition(1.8) and [8,Corollary(2.1.5)] we have the following corollary.

Corollary(1.10):-Let R be a principal ideal ring . Then any R-module M is injective if and only if M is p-pseudo-E(M)-injective.

Proposition(1.11):- Let N be a cyclic submodule of an R-module M. If N is p-pseudo-M-injective, then N is a direct summand of M.

Proof:- Let I_N:N→N be the identity R-homomorphism . Since N is p-pseudo-M-injective R-module, thus there exists an R-homomorphismα:M→N such thatα(a)=I_N(a) for all a$\in$N. Hence ($\alpha \circ i$)(a)=a for all a$\in$N, where i is the inclusion R-homomorphism from N into M. Thus i:N→M is split R-homomorphism and hence N is a direct summand of M [11]. □

An R-module M is called regular if every cyclic R-submodule of M is direct summand of M [11]. Then by proposition(1.11) we have the following corollary.

Corollary(1.12):- If every cyclic R-submodule of an R-module M is p-pseudo-M-injective, then M is a regular R-module.

R.Yue Chi Ming in [13] proved that a ring R is regular if and only if every R-module is p-injective. The following proposition is a generalization of this result.

Proposition(1.13):- The following statements are equivalent for a ring R.

(1) R is a regular ring.

(2) Every R-module is p-pseudo-R-injective,

(3) Every ideal of R is p-pseudo-R-injective R-module.

(4) Every cyclic ideal of R is p-pseudo-R-injective R-module.

Proof:-(1)⇒(2) Let R be a regular ring and M be any R-module. Let $f:Ra\to M$ be any R-monomorphism where Ra be any cyclic ideal of R . Since R is a regular ring and $a\in R$, thus there exists $b\in R$ such that $a=aba$. Put $m=f(ba)$ and defined $g:R\to M$ by $g(x)=xm$ for all $x\in R$. It is clear g is an R-homomorphism. For each $y\in Ra$, $y=ra$ for some $r\in R$, then $g(y)=g(ra)=rg(a)=r(am)=raf(ba)=rf(aba)=rf(a)=f(ra)=f(y)$. Therefore M is p-pseudo-R-injective. **(2)⇒(3)** and **(3)⇒(4)** are obvious. **(4)⇒(1)** by Corollary(1.12). □

Proposition(1.14):- Let M and N be two R-modules. If M is p-pseudo-N-injective, then M is p-pseudo-A-injective for each R-submodule A of N.

Proof:- Let A be any R-submodule of N, B be any cyclic R-submodule of A and $f:B\to M$ be any R-monomorphism. Let $\mathbf{i}_B$ be the inclusion R-homomorphism from B into A and $\mathbf{i}_A$ be the inclusion R-homomorphism from A into N. Since B is a cyclic R-submodule of N and M is p-pseudo-N-injective, thus there exists an R-homomorphism $h:N\to M$ such that $(h\circ\mathbf{i}_A\circ\mathbf{i}_B)(b)=f(b)$, for all $b\in B$. put $g=h\circ\mathbf{i}_A:A\to M$. For each $b\in B$, then $g(b)=(h\circ\mathbf{i}_A)(b)=(h\circ\mathbf{i}_A)(\mathbf{i}_B(b))=(h\circ\mathbf{i}_A\circ\mathbf{i}_B)(b)=f(b)$. Therefore M is p-pseudo-A-injective R-module. □

As an immediate consequence of proposition(1.14) we have the following corollary.

Corollary(1.15):- Let N be any submodule of an R-module M. If N is p-pseudo-M-injective, then N is p-pseudo-injective.

Proposition(1.16):- Any direct summand of p-pseudo-N-injective R-module is p-pseudo-N-injective.

