

On M-N- Fuzzy Ideal Of M-N- Fuzzy Ring

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Abstract

In this paper, we defined the terms M-N- fuzzy ring and studied some basic properties.

Also defined M-N- fuzzy ideal and its some elementary properties are discussed.

Keywords

Fuzzy ring , Fuzzy ideal , Fuzzy ring with operator , M-N- fuzzy ring , M-N- fuzzy ideal , M-N- homomorphism

Introduction

In 1965 Zadeh introduced the concepts about fuzzy set, in 1982 W.J. Liu introduced the concept term of fuzzy ring. In 1985 Ren studied the fuzzy ideal and fuzzy quotient ring . In 1994 W.X.Gu , S.Y.Li and D.G.Chen are introduced further studied in theory of fuzzy groups and gave some new concepts as M- fuzzy groups , and Shaoqan Sun give some new concepts such as fuzzy M- fuzzy ring , M – fuzzy ideal and M - fuzzy quotient .

In this paper we define M-N- fuzzy ring , M-N- ideal and study some their related are properties .

1- Preliminaries

Definition (1.1): Let R be a ring and $\mu: R \rightarrow [0, 1]$, $\mu \neq \emptyset$ then μ is called a fuzzy subring of R if for every $x, y \in R$

1- $\mu(x-y) \geq \min \{\mu(x), \mu(y)\}$.

2. $\mu(xy) \geq \min \{\mu(x), \mu(y)\}$. [1]

Definition (1.2): A fuzzy subset μ of a ring R is called a fuzzy ideal if for every $x, y \in R$

1- $\mu(x-y) \geq \min \{\mu(x), \mu(y)\}$.

2. $\mu(xy) \geq \max \{\mu(x), \mu(y)\}$. [1]

Proposition (1.3): Let μ and σ be two fuzzy ideals of a ring R then $\mu \cap \sigma$ is a fuzzy ideal of R . [2]

Definition (1.4): Let f be a function from a set X to a set Y while μ is fuzzy set of X then the image $f(\mu)$ of μ is the fuzzy set $f(\mu): Y \rightarrow [0,1]$ defined by

$$f(\mu(y)) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases} \quad [3]$$

Definition (1.5): Let f be a function from a set X to a set Y while μ is fuzzy set of X then the inverse image $f^{-1}(\mu)$ of μ under f is the fuzzy set $f^{-1}(\mu): X \rightarrow [0,1]$ defined by $f^{-1}(\mu)(x) = \mu(f(x))$. [3]

Proposition (1.6): Let μ and σ be two fuzzy subrings (fuzzy ideal) of R and R' respectively, $f: R \rightarrow R'$ be a homomorphism, then

1- $f(\mu)$ is a fuzzy subring (fuzzy ideal) of R' .

2- $f^{-1}(\mu)$ is a fuzzy subring (fuzzy ideal) of R . [4]

2-The Main Results

Definition (2.1): Let R be a ring, M, N are sets. If for every $a, b \in R$ and $m \in M, n \in N$

- 1- $(ma)n = m(an)$
- 2- $m(a+b)n = (ma)n + (mb)n = m(an) + m(bn)$
- 3- $m(ab)n = (m(ab))n = (ma)(bn)$

then m is said to be a left operator of R , M is said to be a left operator sets of R , n is said to be a right operator of R , N is said to be a right operator set of R and R is said to be an M - N - ring .

Definition (2.2): If a subring of M - N - ring is also M - N - ring then it is said to be an M - N - subring .

Definition (2.3): Let R and R' are M - N - rings . A homomorphism f from R into R' is said to be an M - N - homomorphism if

$$f(mx) = mf(x) \text{ and } f(xn) = f(x)n \text{ for every } x \in R, m \in M \text{ and } n \in N .$$

Definition (2.4): Let R be an M - N - ring and μ be a fuzzy ring of R if

$\mu(mx) \geq \mu(x)$ and $\mu(xn) \geq \mu(x)$ holds for every $x \in R, m \in M$ and $n \in N$ then μ is said to be an M - N - fuzzy ring of R

Proposition (2.5): Let R be an M - N - ring, μ and σ be two M - N -fuzzy ring of R then $\mu \cap \sigma$ is an M - N - fuzzy ring of R .

