On a New Class of Uniformly Convex Univalent Functions with Negative Coefficient Defined by a Linear Operator

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Abstract: ***The main purpose of this paper is to introduce new class of uniformly convex functions with negative coefficients defined by a linear operator in theopen unit disc U. We obtain some geometric properties, likes coefficient estimates, closure theorems, extreme points, distortion theorems, convolution and radii of starlikeness and convexity.***

Keywords: Univalent function, uniformly convex, linear operator, hadamard product .

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1. Introduction

Let A denote the class of functions of the form :

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ (a_k \geq 0, k \epsilon N) \quad (1)$$

which are analytic and univalent in the open unit disk U={$z \epsilon C$:|z|< 1} .

Definition (1)[3,4]: A function f ϵ A is said to be in the classS_p(uniformly α-starlike functions) if it satisfies the condition:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta,(z\epsilon U,\ \alpha \geq 0)\ (2)$$

Definition (2)[5,6]: A function f ϵ A is said to be in the class UCV(uniformly α-convex functions) if it satisfies the condition **:**

$$Re\left\{\frac{zf''(z)}{f'(z)}+1\right\} > \alpha\left|\frac{zf''(z)}{f'(z)} - 1\right| + \beta,(z\epsilon U,\ \alpha \geq 0)\ (3)$$

For a function f ϵ A given by(1) and g ϵ A defined by

$$g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad ,$$

we define the Hadamard product of f and g by

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k,\quad z\epsilon U. \quad (4)$$

Lemma(1)[2]: Let w= u+iv .Then Re w ≥ α if and only if $|w-(1+\alpha)| \leq |w+(1-\alpha)|$,where α ϵ R .

Lemma(2)[2]: Let w= u+iv and α, β are real numbers.Then Re w ≥ $\alpha|w-1|$+β if and only if Re$\{w(1+\alpha e^{i\theta}) - \alpha e^{i\theta}\}$>β .

Definition(3): For 0 ≤ β < 1, 0 ≤ γ ≤ 1, $\delta \geq 0$, l ≥ 0 , a function f ϵ A is said to be in the class α-UCV(β,γ,a,b) if it satisfies the inequality

$$Re\left\{\frac{(1-\gamma)\left[z\left(D_l^{m,\delta}(a,b)f(z)\right)'' + \left(D_l^{m,\delta}(a,b)f(z)\right)'\right] + \gamma\left[z\left(D_l^{m+1,\delta}(a,b)f(z)\right)'' + \left(D_l^{m+1\delta}(a,b)f(z)\right)'\right]}{(1-\gamma)\left(D_l^{m,\delta}(a,b)f(z)\right)' + \gamma\left(D_l^{m+1,\delta}(a,b)f(z)\right)'}\right\}$$
$$> \alpha\left|\frac{(1-\gamma)\left[z\left(D_l^{m,\delta}(a,b)f(z)\right)'' + \left(D_l^{m,\delta}(a,b)f(z)\right)'\right] + \gamma\left[z\left(D_l^{m+1,\delta}(a,b)f(z)\right)'' + \left(D_l^{m+1,\delta}(a,b)f(z)\right)'\right]}{(1-\gamma)\left(D_l^{m,\delta}(a,b)f(z)\right)' + \gamma\left(D_l^{m+1,\delta}(a,b)f(z)\right)'} - 1\right|$$
$$+\beta, \quad (5)$$

where $D_l^{m,\delta}(a,b)f(z) = Q_l^{m,\delta}(a,b;z) * f(z) = z + \sum_{k=2}^{\infty}\left[\frac{1+\delta(k-1)+l}{1+l}\right]^m \frac{(a)_{k-1}}{(b)_{k-1}} a_k z^k$, is a linear operator defined and introduced by (cf.[1])$D_l^{m,\delta}(a,b): A(n) \to A(n)$and

$Q_l^{m,\delta}(a,b;z) = \sum_{k=2}^{\infty}\left[\frac{1+\delta(k-1)+l}{1+l}\right]^m \frac{(a)_{k-1}}{(b)_{k-1}} z^k$, a,b are positive real numbers, m ϵ Z and $(x)_k$ is the Pochhammer Symbol defined by $(x)_k = \begin{cases} (x+1)\ldots(x+k-1) & ,k\epsilon N \\ 1 & ,k=0 \end{cases}$.

