A NEW EXACT SEQUENCE
A generalization of J.H.C.Whiteheads
" Certain exact sequence "

***Dheia G.S.Al-Khafaji and **Raymond N. Shekoury**
* Department of Math., College of Computer Science and Mathematics , AL-Qadisiya University .
** Department of Math., College of Science , Baghdad University .

ABSTRACT

In this work we introduce and study a new notion in algebraic topology , which we call " a new exact sequence " which is a generalization of " a certain exact sequence " of J.H.C.Whitehead .

Consider the following sequence :

$$\overset{\pm_{p+1,q}}{\ulcorner\!\!\longrightarrow\!\!\urcorner}\qquad\overset{\pm_{p,q}}{\ulcorner\!\!\longrightarrow\!\!\urcorner}$$

$$\to ®_{p+q+1}(k^{p+1},k^{p}) \to ®_{p+q}(k^{p}) \xrightarrow{j_{p,q}} ®_{p+q}(k^{p},k^{p-1}) \xrightarrow{\ \ _{p,q}\ \ } ®_{p+q+1}(k^{p-1}) \to ®_{p+q+1}(k^{p-1},k^{p}) \to$$

denote

$j_{p-1,q} \circ \ _{p,q}$	*by*	$\pm_{p,q}$
$\ker \pm_{p,q} / \mathrm{Im} \pm_{p+1,q}$	*by*	$\mathrm{H}_{p,q}$
$\ker j_{p,q}$	*by*	$\Gamma_{p,q}$
$®_{p+q}(k^{p}) / \mathrm{Im} \ \ _{p+1,q}$	*by*	$\Sigma_{p,q}$

The sequence

$$E_q \ : \ \cdots \to \mathrm{H}_{p+1,q} \xrightarrow{¬_{p+1,q}} \Gamma_{p,q} \xrightarrow{\ddot{}_{p,q}} \Sigma_{p,q} \xrightarrow{«_{p,q}} \mathrm{H}_{p,q} \to \cdots$$

Where $¬_{p+1,q}$, $\ddot{}_{p,q}$ *and* $«_{p,q}$ defined in a natural manner , we call " a new exact sequence " which is denoted by NES .

We obtain some results from NES , which are ;

The our NES it is really exact , the class of all NES,s and the homomorphisms between these sequences forms a category E , there is a functor from the category of cw-complexes into E .

INTRODUCTION

This work is inspired by the paper of "a certain exact sequence " of J.H.C.Whitehead,[w1]. We introduce in it a generalization of the main concept of that paper which we call " a new exact sequence " and denote by NES .

This work contains two sections ; in first section , we construct a new exact sequence and we intraduce some a new notions . In second section , we establish some results about our NES , some of these results purely algebraic and other depend on the topology of space .

Section 1"the generalization"

We mean by the term "complex" in the sequel "connected cw-complex".

Let k be a cw-complex .

Denote $\text{®}_{p+q}(k^p, k^{p-1})$ *by* $C_{p,q}$ *and* $\text{®}_{p+q}(k^p)$ *by* $A_{p,q}$

$\{In\ \ case\ \ q < -p,\ \ then\ \ C_{p,q} = A_{p,q} = 0,\ \ by\ \ the\ \ convevtion\ \ a\ \ mong\ \ a\text{lg}braic\ \ topo\text{logists}\ \ to\ \ assume\ \ \text{®}_n(X,A) = 0\ \ and\ \ \text{®}_n(X) = 0\ \ if\ \ n \le 0\}$

Now, consider the following sequence ;

$$T_q: \quad \cdots \to C_{p+1,q} \xrightarrow{\ \ _{p+1,q}\ \ } A_{p,q} \xrightarrow{\ j_{p,q}\ } C_{p,q} \xrightarrow{\ \ _{p,q}\ \ } A_{p-1,q} \to \cdots$$

where $_{p,q}$, $j_{p,q}$ are the homotopy boundary and the realativizing operators , respectively .

