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Modules with the SL-closed Intersection Property

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Abstract: *A submodule K of an R-module M is called strongly large in M, in case for any $m \in M$, $s \in R$ with $ms \neq 0$ there exists $r \in R$ such that $mr \in K$ and $mrs \neq 0$. A submodule N of an R-module M is called SL-closed if, N has no proper strongly large extension in M [2]. In this paper, we introduce the concept of modules with the SL-closed intersection property. An R-module M is said to have the SL-closed intersection property (briefly SLCIP) if, the intersection of any two SL-closed submodules of M is again SL-closed. Our aim in this paper is to give some characterizations about this concept. Also, we give some properties of such modules.*

Keywords: *SL-closed submodules, SLCIP modules, SL-UC modules.*

I. Introduction

Throughout this paper, by a ring R we always mean an associative ring with identity and M a unital right R-module. A submodule N of a module M is called an essential submodule in M (briefly $N \leq_e M$) if, for any $X \leq M, N \cap X = 0$ implies X=0. A submodule N is called a closed submodule of M (briefly $N \leq^c M$) if, N has no proper essential extensions inside M ; that is the only solution of the relation $N \leq_e K \leq M$ is $K = N$. In this case the submodule K is called closure of N [3]. Notice that every strongly large submodule of a module always essential, this implies that every closed submodule is SL-closed submodule. In this paper, we define and study the classes of modules with the SL-closed intersection property. This work consists of two sections. In section 2, we give several characterizations of modules having the SLCIP. We prove that, a module M has the SLCIP if and only if every (2-generator) submodules of M has the SLCIP. In section 3, we establish the basic properties of these modules. We prove that for any R-monomorphism $f : M \to N$, then M has the SLCIP if and only if the image of M has the SLCIP.

II. Preliminaries

A submodule K of an R-module M is called a strongly large submodule in M, in case for any $m \in M, s \in R$ with $ms \neq 0$ there exits an $r \in R$ such that $mr \in K$ and $mrs \neq 0$. A submodule N of an R-module M is called SL-closed if, N has no proper strongly large extensions in M [2]. We give some properties of strongly large and SL-closed submodules of an R-module M that will be used in our work.

Lemma 1.1. [2]. Let M be an R-module with a submodule N. Then the following statements are equivalent :
(i) N is a strongly large submodule of M ;
(ii) for each $m \in M$ and $s \in R$ with $ms \neq 0$, we have $m(m^{-1}N)s \neq 0$;
(iii) for each $m \in M$, $m(m^{-1}N)$ is a strongly large submodule of mR ;
(iv) for each $m \in M$, $mR \cap N$ is a strongly large submodule of mR .

Lemma 1.2. [2]. The following statements hold for R-modules M, M' :
(i) If $N \leq L \leq M$. Then N is strongly large in M if and only if N is strongly large in L and L is strongly large in M.
(ii) If N_1 is strongly large in $N \leq M$ and L_1 is strongly large in $L \leq M$, then $N_1 \cap L_1$ is strongly large in $N \cap L$ of M.
(iii) If N_i is strongly large in L_i of M $(i = 1,2,3,...,n)$, then $\bigcap_{i=1}^{n} N_i$ is strongly large in $\bigcap_{i=1}^{n} L_i$ of M.

(iv) Let $f: M \rightarrow M^{'}$ be an R-homomorphism and $N \leq L \leq M$. If N is strongly large in L of M then $f(N)$ is strongly large in $f(L)$ of $f(M)$.

(v) Let $f: M \rightarrow M^{'}$ be an R-monomorphism and $L \leq M^{'}$. If L is strongly large in $M^{'}$, then $f^{-1}(L)$ is strongly large in M.

Example 1.3. Clearly, every strongly large submodule of a module is essential, but the converse is not always true, such example: In Z_4 -module Z_4, $(\bar{2})$ an essential submodule but it is not strongly large, since $m = \bar{1} \in Z_4$, $s = \bar{2} \in Z_4$ such that $\bar{1}.\bar{2} = \bar{2} \neq \bar{0}$ but there is only $r = \bar{2} \in Z_4$ such that $\bar{1}.r = \bar{1}.\bar{2} = \bar{2} \in (\bar{\bar{2}})$ with $mrs = \bar{1}.\bar{2}.\bar{2} = \bar{0}$. Notice that every submodule of Z_4 as Z_4 -module is SL-closed.

Lemma 1.4. [2]. Let M be a prime (or torsion-free) R-module. Then essentiality implies strongly largeness in M.

Let M be an R-module. M is said to satisfy condition (*) in case of, for each $(m \neq 0) \in M$ and $r_1, r_2 \in R$, if $r_i \notin ann_R(m)$ for some $(i = 1,2)$ and $r_1 R r_2 \leq ann_R(m)$, then $r_j = 0$ for some $i \neq j$ [2].

Lemma 1.5. [2]. (i) Every faithful prime R-module satisfies the condition (*).
(ii) Every free module over an integral domain satisfies the condition (*).
(iii) Every ring with no nonzero divisors of zero satisfies the condition (*).

Lemma 1.6. [2]. Let M be an R-module with the condition (*). Then N is an essential submodule of M if and only if N is strongly large in M.

Lemma 1.7. [2]. The following statements hold for an R-module M :
(i) Every closed submodule of M is SL-closed. The converse holds if M has the condition (*).
(ii) Every direct summand of M is SL-closed submodule.
(iii) If N is a SL-closed submodule of M and let L be a strongly large submodule in M, then $N \cap L$ is a SL-closed submodule in L.

Lemma 1.8. [2]. Let M be an R-module, $N \leq M$. Then N is strongly large in M if and only if for all $m \in M$, $N \cap mR$ is strongly large in mR.

III. Characterizations of modules with the SLCIP.

In this section, we investigate the necessary and sufficient conditions under which a module has the SLCIP.

Example 2.1. Consider $M = Z \oplus Z_2$ as Z-module, let $A = (1, \bar{0})Z$ and $B = (1, \bar{1})Z$ are submodules of M. clearly, $A \oplus (0, \bar{1})Z = B \oplus (0, \bar{1})Z = M$, thus A and B are direct summands of M, so A,B are SL-closed in M. But $A \cap B = (2, \bar{0})Z$ is not SL-closed in A, B or M. In fact $A \cap B$ is strongly large in both A and B or M.

This leads us to introduce the following definition.