2. Coefficient Estimates

In the following theorem, we obtain the sufficient and necessary condition to be the function f in the class α-UCV(β,γ,a,b) .

Theorem(2.1): Let the function $f(z)$ be defined by (1).Then $f(z)$ is in the class α-UCV(β,γ,a,b) if and only if

$$\sum_{k=2}^{\infty} U_m(k,\delta,\mathrm{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]a_k \le (1+\mathrm{l})(1-\beta)\ , \qquad (6)$$

where $U_m(k,\delta,\mathrm{l},a,b)=\left[\frac{1+\delta(k-1)+l}{1+l}\right]^m\ [1+\delta\gamma(k-1)+$ *l(a)k−1(b)k−1* .

The result (6) is sharp .

Proof: By Definition (3) and let $|z|=1$,we have

$$Re\left\{\frac{(1-\gamma)\left[z\left(D_l^{m,\delta}(a,b)f(z)\right)''+\left(D_l^{m,\delta}(a,b)f(z)\right)'\right]+\gamma\left[z\left(D_l^{m+1,\delta}(a,b)f(z)\right)''+\left(D_l^{m+1,\delta}(a,b)f(z)\right)'\right]}{(1-\gamma)\left(D_l^{m,\delta}(a,b)f(z)\right)'+\gamma\left(D_l^{m+1,\delta}(a,b)f(z)\right)'}\right\}>$$

$$\alpha\left|\frac{(1-\gamma)\left[z\left(D_l^{m,\delta}(a,b)f(z)\right)''+\left(D_l^{m,\delta}(a,b)f(z)\right)'\right]+\gamma\left[z\left(D_l^{m+1,\delta}(a,b)f(z)\right)''+\left(D_l^{m+1,\delta}(a,b)f(z)\right)'\right]}{(1-\gamma)\left(D_l^{m,\delta}(a,b)f(z)\right)'+\gamma\left(D_l^{m+1,\delta}(a,b)f(z)\right)'}-1\right|+\beta\ .$$

ByLemma(2), we have

$$Re\left\{\left[\frac{(1-\gamma)\left[z\left(D_l^{m,\delta}(a,b)f(z)\right)''+\left(D_l^{m,\delta}(a,b)f(z)\right)'\right]+\gamma\left[z\left(D_l^{m+1,\delta}(a,b)f(z)\right)''+\left(D_l^{m+1,\delta}(a,b)f(z)\right)'\right]}{(1-\gamma)\left(D_l^{m,\delta}(a,b)f(z)\right)'+\gamma\left(D_l^{m+1,\delta}(a,b)f(z)\right)'}\right](1+\alpha e^{i\theta})-\alpha e^{i\theta}\right\}\ge\beta$$

Hence

$$Re\left\{\frac{\left[(1-\gamma)\left[z\left(D_l^{m,\delta}(a,b)f(z)\right)''+\left(D_l^{m,\delta}(a,b)f(z)\right)'\right]+\gamma\left[z\left(D_l^{m+1,\delta}(a,b)f(z)\right)''+\left(D_l^{m+1,\delta}(a,b)f(z)\right)'\right]\right](1+\alpha e^{i\theta})}{(1-\gamma)\left(D_l^{m,\delta}(a,b)f(z)\right)'+\gamma\left(D_l^{m+1,\delta}(a,b)f(z)\right)'}-\frac{\alpha e^{i\theta}\left[(1-\gamma)\left(D_l^{m,\delta}(a,b)f(z)\right)'+\gamma\left(D_l^{m+1,\delta}(a,b)f(z)\right)'\right]}{(1-\gamma)\left(D_l^{m,\delta}(a,b)f(z)\right)'+\gamma\left(D_l^{m+1,\delta}(a,b)f(z)\right)'}\right\}\ge\beta(7)$$

Let

$$\boldsymbol{A(z)}=\left[(1-\gamma)\left[z\left(D_l^{m,\delta}(a,b)f(z)\right)''+\left(D_l^{m,\delta}(a,b)f(z)\right)'\right]+\gamma\left[z\left(D_l^{m+1,\delta}(a,b)f(z)\right)''+\left(D_l^{m+1,\delta}(a,b)f(z)\right)'\right]\right](1+\alpha e^{i\theta})-\alpha e^{i\theta}\left[(1-\gamma)\left(D_l^{m,\delta}(a,b)f(z)\right)'+\gamma\left(D_l^{m+1,\delta}(a,b)f(z)\right)'\right].$$