Denote
$j_{p-1,q} \circ \ _{p,q}$ *by* $\pm_{p,q}$,
$Z_{p,q}/B_{p,q}$ *by* $\text{H}_{p,q}$,
$Ker.j_{p,q}$ *by* $\Gamma_{p,q}$, *and*
$A_{p,q}/D_{p,q}$ *by* $\Sigma_{p,q}$.

$$Z_{p,q} = Ker.of \pm_{p,q} ,$$
$$B_{p,q} = \text{Im}.of \pm_{p+1,q} ,\text{where}$$
$$D_{p,q} = \text{Im}.of \ \ _{p+1,q} .$$

$\pm_{p,q} \circ \pm_{p+1,q} = 0$ It is easily to prove that

We will define homomorphisms ;

$$\text{¨}_{p,q} : \Gamma_{p,q} \longrightarrow \Sigma_{p,q} ,$$
$$\text{«}_{p,q} : \Sigma_{p,q} \longrightarrow \text{H}_{p,q} , \quad \text{and}$$
$$\neg_{p+1,q} : \text{H}_{p+1,q} \longrightarrow \Gamma_{p,q} .$$

as follows ;

$$
\begin{array}{lcl}
\ddot{}_{p,q}(x) & = & [x] \quad \textit{for each } x \textit{ in } \Gamma_{p,q} \ , \\
\ll_{p,q}([y]) & = & [j_{p,q}(y)] \quad \textit{for each } y \textit{ in } A_{p,q} \ , \qquad \text{and} \\
\neg_{p+1,q}([z]) & = & {}_{p+1,q}(z) \quad \textit{for each } z \textit{ in } Z_{p+1,q} \ .
\end{array}
$$

where [ ] denote the advise equivalence class of the element inside the bracket .

It is easy to check that $\ll_{p,q}$ & $\neg_{p+1,q}$ are well defined .

Therefore we get the following sequence of groups and homomorphisms ;

$$
E_q \ : \ \cdots \to \mathrm{H}_{p+1,q} \xrightarrow{\neg_{p+1,q}} \Gamma_{p,q} \xrightarrow{\ddot{}_{p,q}} \Sigma_{p,q} \xrightarrow{\ll_{p,q}} \mathrm{H}_{p,q} \to \cdots
$$

we call " a new exact sequence " which is denoted by NES .

It is easy to see that by taking q to be 0 we will obtain the exact sequence of J.H.C.Whitehead . Moreover the results which will obtain with generalize many results of Whitehead .

Now , let $\Im$ be the class of all sequences of the form NES and . , $¸(q) = (²_q, ³_q, °_q): \ E_q \longrightarrow \bar{E}_q$ morphisms

we mean a family of homomorphisms $s(q)$ By a morphism

$$
\begin{array}{ccccccccc}
E_q : \ \cdots \to & \mathrm{H}_{p+1,q} & \xrightarrow{\neg_{p+1,q}} & \Gamma_{p,q} & \xrightarrow{\ddot{}_{p,q}} & \Sigma_{p,q} & \xrightarrow{\ll_{p,q}} & \mathrm{H}_{p,q} & \to \cdots \\
\downarrow s(q) & \downarrow ²_{p+1,q} & & \downarrow ³_{p,q} & & \downarrow °_{p,q} & & \downarrow ²_{p,q} & \\
\bar{E}_q : \ \cdots \to & \bar{\mathrm{H}}_{p+1,q} & \xrightarrow{\bar{\neg}_{p+1,q}} & \bar{\Gamma}_{p,q} & \xrightarrow{\bar{\ddot{}}_{p,q}} & \bar{\Sigma}_{p,q} & \xrightarrow{\bar{\ll}_{p,q}} & \bar{\mathrm{H}}_{p,q} & \to \cdots
\end{array}
$$

such that

$$
\begin{array}{lcl}
\bar{\neg}_{p+1,q} \circ ²_{p+1,q} & = & ³_{p,q} \circ \neg_{p+1,q} \ , \\
\bar{\ddot{}}_{p,q} \circ ³_{p,q} & = & °_{p,q} \circ \ddot{}_{p,q} \quad \textit{and} \\
\bar{\ll}_{p,q} \circ °_{p,q} & = & ²_{p,q} \circ \ll_{p,q} \ .
\end{array}
$$

