$I_{z,m}.cw - complex$

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Abstract

In this paper we introduce a new tool in algebraic topology which is called $I_{z,m}.cw - complex$ (analogous terminology of J.H.C. Whitehead in [1]) .
By studying effect of this complex on the spectral sequences , we obtained some results , such as ;

1- Any complex is an $I_{z,m}.cw - complex \ if \ m \leq -z$
2- Let K & $\overline{K}$ are two cw-complexes , if $K \equiv_z \overline{K}$, then K is an $I_{z,m}.cw - complex \Leftrightarrow \overline{K}$ is an $I_{z,m}.cw - complex$.
3- If K is an $I_{z,m}.cw - complex$, then $\eta^r_{n,m} \cong \pi^r_{n,m} \quad \forall n = 2,3,\cdots,z$ and $j^r_{z+1,m}$ is onto, $\forall r$.
4- Let K be a cw-complex , if K is a $(z-1) - connected$, then K is a $I_{z,m}.cw - complex$, and the converse is not true .

Key words : cw-complexes , homotopy and homology group , new exact sequence , spectral sequences , $I_{z,m}.cw - complex$.

مجمع $I_{z.m}.cw-$

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الخلاصة

في هذا البحث قدمنا أداات جديدة في التبولوجيا الجبرية أسميناه مجمع $I_{z.m}.cw-$ ($I_{z,m}.cw - complex$) ، مُستلهم من بحث منشور في ١٩٤٩ لـ (J.H.C. Whitehead) .
بدراسة تأثير هذا المجمع على المتتالية الطيفية ، حصلنا على بعض النتائج ، منها :

١- أي مجمع يكون مجمع $I_{z.m}.cw-$ اذا كان $m \leq -z$.
٢- أذا كان كلاً من $K, \overline{K}$ مجمع-cw وكان $K \equiv_z \overline{K}$ فأن K مجمع $I_{z.m}.cw-$ $\Leftrightarrow$ $\overline{K}$ مجمع $I_{z.m}.cw-$.
٣- أذا كان K مجمع $I_{z.m}.cw-$ فـأن $\forall n = 2,3,\cdots,z$ $\eta^r_{n,m} \cong \pi^r_{n,m}$ وأن $j^r_{z+1,m}$ تطبيق شامل ، لكل r .
٤- أذا كان K مجمع-cwمترابط من نوع$(z-1)$ فأن K مجمع $I_{z.m}.cw-$ والعكس غير صحيح .

Introduction

There are many tools in algebraic topology , (see [2]) . In this paper we introduce a new tool in algebraic topology which is called $I_{z,m}.cw-complex$ and study the effect of this notion on the spectral sequences .

In this work a complex will mean a pair (K,e^0) , where K is a connected cw-complex and $e^0 \in K^0$ is a 0-cell , which is to be taken as base point for all the homotopy groups which we associate with K . Nevertheless we shall denote complexes by K , $\bar{K}$, L etc. .

This work contains two sections ; in first section , we introduce the spectral sequence and definition of $I_{z,m}.cw-complex$. In second section , we establish some results about our work , some of these results are purely algebraic and others depend on the topology of space , for examples ;

1- If $K \equiv_z \bar{K}$, then K is an $I_{z,m}.cw-complex \Leftrightarrow$ $\bar{K}$ is an $I_{z,m}.cw-complex$

2- If K is an $I_{z,m}.cw-complex$, then for each r we have $\eta^r_{n,m} \cong \pi^r_{n,m}$ $\forall n = 2,3,\cdots,z$, and $j^r_{z+1,m}$ is onto

.

3- If K is a $(z-1)-connected$, then K is an $I_{z,m}.cw-complex$, and the converse is not true .

Section 1

In this section we introduce the main definition $I_{z,m}.cw-complex$.

Let K be a complex . Consider the following sequence which is known to be exact , see [3] ;

$$\cdots \to \pi_{n+m}(K^{n-1}) \xrightarrow{i_{n,m}} \pi_{n+m}(K^n) \xrightarrow{j_{n,m}} \pi_{n+m}(K^n,K^{n-1}) \xrightarrow{\partial_{n,m}} \pi_{n+m-1}(K^{n-1}) \to \cdots$$

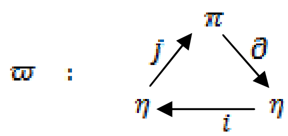
Definition 1.1

We say that a complex K is an $I_{z,m}.cw-complex$ if $i_{n,m}(\pi_{n+m}(K^{n-1})) = 0$ for each $n = 1,2,\cdots,z$.