By simplify A(z) becomes

$$\boldsymbol{A(z)}=\left[(1-\gamma)z\left[\left(D_l^{m,\delta}(a,b)f(z)\right)''\right]+\gamma z\left[\left(D_l^{m+1,\delta}(a,b)f(z)\right)''\right]\right](1+\alpha e^{i\theta})+(1-\gamma)\left(D_l^{m,\delta}(a,b)f(z)\right)'+\gamma\left(D_l^{m+1,\delta}(a,b)f(z)\right)'$$

$$\boldsymbol{B(z)}=(1-\gamma)\left(D_l^{m,\delta}(a,b)f(z)\right)'+\gamma z\left(D_l^{m+1,\delta}(a,b)f(z)\right)'.$$

ByLemma(1), (7)is equivalent to

$$|\boldsymbol{A(z)}+(\boldsymbol{1}-\boldsymbol{\beta})\boldsymbol{B(z)}|\ge|\boldsymbol{A(z)}-(\boldsymbol{1}+\boldsymbol{\beta})\boldsymbol{B(z)}|\quad ,for\ 0\le\boldsymbol{\beta}\le 1\ .$$

$|\boldsymbol{A(z)}+(\boldsymbol{1}-\boldsymbol{\beta})\boldsymbol{B(z)}|=\Big|\Big[-\sum_{k=2}^{\infty}U_m(k,\delta,\mathrm{l},a,b)\frac{1}{(1+l)}k(k-$ *1akzk−11+αeiθ*

$$+1-\sum_{k=2}^{\infty}U_m(k,\delta,\mathrm{l},a,b)\frac{1}{(1+l)}ka_kz^{k-1}+(1-\beta)\left[1-\sum_{k=2}^{\infty}U_m(k,\delta,\mathrm{l},a,b)\frac{1}{(1+l)}ka_kz^{k-1}\right]$$

$=\Big|-\sum_{k=2}^{\infty}U_m(k,\delta,\mathrm{l},a,b)\frac{1}{(1+l)}k(k-1)a_kz^{k-1}-\alpha e^{i\theta}\sum_{k=2}^{\infty}U_m(k,\delta,\mathrm{l},a,b)\frac{1}{(1+l)}k(k-\quad 1)a_kz^{k-1}+1-\sum_{k=2}^{\infty}U_m(k,\delta,\mathrm{l},a,b)\frac{1}{(1+l)}ka_kz^{k-1}+(1-\beta)-(1-\beta)\sum_{k=2}^{\infty}U_m(k,\delta,\mathrm{l},a,b)\frac{1}{(1+l)}ka_kz^{k-1}\ \Big|=\Big|(2-\beta)-\sum_{k=2}^{\infty}U_m(k,\delta,\mathrm{l},a,b)\frac{1}{(1+l)}k[k+1-\beta]a_kz^{k-1}-\alpha e^{i\theta}\sum_{k=2}^{\infty}U_m(k,\delta,\mathrm{l},a,b)\frac{1}{(1+l)}k(k-1)a_kz^{k-1}\Big|$

$\ge(2-\beta)-\sum_{k=2}^{\infty}U_m(k,\delta,\mathrm{l},a,b)\frac{1}{(1+l)}k[k+1-$ *βakzk−1− αk=2∞Umk,δ,l,a,b11+lkk−1akzk−1*

and so

$|\boldsymbol{A(z)}-(\boldsymbol{1}+\boldsymbol{\beta})\boldsymbol{B(z)}|=\Big|\Big[-\sum_{k=2}^{\infty}U_m(k,\delta,\mathrm{l},a,b)\frac{1}{(1+l)}k(k-$ *1akzk−11+αeiθ*

$$+1-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}ka_kz^{k-1} -(1+\beta)\left[1-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}ka_kz^{k-1}\right]$$

$=|-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}k(k-1)a_kz^{k-1}-\alpha e^{i\theta}\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}k(k-1)a_kz^{k-1}+1-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}ka_kz^{k-1}-(1+\beta)+(1+\beta)\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}ka_kz^{k-1}|$

$=\left|-\beta-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}k[k-(1+\right.$ β)akzk−1+ αeiθk=2∞Umk,δ,l,a,b11+lkk−1akzk−1