Proof:- Let M be any p-pseudo-N-injective R-module and A be any direct summand R-submodule of M. Thus there exists an R-submodule A_1 of M such that $M=A\oplus A_1$. let B be any cyclic R-submodule of N and $f:B\to A$ be any R-monomorphism. Define $g:B\to M=A\oplus A_1$ by $g(b)=(f(b),0)$, for all $b\in B$. It is clear that g is an

R-monomorphism and since M is p-pseudo-N-injective R-module, thus there exists an R-homomorphism $h:N\rightarrow M$ such that $h(b)=g(b)$ for all $b\in B$. Let π_A be the natural projection R-homomorphism of $M=A\oplus A_1$ into A . Put $h_1=\pi_A\circ h:N\rightarrow A$ .Thus h_1 is an R-homomorphism and for each $b\in B$, then $h_1(b)=(\pi_A\circ h)(b)=\pi_A(g(b))=\pi_A((f(b),0))=f(b)$. Therefore A is p-pseudo-N-injective R-module. □

By proposition (1.16) and Corollary (1.15) we have the following corollary.

Corollary(1.17):- Any direct summand of p-pseudo-injective R-module is also p-pseudo-injective.

An R-module M satisfies (PC_2), if each cyclic submodule of M which is isomorphic to a direct summand of M is a direct summand of M [17] .The following proposition is a generalization of [10,Theorem(2.7)].

Proposition(1.18):- Any p-pseudo-injective R-module satisfies (PC_2).

Proof:- Let M be a p-pseudo-injective R-module. Let A be any cyclic R-submodule of M which is isomorphic to a direct summand submodule B of M. Since M is p-pseudo-injective, thus M is p-pseudo-M-injective. Since B is a direct summand of M, thus by proposition(1.16) B is p-pseudo-M-injective R-module. Since A is isomorphic to B, thus by remark((1,2),6) A is p-pseudo-M-injective. Since A is a cyclic R-submodule of M, thus by proposition(1.11) A is a direct summand of M. Therefore M satisfies (PC_2). □

§2:- Relationships between p-pseudo-injective modules and other classes of modules

Theorem(2.1):-If $M_1\oplus M_2$ is p-pseudo-injective R-module, then M_i is principally M_j-injective for each $i,j=1,2$, $i\neq j$.

Proof:- Let $M_1\oplus M_2$ be a p-pseudo-injective R-module, we show M_1 is principally M_2-injective. Let A be any cyclic R-submodule of M_2 and $f:A\rightarrow M_1$ be any R-homomorphism. Define $g:A\rightarrow M_1\oplus M_2$ by $g(a)=(f(a),a)$ for all $a\in A$, then g is an

R-monomorphism. Since $M_1 \oplus M_2$ is p-pseudo-$M_1 \oplus M_2$-injective R-module and $(0) \oplus M_2$ is an R-submodule of $M_1 \oplus M_2$, thus by proposition(1.14) $M_1 \oplus M_2$ is p-pseudo-$(0) \oplus M_2$-injective R-module. Since M_2 isomorphic to $(0) \oplus M_2$, thus by remark((1.2),7) $M_1 \oplus M_2$ is p-pseudo-M_2-injective R-module. Thus there exists an R-homomorphism $h:M_2 \to M_1 \oplus M_2$ such that h(a)=g(a) for all $a \in A$. Let $\pi_1:M_1 \oplus M_2 \to M_1$ be the natural projection R-homomorphism of $M_1 \oplus M_2$ to M_1, put $h_1=\pi_1 \circ h:M_2 \to M_1$. Thus for each $a \in A$ we have that $h_1(a)=(\pi_1 \circ h)(a)= \pi_1(g(a))= \pi_1((f(a),a))=f(a)$. Therefore M_1 is principally M_2-injective R-module. Consequently, M_2 is principally M_1-injective.□

The following corollary is immediately from Theorem(2.1).

Corollary(2.2):- If $\bigoplus_{i \in \Gamma} M_i$ is p-pseudo-injective R-module, then M_j is principally M_k-injective for all distinct $j,k \in \Gamma$.

Corollary(2.3):-For any integer $n \geq 2$, M^n is p-pseudo-injective R-module if and only if M is principally quasi-injective.