Proof : Its clear that $\mu \cap \sigma$ is a fuzzy ring of R

Now, for every $x \in R, m \in M$ and $n \in N$

$$\begin{aligned} (\mu \cap \sigma)(mx) &= \mu(mx) \wedge \sigma(mx) \\ &\geq \mu(x) \wedge \sigma(x) \\ &= (\mu \cap \sigma)(x) \end{aligned}$$

Also

$$(\mu \cap \sigma)(xn) = \mu(xn) \wedge \sigma(xn)$$

$$\begin{aligned} &\geq \mu(x) \wedge \sigma(x) \\ &= (\mu \cap \sigma)(x) \end{aligned}$$

Hence $\mu \cap \sigma$ is an M-N- fuzzy ring of R . ■

Proposition (2.6): Let R be an M-N- ring , μ be a fuzzy subset of R then μ is an M-N- fuzzy ring of R iff μ_t is M-N- subring of R for every $t \in [0,1]$ and $\mu_t \neq \emptyset$.

Proof : It is clear μ_t is subring of R while μ_t holds for every $t \in [0,1]$.

For any $x \in \mu_t$, $m \in M$ and $n \in N$

$$\mu(mx) \geq \mu(x) \geq t \text{ and } \mu(xn) \geq \mu(x) \geq t$$

$$mx \in \mu_t \text{ and } nx \in \mu_t$$

hence μ_t is an M- N- subring of R .

Conversely , either if there exists $x \in R$ and $m \in M$ such that $\mu(mx) < \mu(x)$.

$$\text{let } t = \frac{1}{2} (\mu(mx) + \mu(x))$$

$$\text{then } \mu(x) > t > \mu(mx)$$

$mx \notin \mu_t$ so here emerges a contradiction .

if there exists $x \in R$ and $n \in N$ such that $\mu(xn) < \mu(x)$.

$$\text{let } t = \frac{1}{2} (\mu(xn) + \mu(x))$$

$$\text{then } \mu(x) > t > \mu(xn)$$

$xn \notin \mu_t$ so here emerges a contradiction .

hence $\mu(mx) \geq \mu(x)$ and $\mu(xn) \geq \mu(x)$ for all $x \in R$ and $m \in M$, $n \in N$

therefor μ_t is M-N- subring of R for every $t \in [0,1]$. ■

Proposition(2.7): Let R and R' both be M-N- ring and f an M-N- ring homomorphism from R onto R' , if μ' is an M-N- fuzzy ring of R' then $f^{-1}(\mu')$ is an M-N- ring of R .

Proof:

By Proposition(1.6) $f^{-1}(\mu')$ is a fuzzy ring of R .

Now , for every $x \in R$, $m \in M$ and $n \in N$ we have

$$\begin{aligned} f^{-1}(\mu')(mx) &= \mu'(f(mx)) \\ &= \mu'(m f(x)) \\ &\geq \mu'(f(x)) \\ &= f^{-1}(\mu')(x) \end{aligned}$$

also
$$\begin{aligned} f^{-1}(\mu')(xn) &= \mu'(f(xn)) \\ &= \mu'(f(x)n) \\ &\geq \mu'(f(x)) \\ &= f^{-1}(\mu')(x) \end{aligned}$$

hence $f^{-1}(\mu')$ is an M-N- fuzzy ring of R . ■

Proposition(2.8): Let R and R' both be M-N- ring and f an M-N- ring homomorphism from R onto R' , if μ is an M-N- fuzzy ring of R then $f(\mu)$ is an M-N- ring of R' .

Proof:

By Proposition(1.6) $f(\mu)$ is a fuzzy ring of R' .