Definition 2.2. An R-module M is said to have the SL-closed intersection property (briefly SLCIP) if, the intersection of any two SL-closed submodules of M is again SL-closed.

It is clear that the Z-modules Z and Z_2 are modules with the SLCIP. In fact every nonzero submodule of Z, Z_2 as Z-modules is strongly large.

The authors B.Ungor; S.Halicioglu ; M.A. Kamal and A.Harmanci, in [2] introduce the theorem 4.8, " let M be an R-module and $N \leq M$. Then there exists $H \leq M$ such that N is a strongly large submodule in H and H is a SL-closed in M ". A submodule H is called SL-closure of N and it is not necessary unique. We introduce the following definition.

Definition 2.3. An R-module M is called SL-UC module if, every submodule of M has a unique SL-closure.

Theorem 2.4. Let M be an R-module. If M has the SLCIP then M is a SL-UC module.

Proof. Suppose A is a submodule of M and has two SL-closures, so there exists H_1, H_2 SL-closed submodules of M such that A is strongly large in both of H_1 and H_2 ,hence by lemma 2.1, A is strongly large submodule of $H_1 \cap H_2$ in M. Since M has the SLCIP then $H_1 \cap H_2$ is SL-closed in M ,but we have $H_1 \cap H_2$ is strongly large in H_1 .Thus $H_1 \cap H_2 = H_1$, so $H_1 \subseteq H_2$. Similarly, $H_2 \subseteq H_1$. Therefore $H_2 = H_1$. □

Lemma 2.5. Let M be a SL-UC module and N, K, L are submodules of M such that K is the SL-closure of N in M. If N is strongly large in L then $L \subseteq K$.

Proof. Since K is the SL-closure of N in M , we get N is strongly large in K and K is a SL-closed submodules ofM. But M is a SL-UC module, so K is the unique SL-closed submodule of M such that N is strongly large in K .By assumption N is strongly large in L .Thus $L \subseteq K$. □

Theorem 2.6. Let M be an R-module. Then M has the SLCIP if and only if for all submodules N_1, N_2, L_1, L_2 of M with N_1 is strongly large in N_2 and L_1 is strongly large in L_2 implies $N_1 + L_1$ is strongly large in $N_2 + L_2$.

Proof. Since $N_1 + L_1$ and N_1 are submodules of M , so there is H and T a unique SL-closed submodules in M such that $N_1 + L_1$ is strongly large in H and N_1 strongly large in T implies by lemma 2.1, $N_1 = (N_1 + L_1) \cap N_1$ is strongly large in $H \cap T$ which is SL-closed in M. On the other hand, by theorem 2.4 M is a SL-UC module, so $T = H \cap T$ and hence $T \subseteq H$. Now, N_1 is strongly large in N_2 and T is a SL-closure of N_1 in M , so that by above lemma $N_2 \subseteq T$, thus we get $K_1 \subseteq K_2 \subseteq T \subseteq H$ ...(1). Similarly, $L_1 \leq M$ so there is a unique SL-closed T' in M such that L_1 is strongly large in T'. Hence $L_1 = (N_1 + L_1) \cap L_1$ is strongly large in $H \cap T'$ which is SL-closed in M and since M is a SL-UC module, then $T' = H \cap T'$ and hence $T' \subseteq H$. Since L_1 is strongly large in L_2 and T' is a SL-closure of L_1 in M, by apply above lemma $L_2 \subseteq T'$. We have $L_1 \subseteq L_2 \subseteq T' \subseteq H$ ...(2). By (1) and (2) $N_2 + L_2 \subseteq H$ and by lemma 1.2, $N_1 + L_1$ is strongly large in $N_2 + L_2$. Conversely, let A and B be two SL-closed submodule of M. Assume that $A \cap B$ is strongly large in W of M. Since A is strongly large in A, so by assumption $(A \cap B) + A = A$ is strongly large in $A + W$, but A is a SL-closed submodule of M, hence $A = A + W$ and so $W \subseteq A$. By similar proof $W \subseteq B$. Thus $W \subseteq A \cap B$ and hence $A \cap B = W$. □

The right condition of a previous theorem is not hold in general, as example shows:

Consider the Z-module $M = Z \oplus Z_2$. Let $A_1 = A_2 = (2,\bar{0})Z$, $B_1 = (1,\bar{0})Z$ and $B_2 = (1,\bar{1})Z$. Let $b_1 = (r,\bar{0}) \in B_1$, $r \in Z$ and let $n \in Z$ such that $b_1 n \neq 0$. Choose $2 \in Z$, $b_1.2 = (r.2,\bar{0}) \in A_1 = A_2$ and $b_1 2n = (r2n,\bar{0}) \neq (0,\bar{0})$,thus $A_1 = A_2$ is strongly large in B_1. Similarly, let $b_2 = (1,\bar{1})r \in B_2$, $r \in Z$ and let $m \in Z$ such that $b_1 m \neq 0$ .If $(r = even) \in Z$, $b_2 = (r,\bar{0})$, if $(r = odd) \in Z$, $b_2 = (r,\bar{1})$ .Choose $2 \in Z$, then $b_2.2 = (r.2,\bar{0})$ or $b_2.2 = (r.2,\bar{2}) = (r.2,\bar{0})$, thus $2b_2 \in A_1 = A_2$. For $n \in Z$,either $b_2 2n = (r2n,\bar{0})$ or $b_2 2n = (r2n,\overline{2n}) = (r2n,\bar{0})$, so $b_2 2n \neq (0,\bar{0})$, hence $A_1 = A_2$ is strongly large in B_2. On the other hand, $A_1 \cap (0,\bar{1}) = 0$,but $(0,\bar{1})Z \leq B_1 + B_2$,so A_1 is not essential in $B_1 + B_2$ and since $A_1 = A_1 + A_2$, then $A_1 + A_2$ is not essential in $B_1 + B_2$,thus $A_1 + A_2$ is not strongly large in $B_1 + B_2$.

Theorem 2.7. Let M be an R-module. Then M has the SLCIP if and only if for all submodules N , L of M with $N \cap L$ is strongly large in N implies L is strongly large in $N + L$.