Proof:- Let M^n be a p-pseudo-injective R-module. Then by Corollary(2.2) M is principally M-injective and hence M is a principally quasi-injective R-module. Conversely, let M be a principally quasi-injective R-module. Then M^n is principally quasi-injective R-module [2] and hence M^n is p-pseudo-injective R-module . □

In the following theorem we give a new characterization of pointwise injective modules.

Theorem(2.4):- The following statements are equivalent for an R-module M :

(1) M is pointwise injective .

(2) $M \oplus E(M)$ is principally quasi-injective R-module .

(3) $M \oplus E(M)$ is p-pseudo-injective R-module .

proof:-(1)$\Rightarrow$(2) Let M be a pointwise injective R-module. Since E(M) is pointwise injective R-module , thus $M \oplus E(M)$ is pointwise injective [8] and hence $M \oplus E(M)$ is principally quasi-injective R-module**.** **(2)$\Rightarrow$(3)**It is clear.

(3)$\Rightarrow$(1) Let $M\oplus E(M)$ be a p-pseudo-injective R-module. Thus by Theorem(2.1) M is principally E(M)-injective and hence M is p-pseudo-E(M)-injective R-module. Therefore by proposition(1.8) we have that M is pointwise injective R-module. □

By Theorem(2.4) and [8,Proposition(2.1.1)] we have the following corollary.

Corollary(2.5):-Let M be a cyclic R-module. Then M is injective if and only if $M\oplus E(M)$ is p-pseudo-injective R-module .

By Theorem(2.4) and [8, Corollary(2.1.5)] we have the following corollary.

Corollary(2.6):-Let R be a principal ideal ring. Then any R-module M is injective if and only if $M\oplus E(M)$ is p-pseudo-injective R-module .

Since any finitely generated Z-module is not injective[18], thus by Corollary(2.6) we have the following corollary.

Corollary(2.7):-For any finitely generated Z-module M, then $M\oplus E(M)$ is not p-pseudo-injective Z-module .

The following theorem gives a relation between p-pseudo-injective modules and other classes of modules.

Theorem(2.8):- The following statements are equivalent for an R-module M:-

1) M is pointwise injective R-module.

2) M is principally quasi-injective and pointwise ker-injective R-module.

3) M is p-pseudo-injective and pointwise ker-injective R-module.

Proof:-(1)$\Rightarrow$(2) and **(2)$\Rightarrow$(3)** are obvious. **(3)$\Rightarrow$(1)** Let M be a p-pseudo-injective and pointwise ker-injective R-module. Let $\alpha: M\rightarrow E(M)$ be any R-monomorphism. Since M is pointwise ker-injective, thus α is pointwise ker-split [12]. Hence for each $a\in M$ there exist an R-monomorphism $f:M\rightarrow M$ and an R-homomorphism

β_a:E(M)→M such that ($\beta_a \circ \alpha$)(a)=f(a). Since M is p-pseudo-injective R-module and f:M→M is an R-monomorphism, thus by Corollary(1.6) f is p-split. Thus for each $a \in M$ there exists an R-homomorphism g_a:M→M such that ($g_a \circ f$)(a)=a. For each $a \in M$, put $h_a = g_a \circ \beta_a$:E(M)→M, hence $(h_a \circ \alpha)(a)=((g_a \circ \beta_a) \circ \alpha)(a)=(g_a \circ (\beta_a \circ \alpha))(a)= g_a((\beta_a \circ \alpha)(a))=(g_a \circ f)(a)=a$. Then for each $a \in M$, there exists an R-homomorphism h_a:E(M)→M such that ($h_a \circ \alpha$)(a)=a. Thus each R-monomorphism α:M→E(M) is p-split and hence by lemma(1.7) M is pointwise injective R-module. □

Since every semi-simple R-module is p-pseudo-injective, thus by Theorem(2.8) we have the following corollary.

Corollary(2.9):-Every sime-simple pointwise ker-injective R-module is pointwise injective.

By Theorem(2.4) and Theorem(2.8)we get the following corollary.

Corollary(2.10):- The following statements are equivalent for an R-module M.

(1) $M \oplus E(M)$ is p-pseudo-injective R-module.