Remarks 1.2

For each integers n and m , let $\pi_{n,m}$ *be* $\pi_{n+m}(K^n,K^{n-1})$, $\eta_{n,m}$ *be* $\pi_{n+m}(K^n)$, *if* $m \geq -n$, and $\pi_{n+m}(K^n,K^{n-1}) = \pi_{n+m}(K^n) = 0$ *if* $n+m \geq 0$ (see [4]) .

which forms a first exact couple ,



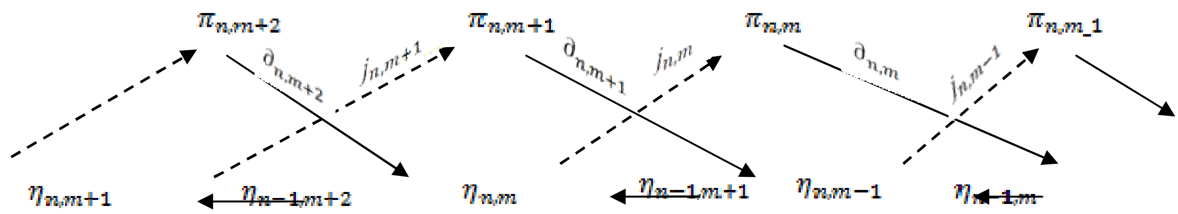
where

∂ is of degree $(-1,0)$

i is of degree $(1,-1)$

j is of degree $(0,0)$

And



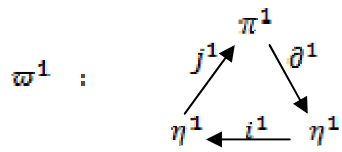
From this exact couple, we obtain a second exact couple, by taking

$$\pi^1_{n,m} = \frac{ker.d_{n,m}}{im.d_{n-1,m}}$$

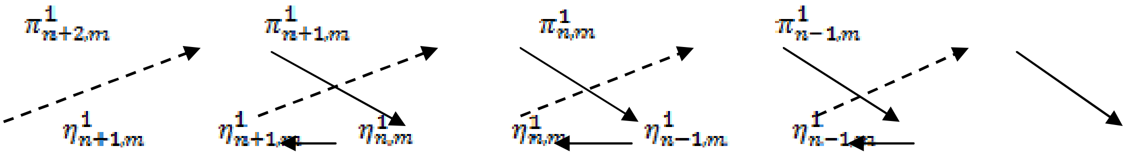
$$\eta^1_{n,m} = \frac{\eta_{n,m}}{im.\partial_{n+1,m}} \quad \text{and} \quad \eta^1_{n,m} = ker.j_{n,m}$$

where $d_{n,m} = j_{n-1,m} \circ \partial_{n,m}$.

Hence we have a second exact couple ;

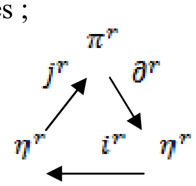


where



The process of derivation can be iterated indefinitely, yielding an infinite sequence of exact couples ;

ϖ^r :



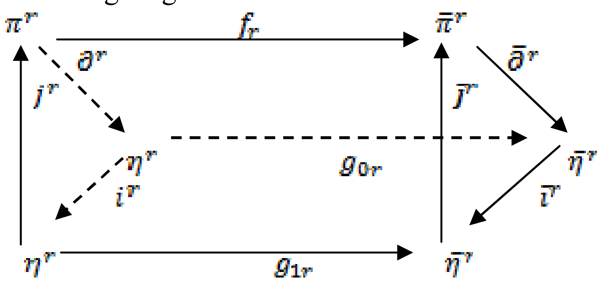
$r = 0,1,2,\cdots$

[1] such that $\varpi^0 = \varpi$, $\varpi^1 = NES$ and ϖ^{r+1} is the derived couple of ϖ^r.

The endomorphism $d^r = j^r \circ \partial^r$ has the property that $d^r \circ d^r = 0$, so that ϖ^r is a chain complex under d^r , whose homology group is ϖ^{r+1}.

In this way we obtain a spectral sequence, (for details see [5]) .

A morphism $\mathcal{F}_r : \varpi^r \longrightarrow \bar{\varpi}^r$ between two exact couples $\varpi^r, \bar{\varpi}^r$, we mean a family of homomorphisms (f_r, g_r) showing in the following diagram



such that

$$\bar{\partial}^r \circ f_r = g_{0r} \circ \partial^r$$
$$\bar{i}^r \circ g_{0r} = g_{1r} \circ i^r$$
$$\bar{j}^r \circ g_{1r} = f_r \circ j^r$$

So that $\mathcal{F}_r$ is a chain morphism and induces a morphism $\mathcal{F}_r^1 = \mathcal{F}_{r+1}$, and f_r^1, g_r^1 defined maps between the derived couples , such that ;

$f_{r+1} : \pi^{r+1} \to \bar{\pi}^{r+1}$ $defined\ by$
$f_{r+1}([z]) = [f_r(z)]$, $\forall [z] \in \pi^{r+1}$.
$g_{0r+1} : \eta^{r+1} \to \bar{\eta}^{r+1}$ $defined\ by$
$g_{0r+1}(x) = g_{0r}(x)$, $\forall x \in \eta^{r+1}$.
$g_{1r+1} : \eta^{r+1} \to \bar{\eta}^{r+1}$ $defined\ by$
$g_{1r+1}([x]) = [g_{1r}(x)]$, $\forall x \in \eta^{r+1}$.