Now , for every $y \in R'$, $m \in M$ and $n \in N$ we have

$$\begin{aligned} f(\mu)(my) &= \sup_{x \in f^{-1}(my)} \mu(x) \\ &\geq \sup_{\substack{f(mx')=my \\ mx' \in R}} \mu(mx') \\ &= \sup_{\substack{f(mx')=my \\ mx' \in R}} \mu(x') \\ &\geq \sup_{\substack{f(x')=y \\ mx' \in R}} \mu(x') \\ &= f(\mu)(y) \end{aligned}$$

also let $z \in R'$, $m \in M$ and $n \in N$

$$\begin{aligned}
 f(\mu)(zn) &= \sup_{r \in f^{-1}(zn)} \mu(r) \\
 &\geq \sup_{\substack{f(r'n)=my \\ r'n \in R}} \mu(r'n) \\
 &= \sup_{\substack{f(r')n=zn \\ r' \in R}} \mu(r') \\
 &\geq \sup_{\substack{f(r')=z \\ r' \in R}} \mu(r') \\
 &= f(\mu)(z)
 \end{aligned}$$

hence $f(\mu)$ is an M-N- fuzzy ring of R' . ■

Definition (2.9): Let R be an M-N- ring, then μ is called M-N- fuzzy ideal of R , if μ is not only M-N- fuzzy ring of R but also a fuzzy ideal of R .

Proposition(2.10): Let R be an M-N- ring, μ and σ be M-N-fuzzy ideal of R then $\mu \cap \sigma$ is an M-N- fuzzy ideal of R .

Proof : By Proposition(2.5) $\mu \cap \sigma$ is an M-N- fuzzy ring of R

by proposition(1.3) $\mu \cap \sigma$ is a fuzzy ideal of R

hence $\mu \cap \sigma$ is an M-N- fuzzy ideal of R . ■

Proposition (2.11): Let R be an M-N- fuzzy ring and μ be a fuzzy subset of R then μ is an M-N- fuzzy ideal of R iff μ_t is M-N- ideal of R for every $t \in [0,1]$ when $\mu_t \neq \emptyset$.

Proof : It is clear μ_t is ideal of R while μ_t holds for every $t \in [0,1]$.

In case of μ being an M-N- fuzzy ideal of R and by Proposition(1.5) we get that μ_t is an M-N- ring, hence μ_t is an M-N- ideal.

Conversely, we know from Proposition(1.5) μ is an M-N- fuzzy subring of R , and immediate that μ is a fuzzy ideal of R .

Hence μ is an M-N- fuzzy ideal of R . ■

Proposition(2.12): Let R and R' both be M-N- ring and f an M-N- ring homomorphism from R onto R' , if μ' is an M-N- fuzzy ideal of R' then $f^{-1}(\mu')$ is an M-N- ideal of R .

Proof: By Proposition(2.7) $f^{-1}(\mu')$ is an M-N- fuzzy ring of R .

Now, to prove $f^{-1}(\mu')$ is a fuzzy ideal, for every $x, y \in R$

$$\begin{aligned} f^{-1}(\mu')(x-y) &= \mu'(f(x-y)) \\ &= \mu'(f(x)-f(y)) \\ &\geq \min \{ \mu'(f(x)), \mu'(f(y)) \} \\ &= \min \{ f^{-1}(\mu')(x), f^{-1}(\mu')(y) \} \end{aligned}$$

$$\begin{aligned} \text{Also } f^{-1}(\mu')(xy) &= \mu'(f(xy)) \\ &= \mu'(f(x) f(y)) \\ &\geq \max \{ \mu'(f(x)), \mu'(f(y)) \} \\ &= \max \{ f^{-1}(\mu')(x), f^{-1}(\mu')(y) \} \end{aligned}$$

Hence $f^{-1}(\mu')$ is a fuzzy ideal of R .

Therefor $f^{-1}(\mu')$ is an M-N- fuzzy ideal. ■

Proposition(2.13): Let R and R' both be M-N- ring and f an M-N- ring homomorphism from R onto R' , if μ is an M-N- fuzzy ideal of R then $f(\mu)$ is an M-N- ideal of R' .

Proof : By Proposition(1.5) $f(\mu)$ is a fuzzy ideal of R' also, by Proposition(2.8) $f(\mu)$ is M-N- fuzzy ring of R' hence $f(\mu)$ is an M-N- fuzzy ideal of R' . ■

Let R be an M-N- ring and μ be an M-N- fuzzy ideal of R . Ren [5] had proved that R / μ was ring.