Proof. Assume that $N \cap L$ is strongly large in N . Since L is strongly large in L and M has the SLCIP then by above theorem $(N \cap L) + L$ is strongly large in $N + L$. Thus L is strongly large in $N + L$.Conversely, let K_1, K_2 be two SL-closed submodule of M. Suppose that $K_1 \cap K_2$ is strongly large in $W \leq M$. Since K_1 is strongly large in K_1, we get $K_1 \cap K_2 = (K_1 \cap K_2) \cap K_1$ is strongly large in $W \cap K_1$, so $W \cap K_1$ is strongly large in $W \leq M$. By hypothesis K_1 is strongly large in $W + K_1$, but K_1 is SL-closed, $K_1 = W + K_1$ and hence $W \subseteq K_1$. Similarly, we have $W \subseteq K_2$. Thus $K_1 \cap K_2 = W$ and hence M has the SLCIP. □

Corollary 2.8. Let M be an R-module. Then M has the SLCIP if and only if every submodule of M has the SLCIP.

Proof. Obviously. □

Theorem 2.9. Let M be an R-module. Then M has the SLCIP if and only if for all SL-closed submodule K of M and $N \leq M$, $K \cap N$ is SL-closed submodule of N.

Proof. Suppose that $K \cap N$ is strongly large in $L \leq N$ .Since K is strongly large in K , then by theorem 2.6 $K = K + (K \cap N)$ is strongly large in $K + L$,but K SL-closed submodule of M implies $K = K + L$ and so $L \subseteq K$,thus $L \subseteq K \cap N$ and hence $L = K \cap N$ .Therefore $K \cap N$ is SL-closed submodule of N. Conversely, let A , B be two SL-closed submodule of M. Assume that $A \cap B$ is strongly large in $W \leq M$. Since A is SL-closed in M and $W \leq M$, by assumption $A \cap W$ is SL-closed in W .Now $A \cap B$ is strongly large in W implies that $A \cap B = A \cap (A \cap B)$ is strongly large in $A \cap W$ .Since B is SL-closed in M ,so $A \cap B$ is SL-closed in A, this is implies $A \cap B = A \cap W$,so $A \cap B$ is SL-closed in W and hence $A \cap B = W$. □

Theorem 2.10. Let M be an R-module has the SLCIP. Then every decomposition $M = A \oplus B$ for all $\phi \in Hom_R(A,B)$, $Ker\phi$ is SL-closed in M.

Proof. Assume that $M = A \oplus B$ has the SLCIP and $\phi : A \to B$ is an R-homomorphism. Define $T = \{a + \phi(a) : a \in A\}$,it is not hard to prove $M = T \oplus B$,so both of A and T is direct summand of M, thus A , T are SL-closed submodule of M and hence $A \cap T$ is SL-closed in M. On the other hand, $Ker\phi = A \cap T$. Therefore $Ker\phi$ is SL-closed in M . □

Remark 2.11. Let M be an R-module, N and K are submodules of M. If N is SL-closed in M and $N \leq K$ then N is SL-closed in K.

Theorem 2.12. Consider the following statements for an R-module M :

(i) M has the SLCIP;
(ii) for each $N \leq M$ and each $\phi \in Hom_R(N,M)$ such that $N \cap \phi(N) = 0$, $Ker\phi$ is SL-closed in N;
(iii)for all $m, m' \in M$ such that $mR \cap m'R = 0$, $ann_R(m) \subseteq ann_R(m')$ and $ann_R(m')/ann_R(m)$

is strongly large in $R/ann_R(m)$ implies $m^{'}=0$.

Then $(i)\Rightarrow(ii)\Rightarrow(iii)$.

Proof. $(i)\Rightarrow(ii)$ Assume that $N\leq M$ and $\phi:N\rightarrow M$ is R-homomorphism with $N\cap\phi(N)=0$. Let $K=N\oplus\phi(N)$, $K\leq M$. Since M has the SLCIP, then by cor. 2.8 K has the SLCIP. Define $\psi:N\rightarrow\phi(N)$ by $\psi(n)=\phi(n)$, for all $n\in N$. It is clear that ψ is well-defined and R-homomorphism so by theorem 2.10, $Ker\psi$ is SL-closed in K, but $Ker\psi=Ker\phi$, thus $Ker\phi$ is SL-closed in $K=N\oplus\phi(N)$ and hence by above remark, $Ker\phi$ is SL-closed in N.
$(ii)\Rightarrow(iii)$ Define $\phi:mR\rightarrow m^{'}R$ by $\phi(mr)=m^{'}r$, for all $r\in R$. Clearly, ϕ is well-defined and R-homomorphism, $mR\cap\phi(mR)=mR\cap m^{'}R=0$, so by hypothesis $Ker\phi$ is SL-closed in mR. Now, let $mr_1\in mR$ and $r_2\in R$ such that $mr_1r_2\neq 0$, so $r_1r_2\notin ann_R(m)$ and hence $r_1r_2+ann_R(m)\neq 0$. But $ann_R(m^{'})/ann_R(m)$ is strongly large in $R/ann_R(m)$, so there exists $s\in R$ such that $0\neq(r_1r_2+ann_R(m))s\in ann_R(m^{'})/ann_R(m)$, hence $r_1r_2s+ann_R(m)=p+ann_R(m)$, $p\in ann_R(m^{'})$, thus $r_1r_2s-p\in ann_R(m)\subseteq ann_R(m^{'})$ implies $r_1r_2s\in ann_R(m^{'})$, hence $m^{'}r_1r_2s=0$ thus $\phi(mr_1r_2s)=0$, so $0\neq mr_1r_2s\in Ker\phi$, and so there exists $t=r_2s\in R$ such that $mr_1t\in Ker\phi$ and $mr_1tr_2=mr_1r_2sr_2\neq 0$, therefore $Ker\phi$ is strongly large in mR and hence $Ker\phi=mR$. Thus $m\in Ker\phi$, that is $m^{'}=\phi(m)=0$. □

Let A and B be two submodules of a module M with $A\cap B=0$. Then B is called an SL-complement of A in M if, B is an SL-closed submodule of M and $A\oplus B$ is strongly large in M. Moreover, a submodule N of a module M is called a SL-complement submodule if, there exists a submodule K of M such that N is an SL-complement of K in M [2]. Notice that every SL-complement is SL-closed.

Theorem 2.13. Let M be an R-module. Then M has the SLCIP if and only if for all strongly large N of M, $(N\cap H)+(N\cap L)$ is strongly large in $H+L$, for all submodules H,L of M.