$\leq\beta+\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}k[k-(1+$ β)akzk−1+ αk=2∞Umk,δ,l,a,b11+lkk−1akzk−1.

$$|\boldsymbol{A}(\boldsymbol{z})+(\boldsymbol{1}-\boldsymbol{\beta})\boldsymbol{B}(\boldsymbol{z})|-|\boldsymbol{A}(\boldsymbol{z})-(\boldsymbol{1}+\boldsymbol{\beta})\boldsymbol{B}(\boldsymbol{z})|\geq$$

$\geq\left[(2-\beta)-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}k[k+1-\right.$ βakzk−1− αk=2∞Umk,δ,l,a,b11+lkk−1akzk−1−β+k=2∞Umk,δ,l,a,b11+lkk−(1+β)akzk−1+αk=2∞Umk,δ,l,a,b11+lkk−1akzk−1

$=2(1-\beta)-2\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}k[k-\beta]a_k-2\alpha\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}k(k-1)a_k$

$=2(1-\beta)-2\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}k[k(1+\alpha)-$ (β+α)]ak≥0

This is equivalent to

$$\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]a_k \leq (1-\beta)(1+l)$$

Conversely suppose that (6) holds, then we must show

Therefore ,

$$Re\left\{\frac{\left[(1-\gamma)\left[z\left(D_l^{m,\delta}(a,b)f(z)\right)''+\left(D_l^{m,\delta}(a,b)f(z)\right)'\right]+\gamma\left[z\left(D_l^{m+1,\delta}(a,b)f(z)\right)''+\left(D_l^{m+1,\delta}(a,b)f(z)\right)'\right]\right](1+\alpha e^{i\theta})}{(1-\gamma)\left(D_l^{m,\delta}(a,b)f(z)\right)'+\gamma\left(D_l^{m+1,\delta}(a,b)f(z)\right)'} -\frac{\alpha e^{i\theta}\left[(1-\gamma)\left(D_l^{m,\delta}(a,b)f(z)\right)'+\gamma\left(D_l^{m+1,\delta}(a,b)f(z)\right)'\right]}{(1-\gamma)\left(D_l^{m,\delta}(a,b)f(z)\right)'+\gamma\left(D_l^{m+1,\delta}(a,b)f(z)\right)'}\right\}\geq\beta$$

By simplify and choosing the values of z on the positive real axis,where $0\leq z=r<1$, the above inequality reduces to

$$Re\left\{\frac{\left[-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}k(k-1)a_kr^{k-1}+r-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}ka_kr^k\right](1+\alpha e^{i\theta})}{r-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}ka_kr^k} -\frac{\left[r-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}ka_kr^k\right](\beta+\alpha e^{i\theta})}{r-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}ka_kr^k}\right\}\geq 0\ .$$

Since $Re(-e^{i\theta})\geq-|e^{i\theta}|=-1$,, the inequality is correct for all $z\epsilon U$, letting $r\to 1^-$yields .

$$Re\left\{\frac{-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}k(k-1)a_k+1-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}ka_k}{1-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}ka_k} -\frac{\alpha\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}k(k-1)a_k+\beta-\beta\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}ka_k}{1-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}ka_k}\right\}\geq 0$$

and so by the mean value theorem, we have

$$(1-\beta\ )-\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}k[k(1+\alpha)-(\beta+\alpha)]a_k\geq 0\ .$$

So we have

$$\sum_{k=2}^{\infty}U_m(k,\delta,l,a,b)\frac{1}{(1+l)}k[k(1+\alpha)-(\beta+\alpha)]a_k \leq 1-\beta\ .$$

Finally, the result is sharp for the function

$$f(z)=z-\frac{(1+l)(1-\beta)}{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}z^k\ ,\quad k\geq 2\,.$$

Corollary(2.1): Let the function$f(z)$ is in the class α-UCV(β,γ,a,b).Then

$$a_k \le \frac{(1+l)(1-\beta)}{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]} \quad , \quad k \ge 2 \ .$$

3. Closure Theorems

In the next theorems, we will prove the closure property for the class α-UCV(β,γ,a,b).

Theorem(3.1): Let the function $f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k$, $(j = 1,2,\dots,s)$ be in the class α-UCV(β,γ,a,b). Then $h(z) = z - \sum_{k=2}^{\infty} b_k z^k$, $(a_{k,j} > 0)$ belong to the class α-UCV(β,γ,a,b) , where $b_k = \frac{1}{s}\sum_{j=1}^{s} a_{k,j}$.