Let $\prod$ be the class of all sequences of the form T_q and morphisms $\int(q) = (h_q, f_q): \ T_q \to \bar{T}_q$, where

$$
T_q : \ \cdots \to C_{p+1,q} \xrightarrow{\ {}_{p+1,q}\ } A_{p,q} \xrightarrow{j_{p,q}} C_{p,q} \to \cdots
$$

and $\int(q)$, we mean a family of homomorphisms ,

$$
\begin{array}{ccccccc}
T_q : \ \cdots \to & C_{p+1,q} & \xrightarrow{\ {}_{p+1,q}\ } & A_{p,q} & \xrightarrow{j_{p,q}} & C_{p,q} & \to \cdots \\
\downarrow \int(q) & \downarrow h_{p+1,q} & & \downarrow f_{p,q} & & \downarrow h_{p,q} & \\
\bar{T}_q : \ \cdots \to & \bar{C}_{p+1,q} & \xrightarrow{\ \bar{}_{p+1,q}\ } & \bar{A}_{p,q} & \xrightarrow{\bar{j}_{p,q}} & \bar{C}_{p,q} & \to \cdots
\end{array}
$$

such that ;

$$
\begin{array}{lcl}
\bar{}_{p+1,q} \circ h_{p+1,q} & = & f_{p,q} \circ {}_{p+1,q} \quad \textit{and} \\
\bar{j}_{p,q} \circ f_{p,q} & = & h_{p,q} \circ j_{p,q} \ .
\end{array}
$$

Notice that , it is easy to show that

$$
\bar{\pm}_{p+1,q} \circ h_{p+1,q} = h_{p,q} \circ \pm_{p+1,q} \ .
$$

Also , we note that

$$
\begin{aligned}
f_{p,q}(\Gamma_{p,q}) &\subset \Gamma^{-}_{p,q} \quad , \\
f_{p,q}(D_{p,q}) &\subset D^{-}_{p,q} \quad and \\
h_{p,q}(B_{p,q}) &\subset B^{-}_{p,q} \quad .
\end{aligned}
$$

Therefor $\int(q)$ *induces morphism* $s(q) = ({}^{2}{}_{q}, {}^{3}{}_{q}, {}^{\circ}{}_{q}) : \ E_q \ \to \ E^{-}_{q}$,
where

$$
\begin{array}{ccccccccccc}
E_q : & \cdots \to & H_{p+1,q} & \xrightarrow{\neg_{p+1,q}} & \Gamma_{p,q} & \xrightarrow{\ddot{}_{p,q}} & \Sigma_{p,q} & \xrightarrow{\ll_{p,q}} & H_{p,q} & \to \cdots \\
\downarrow s(q) & & \downarrow {}^{2}{}_{p+1,q} & & \downarrow {}^{3}{}_{p,q} & & \downarrow {}^{\circ}{}_{p,q} & & \downarrow {}^{2}{}_{p,q} & \\
E^{-}_{q} : & \cdots \to & H^{-}_{p+1,q} & \xrightarrow{\neg^{-}_{p+1,q}} & \Gamma^{-}_{p,q} & \xrightarrow{\ddot{}^{-}_{p,q}} & \Sigma^{-}_{p,q} & \xrightarrow{\ll^{-}_{p,q}} & H^{-}_{p,q} & \to \cdots
\end{array}
$$

according to the rules ,

$$
\begin{aligned}
{}^{2}{}_{p+1,q}([z]) &= [h_{p,q}(z)] & \forall z \in Z_{p+1,q} \quad , \\
{}^{3}{}_{p,q}(x) &= f_{p,q}(x) & \forall x \in \Gamma_{p,q} \quad and \\
{}^{\circ}{}_{p,q}([x]) &= [f_{p,q}(x)] & \forall x \in A_{p,q} \quad .
\end{aligned}
$$

It is easy to show that is a morphism in $s(q)$ $\Im$, which is called the morphism induced by $\int(q)$.