Proof. Let N be strongly large in M and since H is strongly large in H, also L is strongly large in L implies $N\cap H$ and $N\cap L$ are strongly large of H,L respectively. By theorem 2.6, the result is obtained. Conversely, let $A,B\leq M$ Such that $A\cap B$ is strongly large in A. Let P be an SL-complement of $A\cap B$ in M, put $Q=P\oplus(A\cap B)$ is strongly large in M, then by hypothesis $(Q\cap A)+(Q\cap B)$ is strongly large in $A+B$. Since $Q\cap A=[P\oplus(A\cap B)]\cap A$, so if $x\in Q\cap A$ then $x=a+b$, where $a\in P, b\in A\cap B$, thus $a=x-b\in P\cap A=0$, hence $x=b\in A\cap B$, so $(Q\cap A)\subseteq(A\cap B)\subseteq B$, thus $(Q\cap A)+(Q\cap B)\subseteq A+B$ and hence B is strongly large in A. By applying theorem 2.7, M has the SLCIP. □

Definition 2.14. A submodule N of an R-module M is called R-SL-closed, provided $N\cap mR$ is not strongly large in mR for all $m\in M/N$.

Theorem 2.15. The following statements are equivalent for an R-module M.

(i) M has the SLCIP;
(ii) for all SL-closed submodule K of M, $K\cap mR$ is not strongly large in mR for all $m\in M/N$;
(iii) $\overset{+}{N}_{SL}=\{m\in M : N\cap mR$ is strongly large in $mR\}$ is a submodule of M, for all $N\leq M$.

Proof. $(i) \Rightarrow (ii)$ Let K be a SL-closed submodule of M. If $K \cap mR$ is strongly large in mR for some $m \in M/N$, then by theorem 2.7, K is strongly large in $mR + K$ and hence $K = mR + K$, so $mR \subseteq K$, thus $m \in K$ which is a contradiction.

$(ii) \Rightarrow (i)$ Let A and B be any two SL-closed submodules of M. Assume $A \cap B$ is not proper strongly large in $W \leq M$, so there exists $m \in W$ with $m \notin A \cap B$. If $m \notin A$ then $A \cap mR$ is not strongly large in mR which is a contradiction with the lemma 1.8.

$(ii) \Rightarrow (iii)$ Let $N \leq M$. Assume that K is any SL-closure of N, so N is strongly large in K and K is SL-closed in M. Now, let $m \in \overset{+}{N}_{SL}$ then $N \cap mR$ is strongly large in mR, thus $K \cap mR$ is strongly large in mR and hence by (ii) $m \in K$. Therefore $\overset{+}{N}_{SL} \subseteq K$. Conversely, let $m \in K$ so by lemma 1.8, $N \cap mR$ is strongly large in mR and hence $m \in \overset{+}{N}_{SL}$. Thus $\overset{+}{N}_{SL} = K$ and so $\overset{+}{N}_{SL}$ is a submodule of M.

$(iii) \Rightarrow (ii)$ Let K be an SL-closed submodules of M, then $K \subseteq \overset{+}{K}_{SL}$. For any $m \in \overset{+}{K}_{SL}$, $K \cap mR$ is strongly large in mR, so by lemma 1.8, K is strongly large in $\overset{+}{K}_{SL}$, hence $K = \overset{+}{K}_{SL}$. For any M/K, $m \notin K$ and so $m \notin \overset{+}{K}_{SL}$, thus $K \cap mR$ is not strongly large in mR. □

Theorem 2.16. Let M be an R-module. Then M has the SLCIP if and only if for all submodules $K \subseteq L \subseteq M$, if K' is an SL-closure of K, there exists an SL-closure L' of L such that $K' \subseteq L'$.

Proof. Since $K \subseteq L, L = K + L$. But K is strongly large in SL-closed submodule K' of M, thus by theorem 2.6, $L = K + L$ is strongly large in $K' + L$. Now, $K' + L \subseteq M$, so there exists L' an SL-closure submodule of $K' + L$ in M; that is $K' + L$ strongly large in a SL-closed submodule L' of M. Hence L is strongly large in a SL-closed submodule L' of M, so L' is SL-closure of L, also $K' \subseteq K' + L \subseteq L'$. Conversely, let A and B be two SL-closed in M. $K = A \cap B \subseteq B$ and let K' be an SL-closure of K, so by hypothesis, there exists L' SL-closure of B such that $K' \subseteq L'$. Since B is SL-closed in M implies $L' = B$, so $K' \subseteq B$. Also, $A \cap B \subseteq A$ and let K' SL-closure of K, so by hypothesis, there exists L'' SL-closure of A such that $K' \subseteq L''$. Since A is SL-closed in M, so $L'' = A$ and hence $K' \subseteq A$, thus $K' \subseteq L''$. But $K = A \cap B$ is strongly large in K', then $A \cap B = K'$; that is $A \cap B$ SL-closed in M. □

Given submodules K, L of M we define $K\rho_{sl}L$, whenever the submodule $K \cap L$ is strongly large in both of K and L.

Theorem 2.17. Let M be an R-module. Then M has the SLCIP if and only if for all submodules K, K', L, L' of M, $K\rho_{sl}K'$ and $L\rho_{sl}L'$ implies $(K + L)\rho_{sl}(K' + L')$.

Proof. Suppose that $K\rho_{sl}K'$ and $L\rho_{sl}L'$. We have to prove that $(K + L) \cap (K' + L')$ is strongly large in both $(K + L)$ and $(K' + L')$. Since $K\rho_{sl}K'$ then $K \cap K'$ is strongly large in both K and K', also $L\rho_{sl}L'$ implies $L \cap L'$ is strongly large in both L and L'. Since M has the SLCIP, so by theorem 2.6, $(K \cap K') + (L \cap L')$ is strongly large in $(K + K') \cap (L + L')$, but we have $(K \cap K') + (L \cap L') \subseteq (K + L) \cap (K' + L') \subseteq (K + L) + (K' + L')$, so $(K + L) \cap (K' + L')$ is strongly large in $(K + L) + (K' + L')$ and hence $(K + L) \cap (K' + L')$ is strongly large in $K + L$. By similar a way, $(K+L)\cap(K^{'}+L^{'})$ is strongly large in $K^{'}+L^{'}$.Conversely, let K,L be two strongly large in $K^{'},L^{'}$ respectively, where $K,L,K^{'},L^{'}$ are submodules of M. We must to prove $K+L$ is strongly in $K^{'}+L^{'}$.We have $K\cap K^{'}\leq K\leq K^{'}$ and $K\cap K^{'}$ is strongly large in $K^{'}$,then $K\cap K^{'}$ is strongly large in K. Similarly, $L\cap L^{'}$ is strongly large in L. Thus $K\rho_{sl}K^{'}$ and $L\rho_{sl}L^{'}$,hence by hypothesis $(K+L)\cap(K^{'}+L^{'})$ is strongly large in both $(K+L),(K^{'}+L^{'})$. We have $(K+L)\cap(K^{'}+L^{'})\leq(K+L)\leq(K^{'}+L^{'})$ this implies $(K+L)$ is strongly large in $(K^{'}+L^{'})$. By applying theorem 2.6, the result is obtained. □

Theorem 2.18. The following statements are equivalent for an R-module M.