Proof: Since $f_j(z) \epsilon\ \alpha - \mathrm{UCV}(\beta,\gamma,a,b), (j = 1,2,\dots,s)$, then

$$\sum_{k=2}^{\infty} U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]a_{k,j} < (1+l)(1-\beta)$$

Therefore

$$\sum_{k=2}^{\infty} U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]b_k$$

$$= \sum_{k=2}^{\infty} U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]\left(\frac{1}{s}\sum_{j=1}^{s} a_{k,j}\right)$$

$$= \frac{1}{s}\sum_{j=1}^{s}\left[\sum_{k=2}^{\infty} U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]\right]a_{k,j} \le (1+l)(1-\beta)$$

Theorem(3.2): Let $f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k$, $(j = 1,2,\dots,s)$ and $0<c_j<1$ such that $\sum_{j=1}^{s} c_j = 1$. Then $F(z)$ defined by $F(z) = \sum_{j=1}^{s} c_j f_j(z)$ is also in the class α-UCV(β,γ,a,b) .

Proof: Since $f_j \epsilon$ α-UCV(β,γ,a,b) for every $j \epsilon \{1,2,\dots,s\}$, then

$$\sum_{k=2}^{\infty} \frac{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]a_{k,j}}{(1+l)(1-\beta)} \le 1 \ .$$

Since

$$F(z) = \sum_{j=1}^{s} c_j f_j(z) = \sum_{j=1}^{s} c_j\left[z - \sum_{k=2}^{\infty} a_{k,j} z^k \right] = z - \sum_{k=2}^{\infty}\left(\sum_{j=1}^{s} c_j a_{k,j}\right) z^k \ .$$

Then

$$\sum_{k=2}^{\infty} \frac{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]a_{k,j}}{(1+l)(1-\beta)}\left[\sum_{j=1}^{s} c_j a_{k,j}\right]$$

$$= \sum_{j=1}^{s} c_j \left[\frac{\sum_{k=2}^{\infty} U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(1+l)(1-\beta)} a_{k,j}\right] \le \sum_{j=1}^{s} c_j = 1 \ .$$

Hence $F(z) \epsilon$ α-UCV(β,γ,a,b) .□

4. Extreme Points

In the next theorem , we obtain the extreme points of the class α-UCV(β,γ,a,b) .

Theorem(4.1): Let $f_1(z) = z$ and

$$f_k(z) = z - \sum_{k=2}^{\infty} \frac{(1+l)(1-\beta)}{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]} z^k \ , \quad k \ge 2 \ .$$

Then $f \epsilon\ \alpha - \mathrm{UCV}(\beta,\gamma,a,b)$ if and only if it can be expressed as follows :

$$f(z) = \sum_{k=1}^{\infty} \sigma_k f_k(z) \quad , \quad where\ \sigma_k \ge 0 \quad and \quad \sum_{k=1}^{\infty} \sigma_k = 1 \ .$$

Proof: Suppose that $f(z)$ is expressed in the form :

$$f(z) = \sum_{k=1}^{\infty} \sigma_k f_k(z)$$

$$= \sigma_1 z + \sum_{k=2}^{\infty} \sigma_k \left[z - \frac{(1+l)(1-\beta)}{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]} z^k \right]$$

$$= z\left(\sigma_1 + \sum_{k=2}^{\infty} \sigma_k\right) - \sum_{k=2}^{\infty} \frac{(1+l)(1-\beta)}{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}\sigma_k z^k$$

$$= z - \sum_{k=2}^{\infty} d_k z^k \quad , where\ d_k = \frac{(1+l)(1-\beta)}{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}\sigma_k \ .$$

Hence $f \epsilon\ \alpha - \mathrm{UCV}(\beta,\gamma,a,b)$, since

$$\sum_{k=1}^{\infty} \frac{d_k\ U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(1+l)(1-\beta)} = \sum_{k=1}^{\infty} \sigma_k = 1 - \sigma_1 < 1 \qquad (8)$$