Moreover, we will construct $\int(q)$ from the map "which is cellular",
Let $h: \ (K^{p+1}, K^{p}, e^{0}) \longrightarrow (K_1^{p+1}, K_1^{p}, e_1^{0})$, be a cellular map , and
$f = h|_{K^{p}} : \ (K^{p}, e^{0}) \longrightarrow (K_1^{p}, e_1^{0})$ be a cellular map , which is restriction of h .

Then h *and* f induce a homomorphisms h_q *and* f_q , which is defined as following ;

which is represented by a map $x \in C_{p+1,q}$ Let
$\sim : \ (E^{p+q+1}, S^{p+q}, s_0) \to (K^{p+1}, K^{p}, e^{0})$, we define $h_{p+1,q}(x) = [h \circ \sim]$

Let $y \in A_{p,q}$ Which is represented by a map
$£ : \ (S^{p+q}, s_0) \to (K^{p}, e^{0})$, we define $f_{p,q}(y) = [f \circ £]$

It is easy to show that our maps are well defined ,

Moreover , it is easy to show that $\int(q) = (h_q, f_q)$ is a morphism in Π , which is called the morphism induced by h .

Remarks

(i) *Let* $\int(q) = (h_q, f_q) : \ T_q \to T^{-}_{q}$ *and* $\int^{-}(q) = (h^{-}_{q}, f^{-}_{q}) : \ T^{-}_{q} \to T^{=}_{q}$,
be two morphisms in Π , we define the composition of morphisms as following ; $\int^{-}(q) \circ \int(q) = (h^{-}_{q} \circ h_q, f^{-}_{q} \circ f_q)$.

It is easy to show that is a homomorphism in $\int^{-}(q) \circ \int(q)$ Π .

(ii) *Let* $s(q) : \ E_q \to E^{-}_{q}$ *and* $s^{-}(q) : \ E^{-}_{q} \to E^{=}_{q}$, be two morphisms in $\Im$, we define the composition of morphisms as following ;

$$
s^{-}(q) \circ s(q) = ({}^{2}{}^{-}_{q} \circ {}^{2}{}_{q}, {}^{3}{}^{-}_{q} \circ {}^{3}{}_{q}, {}^{\circ}{}^{-}_{q} \circ {}^{\circ}{}_{q}) \ : \ E_q \to E^{=}_{q} \quad .
$$

$s^{-}(q)\circ s(q)$ it is easy to show that is a homomorphism in $\Im$. *

Now , we will define some a new notion which is needs in this work, By a deformation operator , $¢(q):\ C_q(K)\to C_q(K^{-})$ we mean a family of homomorphisms ; $¢_{p+1,q}:C_{p,q}\to C^{-}_{p+1,q}$.

We call two homomorphisms , $\int(q),\int^{*}(q):\ T_q(K)\to T_q(K^{-})$, are homotopic , and write $\int(q)\approx\int^{*}(q)$, if and only if , there is a deformation operator , $¢(q):\ C_q(K)\to C_q(K^{-})$, as shown in the following diagram ;

$$
\begin{array}{cccccccccccc}
 & \downarrow & & \pm_{p+1,q} & & \downarrow\downarrow & & \pm_{p,q} & & \downarrow & \\
\downarrow\ T_q(K): & \cdots\to C_{p+1,q} & \xrightarrow{p+1,q} & A_{p,q} & \xrightarrow{j_{p,q}} & C_{p,q} & \xrightarrow{p,q} & A_{p-1,q} & \xrightarrow{j_{p-1,q}} & C_{p-1,q} & \to\cdots \\
\int(q)\downarrow\downarrow\int^{*}(q) & \downarrow\downarrow & & f_{p,q}\downarrow\downarrow f^{*}_{p,q} & & h_{p,q}\downarrow\downarrow h^{*}_{p,q} & & \downarrow\downarrow & & \downarrow\downarrow & \\
T_q(K^{-}): & \cdots\to C^{-}_{p+1,q} & \xrightarrow{p+1,q} & A^{-}_{p,q} & \xrightarrow{j^{-}_{p,q}} & C^{-}_{p,q} & \xrightarrow{p,q} & A^{-}_{p-1,q} & \xrightarrow{j^{-}_{p-1,q}} & C^{-}_{p-1,q} & \to\cdots
\end{array}
$$