(i) M has the SLCIP;

(ii) for all $K_{\lambda}\leq L_{\lambda}\ (\lambda\in\Lambda)$ are submodules of M such that K_{λ} is SL-closed in L_{λ} for all $(\lambda\in\Lambda)$,then $\bigcap_{\lambda\in\Lambda}K_{\lambda}$ is SL-closed in $\bigcap_{\lambda\in\Lambda}L_{\lambda}$;

(iii) the intersection of any collection of SL-closed submodules of M is SL-closed.

Proof. $(i)\Rightarrow(ii)$ By cor. 2.8, L_{λ} has the SLCIP for all $\lambda\in\Lambda$.Put $L=\bigcap_{\lambda\in\Lambda}L_{\lambda}$ .We have $L\leq L_{\lambda}$ and K_{λ} is SL-closed in L_{λ} for all for all $\lambda\in\Lambda$,so by theorem 2.9 $K_{\lambda}\cap L$ is SL-closed in L, for all $\lambda\in\Lambda$.Since L has the SLCIP, then $\bigcap_{\lambda\in\Lambda}(K_{\lambda}\cap L)$ is a SL-closed submodules of L, this mean $\bigcap_{\lambda\in\Lambda}K_{\lambda}$ is SL-closed in $\bigcap_{\lambda\in\Lambda}L_{\lambda}$. $(ii)\Rightarrow(iii)$ Let K_{λ} is SL-closed in $M=L_{\lambda}$ for all $\lambda\in\Lambda$,then by (ii) ,we get $\bigcap_{\lambda\in\Lambda}K_{\lambda}$ is SL-closed in $M=\bigcap_{\lambda\in\Lambda}L_{\lambda}$. $(iii)\Rightarrow(i)$ Obviously. □

We finish this section by the following theorem.

Theorem 2.19. Let M be an R-module. Then M has the SLCIP if and only if every (2-generator) submodule of M has the SLCIP.

Proof. It follows by cor. 2.8. Conversely, assume that K,L are strongly large submodules in both $K^{'},L^{'}$ of M respectively, to prove $K+L$ is strongly large in $K^{'}+L^{'}$.Let $x\in K^{'}+L^{'}$,so $x=a+b$ where $a\in K^{'},b\in L^{'}$,and let $r\in R$ such that $xr=ar+br\neq0$. Consider $N=\langle a,b\rangle$, we get $K\cap N,L\cap N$ are strongly large in both $K^{'}\cap N$, $L^{'}\cap N$ respectively. Since N has the SLCIP, then by theorem 2.6 $(K\cap N)+(L\cap N)$ is strongly large in $(K^{'}\cap N)+(L^{'}\cap N)$.Since $x\in(K^{'}\cap N)+(L^{'}\cap N)$,there exists $s\in R$ such that $xs\in(K\cap N)+(L\cap N)$ and $xsr\neq0$,thus $xsr\in K+L$ and $xsr\neq0$,so $K+L$ is strongly large in $K^{'}+L^{'}$ and hence by theorem 2.6,we get the result. □

IV. Some Properties of modules with the SLCIP.

In this section, we discuss the concept of modules with the SLCIP and give some properties of such type of these modules.

We start by the following lemma.

Lemma 3.1. Let M be an R-module and $N\subseteq H\subseteq M$ .If H is a SL-closed submodule of M, then H/N is a SL-closed submodules of M/N.

Proof. Suppose that H/N is strongly large in B/N of M/N.For any $b+N\in B/N\ (b\in B)$, $s\in R$ such that $(b+N)s\neq0$,then $bs\notin N$, so $bs\neq0$. So there exists $r\in R$ such that $(b+N)r\in H/N$

and $(b+N)rs \neq 0$,that is $br+N \in H/N$ and $brs \notin N$. Thus $br \in H$ and $brs \neq 0$,hence H is strongly large in B of M, so $H = B.$ Therefore $H/N = B/N$ and so H/N is a SL-closed in M/N .□

Lemma 3.2. Let M be an R-module and $N \subseteq H \subseteq M$ .If H/N and N are SL-closed submodules in both of M/N ,M respectively, then H is SL-closed in M.

Proof. Suppose that H is strongly large in B of M with $N \neq H$ .We claim that H/N is strongly large in B/N .Let $b+N \in B/N$, $s \in R$ such that $(b+N)s \neq 0$,so $bs \notin N$ and hence $bs \neq 0$. Since H is strongly large in B, so there exists $r \in R$ such that $br \in H$ and $brs \neq 0$,so $brs \in H$ and $brs.(1) \neq 0$. Since N is not strongly large in H, thus for all $a \in R$ either $bars \notin N$ or $bars = 0$. Choose $a = 1$, so either $brs \notin N$ or $brs = 0$,but we have $brs \neq 0$,so $brs \notin N$,that is $(b+N)rs \neq 0$. Also, $br \in H$ implies $(b+N)r \in H/N$ and hence H/N is strongly large in B/N .But H/N is SL-closed in M/N ,so $H/N = B/N$ and hence $H = B.$ □

Proposition 3.3. Let M be an R-module and let N be a SL-closed submodule of M. If M has the SLCIP then M/N has the SLCIP.

Proof. Let L_1/N and L_2/N be two a SL-closed submodules of M/N .Since N be SL-closed in M, so by lemma 3.2, L_1 and L_2 are a SL-closed submodules of M and hence $L_1 \cap L_2$ is SL-closed submodule in M. Thus by lemma 3.1, $L_1/N \cap L_2/N = L_1 \cap L_2/N$ is a SL-closed submodules of M/N .

□

Corollary 3.4. Let M be an R-module and let N be a SL-closed submodule of M. Then M has the SLCIP if and only if M/N has the SLCIP, for each N is a SL-closed in M.