Conversely , suppose that $f \epsilon\ \alpha - \mathrm{UCV}(\beta,\gamma,a,b)$, then From(6), we have

$$\sigma_k = \frac{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(1+l)(1-\beta)} a_k \quad , k \le 2 \ and\ 1 - \sum_{k=2}^{\infty} \sigma_k = \sigma_1 \ .$$

Then

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k$$

$$= z - \sum_{k=2}^{\infty} \frac{(1+l)(1-\beta)}{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}\sigma_k z^k$$

$$= z - \sum_{k=2}^{\infty} \sigma_k\left(z - f_k(z)\right)$$

$$= z\left(1 - \sum_{k=2}^{\infty} \sigma_k\right) + \sum_{k=2}^{\infty} \sigma_k f_k(z)$$

$$= \sigma_1 z + \sum_{k=2}^{\infty} \sigma_k f_k(z) = \sum_{k=1}^{\infty} \sigma_k f_k(z) \quad .\square$$

5. Weighted Mean

Definition (5.1): Let $and\ g \in \alpha - UCV(\beta,\gamma,a,b)$, where

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad and \quad g(z) = z - \sum_{k=2}^{\infty} b_k z^k \ .$$

Then the weighted mean $E_i(z)$ of f and g is given by

$$E_i(z) = \frac{1}{2}[(1-i)f(z) + (1+i)g(z)], \qquad 0 < i < 1 \ .$$

In the theorem below ,we will show the weighted mean for this class:

Theorem(5.2): If f and $g$be in the class $\alpha - UCV(\beta,\gamma,a,b)$, then the weighted mean of f and g is also in the class$\alpha - UCV(\beta,\gamma,a,b)$.

Proof: By Definition (5.1), we have

$$E_i(z) = \frac{1}{2}[(1-i)f(z) + (1+i)g(z)]$$

$$E_i(z) = \frac{1}{2}\left[(1-i)\left(z - \sum_{k=2}^{\infty} a_k z^k\right) + (1+i)\left(z - \sum_{k=2}^{\infty} b_k z^k\right)\right]$$

$$= \frac{1}{2}\left[z - \sum_{k=2}^{\infty}(1-i)a_k z^k + z - \sum_{k=2}^{\infty}(1+i)b_k z^k\right]$$

$$= z - \sum_{k=2}^{\infty}\frac{1}{2}[(1-i)a_k + (1+i)b_k]z^k$$

Since f and g are in the class $\alpha - UCV(\beta,\gamma,a,b)$ so by Theorem(2.1), we get

$$\sum_{k=2}^{\infty}\frac{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(1+l)(1-\beta)}a_k \le \mathbf{1} \ ,$$

and

$$\sum_{k=2}^{\infty}\frac{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(1+l)(1-\beta)}b_k \le \mathbf{1} \ .$$

Then

$$\sum_{k=2}^{\infty}\frac{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(1+l)(1-\beta)}\left(\frac{1}{2}[(1-i)a_k + (1+i)b_k]\right)$$

$$= \frac{1}{2}\sum_{k=2}^{\infty}\frac{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(1+l)(1-\beta)}(1-i)a_k + \frac{1}{2}\sum_{k=2}^{\infty}\frac{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(1+l)(1-\beta)}(1+i)b_k$$

$$\le \frac{1}{2}(1-i) + \frac{1}{2}(1+i) = 1 \quad .\square$$

6. Radii of starlikeness and convexity

In the next theorems , we obtain the radii of starlikeness and convexity for the class α-UCV(β,γ,a,b) .

Theorem(7.1): Let the function $f(z)$ defined by (1) be in the class α-UCV(β,γ,a,b). Then $f(z)$ is starlikeness of order ρ $(0 \le \rho < 1)$ in disk $|z| < r_1(\beta,\gamma,a,b,\rho)$, where

$$r_1(\beta,\gamma,a,b,\rho) = \inf_{k\ge2}\left\{\frac{(1-\rho)U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(k-\rho)(1+l)(1-\beta)}\right\}^{\frac{1}{k-1}} .$$

The result is sharp for the function

$$f(z) = z - \frac{(1+l)(1-\beta)}{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}z^k \quad (9)$$

Proof: We must show that $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \rho$ for $|z| < r_1(\beta,\gamma,a,b,\rho)$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1 - \sum_{k=2}^{\infty}a_k|z|^{k-1}} \qquad (10)$$