such that

$$f^{*}_{p,q}-f_{p,q} = {}^{-}_{p+1,q}\circ ¢_{p+1,q}\circ j_{p,q} \qquad and$$

$$h^{*}_{p,q}-h_{p,q} = \pm^{-}_{p+1,q}\circ ¢_{p+1,q}+¢_{p,q}\circ\pm_{p,q} \quad .$$

We shall describe a homomorohism $\int(q):\ T_q\to T^{-}_q$ as an (algebraic) equivalence , iff ,there is a homomorphism $\int^{-}(q):\ T^{-}_q\to T_q$ such that $\int^{-}(q)\circ\int(q)\approx I$ and $\int(q)\circ\int^{-}(q)\approx I$, where I denotes the identical isomorphism both in T_q and in T^{-}_q .

We shall describe T_q as equivalent to T^{-}_q And shall write $T_q\equiv T^{-}_q$, iff , there is an equivalence $\int(q):\ T_q\to T^{-}_q$.

We shall describe $s(q)$ as an isomorphism , iff , $^{2}{}_{p,q}$, $^{3}{}_{p,q}$ *and* $^{\circ}{}_{p,q}$ are an isomorphisms for each p . We shall describe E_q as isomorphic to E^{-}_q ,and shall write $E_q\cong E^{-}_q$, iff , there is an isomorphism $s(q):\ E_q\to E^{-}_q$.

Section 2 "Results and Conclusion"

We obtain some results about our " NES" . Some of these results purely algebraic and others depend on the topology of space .We will write some of these results without proof .

Theorem 1

The new exact sequence E_q is an exact sequence

Proof

We will prove this theorem in three stages;

$(i) \quad \mathrm{Im}.«_{p,q} = Ker.¬_{p,q}$,

$(ii) \quad \mathrm{Im}.¨_{p,q} = Ker.«_{p,q}$,

$(iii) \quad \mathrm{Im}.¬_{p+1,q} = Ker.¨_{p,q}$.

Stage (i) ;

$(i1)$ *Let* $a \in \Sigma_{p,q}$ *and* $a = [x]$, *where* $x \in A_{p,q}$, *then*

$$¬_{p,q}(«_{p,q}(a)) = ¬_{p,q}([j_{p,q}(x)]) \quad (by \ def.)$$
$$= \ _{p,q}(j_{p,q}(x)) \quad (by \ def.)$$
$$= (\ _{p,q} \circ j_{p,q})(x)$$
$$= 0 \quad \bullet$$

$(i2)$ *Let* $b \in \mathrm{H}_{p,q}$ *and* $b = [z]$, *where* $z \in Z_{p,q}$.

Assume that $¬_{p,q}(b) = 0 \Rightarrow \ _{p,q}(z) = 0 \Rightarrow z \in Ker. \ _{p,q} = \mathrm{Im}.j_{p,q}$

so that $z = j_{p,q}(y)$ *for some* $y \in A_{p,q}$,

so we let $«_{p,q}([y]) = [j_{p,q}(y)] = [z] = b \quad \bullet$

Thus from $(i1) \& (i2)$ *we get* (i) $*$

Stage (ii) ;

$(ii1)$ *Let* $a \in \Sigma_{p,q}$ *and* $a = [x]$, *where* $x \in A_{p,q}$.