(2) M is p-pseudo-injective and pointwise ker-injective R-module.

The following proposition gives a condition on which p-pseudo-injective module is principally quasi-injective.

Proposition(2.11):-Any uniform p-pseudo-injective R-module is principally quasi-injective.

Proof:-Let M be any uniform p-pseudo-injective R-module. Let f:N→M be any R-homomorphism where N be any cyclic R-submodule of M. If ker(f)=(0), thus f is R-monomorphism. Since M is p-pseudo-injective, thus there exists an R-homomorphism f_1:M→M such that $f_1(n)=f(n)$ for all $n \in N$.Thus M is principally quasi-injective R-module. If ker(f)≠(0).Since $\ker(f) \cap \ker(i_N+f)=(0)$ where i_N is the inclusion R-homomorphism from N into M and M is a uniform R-module, thus $\ker(i_N+f)=(0)$.Hence i_N+f is an R-monomorphism. Since M is p-pseudo-injective

R-module, thus there exists an R-homomorphism $h:M\to M$ such that $h(n)=(i_N+f)(n)$, for all $n\in N$. Put $g=h-I_M:M\to M$. g is an R-homomorphism and for each $n\in N$ we have that $g(n)=(h-I_M)(n)=h(n)-I_M(n)=(i_N+f)(n)-i_N(n)=f(n)$. Therefore M is principally quasi-injective R-module. □

Remark(2.12):-Direct sum of two p-pseudo-injective R-modules need not be p-pseudo injective, for example ; let p be a prime number, then Z_p and $E(Z_p)$ are p-pseudo injective Z-modules but by Corollary(2.7) $Z_p\oplus E(Z_p)$ is not p-pseudo- injective Z-module.

The following proposition gives a condition on which direct sum of any two p-pseudo-injective R-modules is p-pseudo-injective.

Proposition(2.13):- The following statements are equivalent for a ring R:-

(1) Direct sum of any two p-pseudo-injective R-modules is p-pseudo-injective.

(2) Evrey p-pseudo-injective R-module is pointwise injective.

Proof:-(1)⇒(2) Let M be any p-pseudo-injective R-module. By hypothesis $M\oplus E(M)$ is p-pseudo-injective R-module. Thus by Theorem(2.4) we have that M is pointwise injective R-module**.** **(2)⇒(1)**Let M_1 and M_2 be any two p-pseudo-injective R-modules. By hypothesis M_1 and M_2 are pointwise injective R-modules.Thus $M_1\oplus M_2$ is pointwise injective [8]and hence $M_1\oplus M_2$ is p-pseudo-injective R-modue.□

Faith and Utumi in [6] are proved that a ring R is a semi-simple Artinian if and only if every R-module is quasi-injective. In the following corollary we give a new characterization of semi-simple Artinian ring in terms of p-pseudo-injective R-modues which is a generalization of Faith's and Utumi's result.

Corollary(2.14):- The following statements are equivalent for a ring R:-

(1) R is a semi-simple Artinian ring.

(2) Every R-module is p-pseudo-injective.

(3) Every cyclic R-module is p-pseudo-injective and direct sum of any two p-pseudo-injective R-modules is p-pseudo-injective.

Proof:- (1)⇒(2) and **(2)⇒(3)** are obvious. **(3)⇒(1)**By using proposition(2.13) and [8,Theorem(1.2.12)]. □

As an immediate consequence of proposition(2.13) we have the following corollary.

Corollary(2.15):-If the direct sum of any two p-pseudo-injective R-modules is p-pseudo-injective, then every principally quasi-injective R-module (so simple R-module) is pointwise injective.

Corollary(2.16):-If the direct sum of any two p-pseudo-injective R-modules is p-pseudo-injective, then R is a regular ring.

Proof:-Let M be any simple R-module , thus by Corollary(2.15) M is pointwise injective R-module .Since M is a cyclic , thus M is injective R-module[8] .Hence every simple R-module is injective and this implies that R is a regular ring [11]. □

In the following theorem we give a new characterization of semi-simple Artinian ring which is a generalization of Osofsky's result in [7,p.63].