(10) is bounded above by $1-\rho$ if

$$\sum_{k=2}^{\infty}\frac{(k-\rho)}{(1-\rho)}a_k|z|^{k-1} \le 1 \ . \qquad (11)$$

Also from Theorem (2.1), if $f \in \alpha - UCV(\beta,\gamma,a,b)$, then

$$\sum_{k=2}^{\infty}\frac{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(1+l)(1-\beta)}a_k \le \mathbf{1} \qquad (12)$$

In view of (12), we notice that (11) holds true if

$$\frac{(k-\rho)}{(1-\rho)}|z|^{k-1} \le \frac{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(1+l)(1-\beta)} .$$

That is , if

$$|z| \le \left\{\frac{(1-\rho)U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(k-\rho)(1+l)(1-\beta)}\right\}^{\frac{1}{k-1}} .$$

Setting$|z| = r_1$, we get the desired result .□

Theorem(7.2): Let the function $f(z)$ defined by (1) be in the class α-UCV(β,γ,a,b). Then $f(z)$ is convex of order ρ $(0 \le \rho < 1)$ in disk $|z| < r_2(\beta,\gamma,a,b,\rho)$, where

$$r_2(\beta,\gamma,a,b,\rho) = \inf_{k\ge2}\left\{\frac{(1-\rho)U_m(k,\delta,l,a,b)[k(1+\alpha)-(\beta+\alpha)]}{(k-\rho)(1+l)(1-\beta)}\right\}^{\frac{1}{k-1}} .$$

The result is sharp for the function given by (9) .

Proof: It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 - \rho \ , \quad |z| < r_2(\beta,\gamma,a,b,\rho)$$

we have

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{k=2}^{\infty}k(k-1)a_k z^{k-1}}{1 - \sum_{k=2}^{\infty}k a_k z^{k-1}}\right| \le \frac{\sum_{k=2}^{\infty}k(k-1)a_k|z|^{k-1}}{1 - \sum_{k=2}^{\infty}k a_k|z|^{k-1}} \quad (13)$$

(13) is bounded above by $1-\rho$ if

$$\sum_{k=2}^{\infty}\frac{k(k-\rho)}{(1-\rho)}a_k|z|^{k-1} \le 1 \ . \qquad (14)$$

Also from Theorem (2.1), if $f \in \alpha - UCV(\beta,\gamma,a,b)$, then we have (12)

In view of (12), we notice that (14) holds true if

$$\frac{k(k-\rho)}{(1-\rho)}|z|^{k-1} \le \frac{U_m(k,\delta,l,a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(1+l)(1-\beta)} .$$

That is , if

$$|z| \leq \left\{\frac{(1-\rho)U_m(k,\delta,\mathfrak{l},a,b)[k(1+\alpha)-(\beta+\alpha)]}{(k-\rho)(1+l)(1-\beta)}\right\}^{\frac{1}{k-1}} .$$

Setting $|z| = r_2$, we get the desired result . □

7. Distortion Theorems

In the next theorems , we obtain the growth and distortion bounds for the function $f \in$ α-UCV(β,γ,a,b) .

Theorem(8.1): Let the function $f(z)$ defined by (1) be in the class α-UCV(β,γ,a,b).Then

$$|z| - \frac{(1+l)(1-\beta)}{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]}|z|^2 \leq |f(z)| \leq |z| + \frac{(1+l)(1-\beta)}{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]}|z|^2 \quad , \quad |z|< 1$$

The result is sharp for the function

$$f(z) = z - \frac{(1+l)(1-\beta)}{U_m(2,\delta,\mathfrak{l},a,b)2[2(1+\alpha)-(\beta+\alpha)]}z^2 \quad (15)$$

Proof: Since $f(z) \in$ α-UCV(β,γ,a,b), then

$$\sum_{k=2}^{\infty} U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]a_k \leq (1+\mathfrak{l})(1-\beta)$$

$$U_m(2,\delta,\mathfrak{l},a,b)2[2(1+\alpha)-(\beta+\alpha)]\sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]a_k \leq (1+\mathfrak{l})(1-\beta).$$

Then

$$\sum_{k=2}^{\infty} a_k \leq \frac{(1+l)(1-\beta)}{2[2(1+\alpha)-(\beta+\alpha)]U_m(2,\delta,\mathfrak{l},a,b)} .$$