Remark 3.5. The direct sum of two modules with the SLCIP need not be have SLCIP,(see example 2.1).

Now, we give certain condition under which a direct sum of modules with, SLCIP has the SLCIP. we need some lemma's.

Lemma 3.6. Let $M = M_1 \oplus M_2$ be an R-module such that $ann_R M_1 + ann_R M_2 = R$ .If N_i is strongly large in $L_i \leq M_i$ $(i=1,2)$,then $N_1 \oplus M_2$ is strongly large in $L_1 \oplus M_2$ and $M_1 \oplus N_2$ is strongly large in $M_1 \oplus L_2$.

Proof. To prove $N_1 \oplus M_2$ is strongly large in $L_1 \oplus M_2$ .Let $(l_1, m_2) \in L_1 \oplus M_2$ and $s \in R$ such that $(l_1, m_2)s \neq (0,0)$ where $l_1 \in L_1$ and $m_2 \in M_2$ .We have the two cases: if $l_1 s \neq 0$,so there exists $r \in R$ such that $l_1 r \in N_1$ and $l_1 sr \neq 0$,thus $(l_1, m_2)r = (l_1 r, m_2 r) \in N_1 \oplus M_2$ and $(l_1, m_2)rs = (l_1 rs, m_2 rs) \neq (0,0)$ .If $m_2 s \neq 0$ .Since $ann_R M_1 + ann_R M_2 = R$, $a + b = 1$ for some $a \in ann_R M_1$, $b \in ann_R M_2$ .We get $m_2.1 = m_2.a + m_2.b$,then $m_2 = m_2.a$,thus $(l_1, m_2)a = (l_1 a, m_2 a) = (0, m_2) \in N_1 \oplus M_2$ .Also, $(l_1, m_2)as = (l_1 as, m_2 as) = (0, m_2 s) \neq (0,0)$ .Therefore by two cases, we get $N_1 \oplus M_2$ is strongly large in $L_1 \oplus M_2$ .Similarly, $M_1 \oplus N_2$ is strongly large in $M_1 \oplus L_2$. □

Lemma 3.7. Let $M = M_1 \oplus M_2$ be an R-module such that $ann_R M_1 + ann_R M_2 = R$ .If $N_1 \oplus N_2$ is a SL-closed submodule of $M_1 \oplus M_2$ then N_i is a SL-closed submodule of M_i , for $(i=1,2)$.

Proof. Suppose that N_1 is strongly large in $L_1 \leq M_1$ and N_2 is strongly large in $L_2 \leq M_2$. By above lemma, $N_1 \oplus M_2$ is strongly large in $L_1 \oplus M_2 \leq M$ and $M_1 \oplus N_2$ is strongly large in $M_1 \oplus L_2 \leq$ M and hence $(N_1 \oplus M_2) \cap (M_1 \oplus N_2)$ is strongly large in $(L_1 \oplus M_2) \cap (M_1 \oplus L_2)$, so $N_1 \oplus N_2$ is strongly large in $L_1 \oplus L_2 \leq M_1 \oplus M_2$, but $N_1 \oplus N_2$ is a SL-closed submodule of $M_1 \oplus M_2$, so $N_1 \oplus N_2 = L_1 \oplus L_2$. Thus $N_1 = L_1$ and $N_2 = L_2$. □

Lemma 3.8. Let M_α be an R-modules, $\alpha \in \Lambda$. If A_α is a SL-closed submodule of M_α then $\underset{\alpha\in\Lambda}{\oplus} A_\alpha$ is a SL-closed submodule of $\underset{\alpha\in\Lambda}{\oplus} M_\alpha$.

Proof. Suppose that $\underset{\alpha\in\Lambda}{\oplus} A_\alpha$ is strongly large in $X \leq \underset{\alpha\in\Lambda}{\oplus} M_\alpha$. For any $\alpha \in \Lambda$, $A_\alpha = (\underset{\alpha\in\Lambda}{\oplus} A_\alpha) \cap M_\alpha$ is strongly large in $X \cap M_\alpha \leq M_\alpha$. But A_α is a SL-closed submodule of M_α, so $A_\alpha = X \cap M_\alpha$ for all $\alpha \in \Lambda$. The ith component x_i of x is in $X \cap M_{\alpha_i} = A_{\alpha_i}$, hence $x \in \underset{\alpha\in\Lambda}{\oplus} A_\alpha$, that is $X \leq \underset{\alpha\in\Lambda}{\oplus} A_\alpha$, so $X = \underset{\alpha\in\Lambda}{\oplus} A_\alpha$ and hence $\underset{\alpha\in\Lambda}{\oplus} A_\alpha$ is SL-closed in $\underset{\alpha\in\Lambda}{\oplus} M_\alpha$. □

Proposition 3.9. Let $M = M_1 \oplus M_2$ be an R-module such that $ann_R M_1 + ann_R M_2 = R$. Then M_1 and M_2 has the SLCIP if and only if $M_1 \oplus M_2$ has the SLCIP.

Proof. Assume that A, B are two SL-closed submodules of M. Since $ann_R M_1 + ann_R M_2 = R$, then by the same way of the proof of [4, prop 4.2], $A = A_1 \oplus B_1$ and $B = A_2 \oplus B_2$ where A_1, A_2 are submodules in M_1 and B_1, B_2 are submodules in M_2. By some properties of SL-closed submodule, we have A_1, A_2 are SL-closed in M_1 and B_1, B_2 are SL-closed in M_2. Thus $A_1 \cap A_2$ and $B_1 \cap B_2$ are SL-closed in both of M_1, M_2 respectively. By above lemma, $A \cap B = (A_1 \oplus B_1) \cap (A_2 \oplus B_2) = (A_1 \cap A_2) \oplus (B_1 \cap B_2)$ is SL-closed in $M_1 \oplus M_2$. Conversely, it follows directly by cor. 2.8. □

Now, we will present another result with different conditions under which the direct sum of modules with the SLCIP is also has the SLCIP, but first we need to introduce the following lemma.

Lemma 3.10. Let M_i be an R-modules, $i \in I$. Then $M = \underset{i\in I}{\oplus} M_i$ satisfies (*) if and only if M_i is satisfies (*), for all $i \in I$.