Theorem(2.17):-The following statements are equivalent for a ring R :-

(1) R is a semi-simple Artinian ring .

(2) For each R-module M , if N_1 and N_2 are p-pseudo-injective R-submodules of M , then $N_1 \cap N_2$ is a p-pseudo-injective R-module .

(3) For each R-module M , if N_1 and N_2 are principally quasi-injective R-submodules of M, then $N_1 \cap N_2$ is a p-pseudo-injective R-module.

(4) For each R-module M , if N_1 and N_2 are quasi-injective R-submodules of M, then $N_1 \cap N_2$ is a p-pseudo-injective R-module.

(5) For each R-module M , if N_1 and N_2 are injective R-submodules of M, then $N_1 \cap N_2$ is a p-pseudo-injective R-module.

proof:- (1)⇒(2). It follows from corollary(2.14). **(2)⇒(3)** , **(3)⇒(4)** and **(4)⇒(5)** are obvious. **(5)⇒(1)**Let M be any R-module and E=E(M) is the injective envelope of M ,let $Q=E\oplus E$, $K=\{(x,x)\in Q \mid x\in M\}$ and let $\overline{Q}=Q/K$ .Also, put $M_1=\{y+K\in \overline{Q} \mid y\in E\oplus(0)\}$ and $M_2=\{y+K\in \overline{Q} \mid y\in(0)\oplus E\}$. It is clear that $\overline{Q}= M_1+ M_2$.Define $\alpha_1:E\to M_1$ by $\alpha_1(y)=(y,0)+K$, for all $y\in E$ and $\alpha_2:E\to M_2$

by $\alpha_2(y) = (0,y) + K$, for all $y \in E$. Since $(E \oplus (0)) \cap K = (0)$ and $((0) \oplus E) \cap K = (0)$, thus we have α_1 and α_2 are R-isomorphisms. Since E is an injective R-module , therefore M_i is injective R-submodule of $\overline{Q}$, for i=1,2 [7] . Thus by (5) , we have $M_1 \cap M_2$ is a p-pseudo-injective R-module. Define $f: M \to M_1 \cap M_2$ by $f(m) = (m,0) + K$, for all $m \in M$. Since $M_1 \cap M_2 = \{y + K \in \overline{Q} \mid y \in M \oplus (0)\}$, thus it is easy to prove that f is an R-isomorphism. Thus M is a p-pseudo-injective R-module, by remark ((1.2),6) . Hence every R-module is p-pseudo-injective and this implies that R is a semi-simple Artinian ring , by Corollary(2.14) . □

Proposition(2.18):- The following statements are equivalent for a ring R :-

(1) Every p-injective R-module is pointwise injective.

(2) Every p-injective R-module is principally quasi-injective.

(3) Every p-injective R-module is p-pseudo-injective.

Proof:- (1)⇒(2) and **(2)⇒(3)** are obvious. **(3)⇒(1)**Let M be any p-injective R-module and E(M) be the injective envelope of M. Then $M \oplus E(M)$ is p-injective and hence by hypothesis $M \oplus E(M)$ is p-pseudo-injective R-module. Therefore M is pointwise injective R-module, by Theorem(2.4). □

In the following theorem we give a new characterization of semi-simple Artinian ring .

Theorem(2.19):-The following statements are equivalent for a ring R :-

(1) R is a semi-simple Artinian ring .

(2) For each R-module M , M is p-injective if and only if M is p-pseudo-injective.

(3) For each R-module M , M is p-injective if and only if M is principally quasi-injective.

Proof:- (1)⇒(2) It is obvious. **(2)⇒(3)**Let M be a p-injective R-module. By hypothesis M is p-pseudo-injective . Thus every p-injective R-module is p-pseudo-injective and hence by proposition(2.18) we have that every p-injective R-module is principally quasi-injective. Hence M is principally quasi-injective R-module .Conversely, is clear.