Proposition(2.14): Let R be an M-N- ring and μ be an M-N- fuzzy ideal of R then R / μ is an M-N- fuzzy ring which is called the M-N- fuzzy quotient ring of R with respect to μ .

Proof: For every $m \in M$, $n \in N$ and $x \in R$, $m \in M$ and $n \in N$ $x + \mu$, $y + \mu \in R / \mu$

we define $m(x + \mu) n = mxn + \mu$

now we can prove

$$\begin{aligned}
 m[(x + \mu) + (y + \mu)] &= m[(x + y) + \mu] n \\
 &= m(x + y) n + \mu \\
 &= (mxn + myn) + \mu \\
 &= (mxn + \mu) + (myn + \mu) \\
 &= m(x + \mu)n + m(y + \mu)n
 \end{aligned}$$

Also

$$\begin{aligned}
 m[(x + \mu) \cdot (y + \mu)] &= m[(x y) + \mu] n \\
 &= m(x y) n + \mu \\
 &= (mx) (yn) + \mu \\
 &= (mx + \mu) (yn + \mu) \\
 &= m(x + \mu) (y + \mu)n
 \end{aligned}$$

Hence R / μ is an M-N- ring .

Now , we define a fuzzy set on R / μ

let σ be an M-N- fuzzy ring of R then σ / μ is a fuzzy set of R / μ defined as follows

$\sigma / \mu : R / \mu \rightarrow [0,1]$ satisfying

$$\sigma / \mu (a + \mu) = \sup_{x+\mu=a+\mu} \sigma (x)$$

let $V = \sigma / \mu$, $c = a + \mu$ and $d = b + \mu$ for every $a, b \in R$

$$1- V(c- d) = V((a + \mu) - (b + \mu))$$

$$= V((a-b) \mu) = \sup_{x+\mu=(a-b)+\mu} \sigma (x)$$

$$= \sup_{y+\mu=a+\mu, z+\mu=b+\mu} \sigma (y - z)$$

$$\geq \sup_{y+\mu=a+\mu, z+\mu=b+\mu} \min\{ \sigma (y) , \sigma(z) \}$$

$$= \min \{ \sup_{y+\mu=a+\mu} \sigma(y), \sup_{z+\mu=a+\mu} \sigma(z) \}$$

$$= \min \{ V(c), V(d) \}$$

$$\sigma / \mu ((a + \mu) - (b + \mu)) \geq \min \{ \sigma / \mu (a + \mu), \sigma / \mu (b + \mu) \}$$

$$2- V(cd) = V((a + \mu)(b + \mu)) = V(ab + \mu)$$

$$= \sup_{x+\mu=(ab+\mu)} \sigma(x)$$

$$= \sup_{y+\mu=a+\mu, z+\mu=b+\mu} \sigma(yz)$$

$$\geq \sup_{y+\mu=a+\mu, z+\mu=b+\mu} \min \{ \sigma(y), \sigma(z) \}$$

$$= \min \{ \sup_{y+\mu=a+\mu} \sigma(y), \sup_{z+\mu=b+\mu} \sigma(z) \}$$

$$= \min \{ V(c), V(d) \}$$

$$\sigma / \mu((a + \mu)(b + \mu)) = \sigma / \mu(a + \mu) \sigma / \mu(b + \mu)$$

$$3- \sigma / \mu(m(a + \mu)) = \sigma / \mu (ma + \mu)$$

$$= \sup_{x+\mu=ma+\mu} \sigma(x)$$

$$\geq \sup_{mc+\mu=ma+\mu} \sigma(mc)$$

$$\geq \sup_{c+\mu=a+\mu} \sigma(mc)$$

$$\geq \sup_{c+\mu=a+\mu} \sigma(c)$$

$$= \sigma / \mu(a + \mu)$$

$$\text{Similarly } \sigma / \mu((a + \mu)n) \geq \sigma / \mu(a + \mu)$$

Hence σ / μ M-N- fuzzy ring of R / μ . ■

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