Assume that $«_{p,q}(a) = 0$ *thus* $[j_{p,q}(x)] = 0$ $(from \ def.)$

but $\mathrm{H}_{p,q} = Z_{p,q}/B_{p,q}$ *then* $j_{p,q}(x) \in \ _{p,q}$

so that $\pm_{p+1,q}(y) = j_{p,q}(x)$ *for some* $y \in C_{p+1,q}$,

thus $j_{p,q}(\ _{p+1,q}(y) = j_{p,q}(x) \Rightarrow x - \ _{p+1,q}(y) \in Ker.j_{p,q} = \Gamma_{p,q}$

so we let $¨_{p,q}(x - \ _{p+1,q}(y)) = [x] = a \quad \bullet$

$(ii2)$ *Let* $x \in \Gamma_{p,q}$, *then*

$$«_{p,q}(¨_{p,q}(x)) = «_{p,q}([x]) \quad (by \ def.)$$
$$= [j_{p,q}(x)] \quad (by \ def.)$$
$$= 0 \quad \bullet$$

Thus from $(ii1)$ & $(ii2)$ *we get* (ii) $*$

Stage (iii)

$(iii1)$ *Let* $b \in \mathrm{H}_{p+1,q}$ *and* $b = [z]$, *where* $z \in Z_{p+1,q}$, *then*

$$¨_{p,q}(¬_{p+1,q}(b)) = ¨_{p,q}(\ _{p+1,q}(z)) \quad (by \ def.)$$
$$= [\ _{p+1,q}(z)] \quad (by \ def.)$$
$$= 0 \ . \quad (since \ in \ \Sigma_{p,q}) \quad \bullet$$

$(iii2)$ *Let* $x \in \Gamma_{p,q}$ *and* $¨_{p,q}(x) = 0$, *then* $[x] = 0 \Rightarrow x \in \mathrm{Im}. \ _{p+1,q}$

so that $\ _{p+1,q}(y) = x$ *for some* $y \in C_{p+1,q}$,

so we let $¬_{p+1,q}([y]) = \ _{p+1,q}(y) = x \quad \bullet$

Thus from $(iii1)$ & $(iii2)$ *we get* (iii) $*$ We are done *

Theorem 2

The class Π mention in section 1, is a category .

Theorem 3

The class $\Im$ mention in section 1, is a category .

Theorem 4

. Π There is a functor from the category of cw-complexs into

Proof

The prove directly from our remark (i) "section 1" and theorem 2 .
Denote this functor by , thus $\int$ $\int$: $CW \to \Pi$. where CW denote the category of cw-complexs .

Theorem 5

There is a functor from the category into the category Π $\Im$.

Proof

The prove directly from our remark (ii) "section 1" and theorem 3 .
Denote this functor by , thus s s: $\Pi \to \Im$.

Theorem 6

There is a functor from the category CW into the category $\Im$.

Proof

The composition of two functors is again a functor ,(see [SW]) .
Hence from (theorem 4) and (theorem 5) , it follows that $s \circ \int$ as a functor .

. E where $E(K) = E_q(K)$ and $E(h) = F(q)$ Denote this fumctor by

Lemma 1

The relation (i) $\equiv$ mention in section 1, is an equivalence relation .
The relation (ii) $\cong$ Mention in section 1, is an equivalence relation .

Lemma 2 If

$\int(h) \approx \int(h^-)$: $T_q(K) \to T_q(K^-)$ are homotopic, and $\int(h^`)$: $T_q(K^-) \to T_q(K^=)$ be a homomorphism , then $\int(h^`) \circ \int(h) \approx \int(h^`) \circ \int(h^-)$: $T_q(K) \to T(K^=)$.

Lemma 3

Let be homotopic $(h, \ h^- \ : \ K \to K^- \ \ h \approx h^-)$, then
$\int(h) \approx \int(h^-)$: $T_q(K) \to T_q(K^-)$.

Lemma 4

Let $\int(q) \approx \int^-(q)$: $T_q(K) \to T_q(K^-)$, then
$F(q) = F^-(q)$: $E_q(K) \to E_q(K^-)$.

Theorem 7

Let be homotopic h, h^- : $K \to K^-$ $h \approx h^-$, then
$F(q) = F^-(q)$: $E_q(K) \to E_q(K^-)$.

Proof

The prove direct from (lemma 3) and (lemma 4) .

Theorem 8

If $K \equiv K^{-}$, *then* $E_q(K) \cong E_q(K^{-})$.

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