(3)⇒(1) Let M be any simple R-module, then M is principally quasi-injective. By hypothesis, M is p-injective. Thus every simple R-module is p-injective. Since R is a commutative ring, then R is a regular ring[13] and hence every R-module is p-injective[13]. Thus by hypothesis we have that every R-module is principally quasi-injective and hence every R-module is p-pseudo-injective. Therefore R is a semi-simple Artinian ring , by Corollary(2.14). □

§3:-Endomorphism rings of p-pseudo-injective modules

It is easy to prove the following lemma.

lemma(3.1):-Let M be an R-module**,** $S=End_R(M)$ and $W(S)=\{\alpha\in S \mid ker(\alpha)\subset^e M\}$, thus W(S) is a two sided ideal of S.

Theorem(3.2):-Let M be a p-pseudo-injective R-module , $S=End_R(M)$ and let $W(S)=\{\alpha\in S \mid ker(\alpha)\subset^e M\}$ .Then

(1) S/W(S) is a regular ring.

(2) $J(S)\subseteq W(S)$.

proof(1):-Let $\lambda+W(S)\in S/W(S)$; $\lambda\in S$. Put $K=ker(\lambda)$ and let L be the relative complement of K in M . Define $\theta:\lambda(L)\to M$ by $\theta(\lambda(x))= x$, for all $x\in L$ .It is easy to prove that θ is a well-defined R-monomorphism .Since M is a p-pseudo-injective R-module, thus by Corollary(1.4) we have that for each $a=\lambda(x)\in\lambda(L)$,$(x\in L)$,there exists an R-homomorphism $\alpha:M\to M$ such that $\alpha(a)=\theta(a)$. If $u=x+y\in L\oplus K$ ($x\in L$ and $y\in K$), thus $(\lambda-\lambda\alpha\lambda)(u) = \lambda(x)-(\lambda\alpha\lambda)(x) = \lambda(x)-\lambda(\alpha(\lambda(x)))=\lambda(x)-\lambda(\alpha(a))=\lambda(x)-\lambda(\theta(a))=\lambda(x)-\lambda(\theta(\lambda(x)))=\lambda(x)-\lambda(x)=0$, and this implies that $u\in ker(\lambda-\lambda\alpha\lambda)$ and hence $L\oplus K\subseteq ker(\lambda-\lambda\alpha\lambda)$. Since $L\oplus K$ is an essential R-submodule of M [7], thus $ker(\lambda-\lambda\alpha\lambda)$ is an essential R-submodule of M [11] ,so $\lambda-\lambda\alpha\lambda\in W(S)$, in turn $\lambda+W(S)=(\lambda\alpha\lambda)+W(S)$. Therefore S/W(S) is a regular ring .

proof(2):- Let $\alpha\in J(S)$. Since by (1) S/W(S) is a regular ring, thus there exists $\lambda\in S$ such that $\alpha-\alpha\lambda\alpha\in W(S)$.Put $\beta=\alpha-\alpha\lambda\alpha$. Since J(S) is a two sided ideal of S, thus $-\alpha\lambda\in J(S)$. Since J(S) is quasi-regular, then$(I_M-\alpha\lambda)^{-1}$ exists where I_M is the identity R-homomorphism from M to M. Hence $(I_M-\alpha\lambda)^{-1}(I_M-\alpha\lambda)=I_M$ .Since

$(I_M-\alpha\lambda)^{-1}(\alpha-\alpha\lambda\alpha)=\alpha$, thus $(I_M-\alpha\lambda)^{-1}\beta=\alpha$. Since $\beta\in W(S)$, $(I_M-\alpha\lambda)^{-1}\in S$ and W(S) is a two sided ideal of S by lemma(3.1), thus $\alpha\in W(S)$. Therefore $J(S)\subseteq W(S)$. □

It is easy to prove the following corollary.

Corollary(3.3):- Let M be a p-pseudo-injective R-module, $S=End_R(M)$ and $W(S)=\{\alpha\in S \mid \ker(\alpha)\subset^e M\}$. Then $H\cap K = HK + W(S)\cap(H\cap K)$, for each two-sided ideals H and K of S. In particular, $K=K^2 + W(S)\cap K$ for each two-sided ideal K of S.