Proof. Take M_k be an R-module, $(k \geq 1) \in I$. Let $m_k \in M_k$ and $r_1, r_2 \in R$ such that $m_k r_1 r_2 = 0$ and $m_k r_2 \neq 0$, hence $(0,0,...,m_k,0,0,...)r_1 r_2 = (0,0,0....)$ and $(0,0,...,m_k,0,0,...)r_2 \neq (0,0,0....)$. Since $M = \underset{i\in I}{\oplus} M_i$ has the condition (*), so $r_1 = 0$. Thus M_i satisfies (*), for all $i \in I$. Conversely, let $r_1, r_2 \in R$ and $m = \underset{i\in I}{\oplus} m_i \in M$ such that $m r_1 r_2 = 0$ and $m r_2 \neq 0$, where $m_i \in M_i$ for all $i \in I$. Then $m_i r_1 r_2 = 0$ (for $i \in I$) and $m_j r_2 \neq 0$ for some $j \in I$, but M_j satisfies the condition (*), then $r_1 = 0$. Therefore $M = \underset{i\in I}{\oplus} M_i$ satisfies (*). □

Theorem 3.11. Let $M = \underset{i\in I}{\oplus} M_i$ be an R-module satisfies (*) such that every SL-closed submodule of M is fully invariant. Then M_i has the SLCIP (for all $i \in I$) if and only if M has the SLCIP.

Proof. Let A, B be two SL-closed submodules of M. By assumption A, B are fully invariant submodules, so $A = \oplus_{i\in I}(A \cap M_i)$, $B = \oplus_{i\in I}(B \cap M_i)$. Put $A_i = A \cap M_i$, $B_i = B \cap M_i$, $i \in I$. Hence $A = \oplus_{i\in I} A_i$, $B = \oplus_{i\in I} B_i$. As A, B are SL-closed in M and M satisfies (*), then A, B are closed in M and so A_i and B_i are closed in M_i, for all $i \in I$. Hence A_i, B_i are SL-closed in M_i. But M_i satisfies SLCIP, for all $i \in I$, so that $A_i \cap B_i$ is SL-closed in M, for all $i \in I$. Now, by lemma 3.8, $\oplus_{i\in I}(A_i \cap B_i)$ is SL-closed in $\oplus_{i\in I} M_i = M$. The converse follows directly by cor. 2.8. □

Proposition 3.12. Let $M = M_1 \oplus M_2$ be an R-module has the SLCIP then there exists $\phi \in End_R(M)$ such that $Ker\phi$ is SL-closed in M.

Proof. Consider $\alpha : M_1 \to M_2$ and $\beta : M_2 \to M_1$ be two R-homomorphisms. Define $\phi : M \to M$ by $\phi(m_1 + m_2) = \alpha(m_1) + \beta(m_2)$ for all $m_1 \in M_1, m_2 \in M_2$. It is clear that $\phi \in End_R(M)$. Notice that $Ker\phi = Ker\alpha \oplus Ker\beta$, to show this: let $a + b \in Ker\phi$ then $\alpha(a) + \beta(b) = \phi(a+b) = 0$, so $\alpha(a) = 0$ and $\beta(b) = 0$, thus $a + b \in Ker\alpha + Ker\beta$. Conversely, let $x + y \in Ker\alpha + Ker\beta$, then $x \in Ker\alpha$ and $y \in Ker\beta$, so $\phi(x + y) = \alpha(x) + \beta(y) = 0 + 0 = 0$ and hence $x + y \in Ker\phi$. On the other hand $Ker\alpha \cap Ker\beta \subseteq M_1 \cap M_2 = 0$, thus $Ker\phi = Ker\alpha \oplus Ker\beta$. Since M has SLCIP then by theorem 2.10, $Ker\alpha$ and $Ker\beta$ are SL-closed submodules of M_1, M_2 respectively. Then by lemma 3.8, $Ker\alpha \oplus Ker\beta$ is SL-closed in $M_1 \oplus M_2$; that is $Ker\phi$ SL-closed in M.

□

Note: If the factor module of a module M has the SLCIP, then M need not be have the SLCIP. Consider the following example; it is well known that $M = Z \oplus Z_2$ does not have the SLCIP as Z-module, but $Z \oplus Z_2 / Z_2 \cong Z$ has the SLCIP.

Recall that an R-module M is said to be strongly extending if for every submodule N of M, there exists a decomposition $M = K \oplus L$ such that N is a strongly large submodule of K. Equivalently, a module M is SL-extending if every SL-closed is a direct summand. Since each strongly large submodule is large, every strongly extending module is extending [1]. A module M is said to have the SIP if, the intersection of any two direct summands of M is again direct summand.

Remark 3.13. A homomorphic image of module satisfies SLCIP need not be have SLCIP. Consider the following example; The Z-homomorphism $\psi : Z \oplus Z \to Z \oplus Z_2$ defined by $\psi(n, m) = (n, \bar{m})$, for all $n, m \in Z$. By [5, Ex. 5] $Z \oplus Z$ has the SIP, but by [4, prop 3.9] $Z \oplus Z$ is strongly extending, so it is easy to say that $Z \oplus Z$ has the SLCIP. But $\mathrm{Im}\psi = Z \oplus Z_2$ does not be have SLCIP.
Now, we shall give a condition under which the image of modules with the SLCIP is also has the SLCIP, we need the following lemma.

Lemma 3.14. Let M and N be an R-modules with $A \subseteq B \subseteq M$ such that $f : M \to N$ be an R-monomorphism. Then A is SL-closed in B if and only if $f(A)$ is SL-closed in $f(B)$.

Proof. Assume that A is SL-closed in B of M. Let $f(A)$ be strongly large in $W \leq f(B)$ then by [5, cor. 2.16] $A = f^{-1}(f(A))$ is strongly large in $f^{-1}(W) \leq B$, so $A = f^{-1}(W)$ and hence $f(A) = W$. Conversely, suppose $f(A)$ is SL-closed in $f(B)$ of $\mathrm{Im} f$. Let A be strongly large in $P \leq B$, then by [5,prop 2.14] $f(A)$ is strongly large in $f(P) \leq f(B)$ and hence $f(A) = f(P)$. Since f is a monomorphism then $A = P$. □

Proposition 3.15. Suppose that M and N are two R-modules such that $f: M \to N$ be an R-monomorphism. Then M has the SLCIP if and only if the image of M satisfies SLCIP.