Now

$$|f(z)| \geq |z| - |z|^2 \sum_{k=2}^{\infty} a_k \geq |z| - |z|^2 \frac{(1+l)(1-\beta)}{2[2(1+\alpha)-(\beta+\alpha)]U_m(2,\delta,\mathfrak{l},a,b)} ,$$

and

$$|f(z)| \leq |z| + |z|^2 \sum_{k=2}^{\infty} a_k \leq |z| + |z|^2 \frac{(1+l)(1-\beta)}{2[2(1+\alpha)-(\beta+\alpha)]U_m(2,\delta,\mathfrak{l},a,b)} .□$$

Theorem (8.2): Let the function $f(z)$ defined by (1) be in the class α-UCV(β,γ,a,b).Then

$$1 - \frac{2(1+l)(1-\beta)}{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]}|z| \leq |f'(z)| \leq 1 + \frac{2(1+l)(1-\beta)}{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]}|z|$$

The result is sharp for the function given by (15) .

Proof: The proof similar to the Theorem(8.1) .

8. Convolution

In the following theorem , we obtain Hadamard product property for the class α-UCV(β,γ,a,b) .

Theorem(9.1): Let $f, g \in$ α-UCV(β,γ,a,b).Then $f * g$ is also in the classα-UCV(τ,γ,a,b) , and

$$\boldsymbol{\tau} \leq \frac{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]^2 - (1+l)(1-\beta)^2[k(1+\alpha)-\alpha]}{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]^2 - (1+l)(1-\beta)^2}$$

Proof: Since $f, g \in$ α-UCV(β,γ,a,b) , we have

$$\sum_{k=2}^{\infty} \frac{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(1+l)(1-\beta)} a_k \leq 1 ,$$

and

$$\sum_{k=2}^{\infty} \frac{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(1+l)(1-\beta)} b_k \leq 1 .$$

We have to find the largest $\boldsymbol{\tau}$ such that

$$\sum_{k=2}^{\infty} \frac{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\tau+\alpha)]}{(1+l)(1-\boldsymbol{\tau})} a_k b_k \leq 1 .$$

By Cauchy-Schwarz inequality , we get

$$\sum_{k=2}^{\infty} \frac{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(1+l)(1-\beta)} \sqrt{a_k b_k} \leq 1 . \quad (16)$$

We want to show that

$$\frac{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\tau+\alpha)]}{(1+l)(1-\boldsymbol{\tau})} a_k b_k \leq \frac{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]}{(1+l)(1-\beta)} \sqrt{a_k b_k} .$$

This equivalently to

$$\sqrt{a_k b_k} \leq \frac{(1-\boldsymbol{\tau})[k(1+\alpha)-(\beta+\alpha)]}{(1-\beta)[k(1+\alpha)-(\tau+\alpha)]} .$$

From (16), we get

$$\sqrt{a_k b_k} \leq \frac{(1+l)(1-\beta)}{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]} .$$

Thus it is enough to show that

$$\frac{(1+l)(1-\beta)}{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]} \leq \frac{(1-\boldsymbol{\tau})[k(1+\alpha)-(\beta+\alpha)]}{(1-\beta)[k(1+\alpha)-(\tau+\alpha)]} ,$$

which is simplified to

$$\boldsymbol{\tau} \leq \frac{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]^2 - (1+l)(1-\beta)^2[k(1+\alpha)-\alpha]}{U_m(k,\delta,\mathfrak{l},a,b)k[k(1+\alpha)-(\beta+\alpha)]^2 - (1+l)(1-\beta)^2}$$

References

[1] A. A. Amer and M. Darus, A Distortion theorem for a certain class of Bazilevicfunction , International Journal of Mathematical Analysis, Vol. 6, No. 12, 2012, pp. 591-597 .
[2] E. S. Aqlan , Some Problems Connected with Geometric Function Theory , Ph. D. thesis (2004), Pune University, Pune .
[3] A. W. Goodman ,On uniformly convex functions , Ann. Polon. Math., 56 (1991), 87-92
[4] A. W. Goodman ,On uniformly starlike functions , J.Math. Anal.& Appl. 155 (1991), 364-370 .
[5] F. Ronning , Uniformly convex functions and a corresponding class of starlike functions , Proc. Amer. Math. Soc. , 118 (1993), 189-196 .
[6] F. Ronning , Integral representation for bounded starlike functions , Annal. Polon.Math. , 60 (1995), 289-297 .