The following proposition is a generalization of [10,proposition(2.5)].

Proposition(3.4):- If M is p-pseudo-injective R-module and $S=End_R(M)$, then SA=SB, for any isomorphic R-submodules A,B of M.

Proof:- Since A isomorphic to B, then there exists an R-isomorphism $\alpha:A\rightarrow B$.Let $b\in B$, since α is R-epimorphism , thus there exists an element $a\in A$ such that $\alpha(a)=b$. It is clear that $ann_R(a)=ann_R(b)$. Since M is p-pseudo-injective R-module, then by corollary(1.4) $Sb\subseteq Sa$ and so $Sb\subseteq SA$ for all $b\in B$. then $SB\subseteq SA$. Similarly we can prove that $SA\subseteq SB$. Therefore SA=SB. □

As an immediate consequence of proposition(3.4)we have the following corollary.

Corollary(3.5):-If R is p-pseudo-injective ring and A,B any two isomorphic ideals of R , then A=B.

A ring R is called terse if every two distinct ideals of R are not isomorphic[20].

Proposition(3.6):- The following statements are equivalent for a ring R :-

(1) R is p-pseudo-injective ring.

(2) R is terse ring.

(3) $ann_R(x)=ann_R(y)$ implies Rx=Ry for each x,y in R.

Proof:-(1)$\Rightarrow$(2) Let R be a p-pseudo-injective ring. Let A and B are any two distinct ideals of R, thus by Corollary(3.5) A and B are not isomorphic. Therefore R is a terse ring. **(2)$\Rightarrow$(3)**[1,Theorem(2.12)].

(3)$\Rightarrow$(1) Let $x,y\in R$ such that $ann_R(x)=ann_R(y)$.By hypothesis we have Rx=Ry .We will prove that $Sx\subseteq Sy$. Let $a\in Sx$,thus there exists $f\in S$ such that a=f(x).Since

$x\in Rx=Ry$, thus there exists $r\in R$ such that $x=ry$. Define $g:R\rightarrow R$ by $g(m)=rf(m)$ for all $m\in R$. Thus $g\in S$ and $g(y)=rf(y)=f(ry)=f(x)=a$. Since $g(y)\in Sy$,thus $a\in Sy$. Hence $Sx\subseteq Sy$ and thus by Corollary(1.4) we have that R is a p-pseudo-injective ring . □

As an immediate consequence of proposition(3.6) and [1,Theorem(2,12)] we have the following corollary.

Corollary(3.7):- The following statements are equivalent for a ring R :-

(1) R is p-pseudo-injective ring .

(2) R is fully p-stable ring .

(3) Distinct cyclic ideals of R are not isomorphic.

As an immediate consequence of [1,Theorem(2,8)] and proposition(3.6) we have the following corollary.

Corollary(3.8):- The following statements are equivalent for a ring R :-

(1) R is fully stable ring.

(2) R is p-pseudo-injective ring and $Rx\cong Hom_R(Rx,R)$ for each $x\in R$.

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الموديولات الاغمارية الكاذبة رئيسياً

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الخلاصة:-

مفهومي الموديولات الاغمارية الكاذبة والموديولات شبه الاغمارية رئيسياً قد عممت في هذا البحث الى الموديولات الاغمارية الكاذبة رئيسياً. جملة من التشخيصات والخواص للموديولات الأغمارية الكاذبة رئيسياً قد أعطيت.العلاقة بين الموديولات الاغمارية الكاذبة رئيسياً وأصناف أخرى من الموديولات قد أعطيت، فمثلا برهنا انه لكل عدد صحيح موجب 2≤n فان الموديول M^n على الحلقة R يكون اغماري كاذب رئيسياً إذا وفقط إذا كان الموديول M شبه اغماري رئيسياً . أعطينا جملة من التميزات للحلقات الارتينية شبه البسيطة بدلالة هذا النوع من الموديولات. حلقة التشاكلات الموديولية للموديولات الأغمارية الكاذبة رئيسياً قد درست.