Proof. Suppose that M has the SLCIP. Let L_1 and L_2 be two SL-closed submodules of $\operatorname{Im} f$, so $L_1 = f(K_1), L_2 = f(K_2)$ for some submodules K_1, K_2 of M. Then by above lemma K_1, K_2 are SL-closed in M and hence $K_1 \cap K_2$ is SL-closed in M .Again by above lemma, $f(K_1) \cap f(K_2) = f(K_1 \cap K_2)$ is SL-closed in $\operatorname{Im} f$; that is $L_1 \cap L_2$ SL-closed in $\operatorname{Im} f$. Conversely, suppose that $\operatorname{Im} f$ has the SLCIP. Let A, B are SL-closed submodules of M, then by above lemma $f(A)$, $f(B)$ are SL-closed submodules of $\operatorname{Im} f$,so $f(A \cap B) = f(A) \cap f(B)$ is SL-closed in $\operatorname{Im} f$ and hence $A \cap B$ is SL-closed in M. □

Proposition 3.16. Let M be an R-module, $\overline{R} = R/ann_R M$ .Then M has the SLCIP as R-module if and only if M has the SLCIP as $\overline{R}-$module .

Proof. Obviously. □

Now, we will investigate the behavior of SL-closed submodules and modules with the SLCIP under localization. We need the following lemma's.

Lemma 3.17. Let M be an R-module satisfies (*) with $N, W \leq M$.Let S be a multiplicative closed subset of R, if N is strongly large in W of M then $S^{-1}N$ is strongly large in $S^{-1}W$ of $S^{-1}M$.

Proof. Let $\bar{m} \in S^{-1}W$, $\bar{m} = m/s_1 = w/s$ for some $w \in W$,so there exists $s_2 \in S$ such that $ms_2 s = ws_2 s_1 \in W$ .Put $s_2 s = t \in S$,then $mt \in W$ .Let $a/b \in S^{-1}R$ such that $m/s_1 . a/b \neq 0$, thus for all $l \in S$, $mla \neq 0$,hence $mta \neq 0$ .Since N is strongly large in W, so there exists $r \in R$ such that $mrt \in N$ and $mrta \neq 0$,so $m/s_1 . r/1 = m/s_1 . rt/t = mrt/s_1 t \in S^{-1}N$ .We claim that $m/s_1 . rt/t . a/b \neq 0$. If $m/s_1 . rt/t . a/b = 0$; that is $mra/s_1 b = 0$, so there exists $t_1 \in S$ such that $mrt_1 a = 0$, $rt_1 \in ann_R(ma)$ but $t_1 \notin ann_R(ma)$ and M satisfies (*), so $r = 0$ this is a contradiction. □

Lemma 3.18. Let M be an R-module $N \leq M$ and let S be a multiplicative closed subset of R. Then N is closed in M if and only if $S^{-1}N$ is closed in $S^{-1}M$,provided $S^{-1}A = S^{-1}B$ iff $A = B$.

Proof. Assume that $S^{-1}N \leq_e S^{-1}W \leq S^{-1}M$. We claim that $N \leq_e W$. Let $K \leq W$, $N \cap K = 0$, so $S^{-1}N \cap S^{-1}K = S^{-1}(N \cap K) = S^{-1}(0)$, thus $S^{-1}K = S^{-1}(0)$ and by assumption $K = 0$. Hence $N \leq_e W$ and so $N = W$ then $S^{-1}N = S^{-1}W$. Conversely, suppose that $N \leq_e K \leq M$. We claim that $S^{-1}N \leq_e S^{-1}K \leq S^{-1}M$. Let $S^{-1}H \leq S^{-1}K$ such that $S^{-1}N \cap S^{-1}H = S^{-1}(0)$, $S^{-1}(N \cap H) = S^{-1}(0)$ and hence by assumption $N \cap H = 0$, thus $H = 0$, $S^{-1}H = S^{-1}(0)$. Therefore $S^{-1}N \leq_e S^{-1}K$, so $S^{-1}N = S^{-1}K$, and by assumption $N = K$. □

Lemma 3.19. Let M be an R-module and let S be a multiplicative closed subset of R. If M satisfies (*) then $S^{-1}M$ satisfies (*) as $S^{-1}R$ -module.

Proof. Let $\frac{a_1}{s_1}, \frac{a_2}{s_2} \in S^{-1}R$ and $(\frac{m}{s} \neq 0) \in S^{-1}M$ such that $\frac{m}{s}.\frac{a_1}{s_1}.\frac{a_2}{s_2} = 0$ and $\frac{m}{s}.\frac{a_1}{s_1} \neq 0$,so there exists $t \in S$ such that $ma_1a_2t = 0$ and for all $l \in S$, $ma_1l \neq 0$.We have $ma_1ta_2 = 0$ and $ma_1t \neq 0$, so by (*) of M as R-module, $a_2 = 0$ and hence $\frac{a_2}{s_2} = 0$. □

From lemma's [3.17, 3.18, 1.7 and 3.19] we get the following proposition.

Proposition 3.20. Let M be an R-module satisfies (*) and let S be a multiplicative closed subset of R. Then N is a SL-closed submodule of M as R-module if and only if $S^{-1}N$ is a SL-closed submodule of $S^{-1}M$ as $S^{-1}R$ -module, provided $S^{-1}A = S^{-1}B$ iff $A = B.$

Theorem 3.21. Let M be an R-module satisfies (*) and let S be a multiplicative closed subset of R. Then M has the SLCIP as R-module if and only if $S^{-1}M$ has the SLCIP as $S^{-1}R$ -module, provided $S^{-1}A = S^{-1}B$ iff $A = B.$

Corollary 3.22. Let M be an R-module satisfies (*) and let S be a multiplicative closed subset of R. Then M has the SLCIP as R-module if and only if M_P has the SLCIP as R_P -modules, for all maximal ideal P of R.

References

[1]. B. Ungor ; S. Halicioglu, Strongly extending modules, Hacettepe J. math and statistics, 42(5), 2013, 465-478.

[2]. B. Ungor ;S. Halicioglu ;M.A. Kamal ; A. Harmanci, Strongly large module extensions, An.stiint.Univ. Al, I.Cuza Iasi.math.(S.N.)59,No.2, 2013, 431-452.

[3]. K.R. Goodearl, Ring theory, Nonsingular rings and modules, Dekker, Newyork, 1967.

[4]. M.S. Abbas, On fully stable modules, Ph.D. Thesis, Univ. Of Baghdad, Iraq, 1990.

[5]. M. Alkan ; A. Harmanci, On summand sum and summand intersection property of modules, Turk. J. math, 26(2002), 131-147.