Now, let $\mathcal{F}_r : \varpi^r \longrightarrow \bar{\varpi}^r$ and $\mathcal{F}_r^* : \bar{\varpi}^r \longrightarrow \bar{\bar{\varpi}}^r$ are two morphisms , the composition of morphisms can be given by ; $\mathcal{F}_r^* \circ \mathcal{F}_r = (f_r^* \circ f_r , g_r^* \circ g_r) : \varpi^r \longrightarrow \bar{\bar{\varpi}}^r$.

We shall describe $\mathcal{F}_r : \varpi^r \longrightarrow \bar{\varpi}^r$ is an isomorphism , if and only if , f_r , g_{0r} & g_{1r} are isomorphisms . We shall describe ϖ^r as isomorphic to $\bar{\varpi}^r$ and write $\varpi^r \cong \bar{\varpi}^r$, if and only if , there is an isomorphism $\mathcal{F}_r : \varpi^r \longrightarrow \bar{\varpi}^r$.

Section 2 (Results and Conclusions)

Lemma 2.1

Any complex is an $I_{z,m}.cw - complex$ $if\ m \leq -z$.

Proof

Since $m \leq -z \Rightarrow \pi_{n+m}(K^{n-1}) = 0$ $\forall n = 1,2,\cdots,z$
$\Rightarrow i_{n,m}(\pi_{n+m}(K^{n-1})) = 0$ $\forall n = 1,2,\cdots,z$ ■

Lemma 2.2

If $K = \{e^0\} \Rightarrow K$ is an $I_{z,m}.cw - complex$.

Proof

Since $\pi_{n+m}(K) = \pi_{n+m}(e^0) = 0$ $\forall n, m$
$\Rightarrow i_{n,m}(\pi_{n+m}(K)) = 0$ $\forall n, m$ ■

Lemma 2.3

Let K be a complex , and $j_{n,m} : \pi_{n+m}(K^n) \longrightarrow \pi_{n+m}(K^n, K^{n-1})$.

Then

$ker.j_{n,m} = 0$ $\forall n = 2,3,\cdots,z$ iff
K $is\ an$ $I_{z,m}.cw - complex$.

Proof

We know that $im.i_{n,m} = ker.j_{n,m}$.

Then $ker.j_{n,m} = 0$ $\forall n = 2,3,\cdots,z$
iff $i_{n,m}(\pi_{n+m}(K^n)) = 0$ $\forall n = 2,3,\cdots,$
iff k $is\ a$ $I_{z,m}.cw - complex$ ■

Theorem 2.4

If $K \equiv \bar{K}$ ($\equiv$ of the same homotopy type) , then

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[1] I mean " the new exact sequence" NES , for details see [3] .

K $is\ an\ I_{z,m}.cw-complex \iff \bar{K}\ is\ an\ I_{z,m}.cw-complex$

Proof

Since $K \equiv \bar{K}$ then there are two maps $f: K \to \bar{K}$ and $g: \bar{K} \to K$ such that $g \circ f \approx 1_K$ & $f \circ g \approx 1_{\bar{K}}$

Hence , f & g induce a homomorphisms

$\alpha_n: i_{n,m}(\pi_{n+m}(K)) \to i_{n,m}(\pi_{n+m}(\bar{K}))$ $\forall n$

$\beta_n: i_{n,m}(\pi_{n+m}(\bar{K})) \to i_{n,m}(\pi_{n+m}(K))$ $\forall n$

And $f \circ g$ induce an automorphism of $i_{n,m}(\pi_{n+m}(\bar{K}))$ for each n .

Thus $\alpha_n \circ \beta_n$ is an automorphism , thus α_n maps $i_{n,m}(\pi_{n+m}(K))$ onto $i_{n,m}(\pi_{n+m}(\bar{K}))$, that is

$\alpha_n\left(i_{n,m}(\pi_{n+m}(K))\right) = i_{n,m}(\pi_{n+m}(\bar{K}))$.

Now , if K is an $I_{z,m}.cw-complex$, it follows that

$i_{n,m}(\pi_{n+m}(K)) = 0$ $\forall n \implies i_{n,m}(\pi_{n+m}(\bar{K})) = 0$ $\forall n$

Therefore $\bar{K}$ is an $I_{z,m}.cw-complex$.

Conversely, $g \circ f$ induce an automorphism $\beta_n \circ \alpha_n$ for each n .

Thus β_n maps $i_{n,m}(\pi_{n+m}(\bar{K}))$ onto $i_{n,m}(\pi_{n+m}(K))$, that is

$\beta_n\left(i_{n,m}(\pi_{n+m}(\bar{K}))\right) = i_{n,m}(\pi_{n+m}(K))$.

Now , if $\bar{K}$ is an $I_{z,m}.cw-complex$, it follows that

$i_{n,m}(\pi_{n+m}(\bar{K})) = 0$ $\forall n \implies i_{n,m}(\pi_{n+m}(K)) = 0$ $\forall n$

Therefore K is an $I_{z,m}.cw-complex$ ∎

Theorem 2.5

If $K \equiv_z \bar{K}$ ($\equiv_z$ $z-homotopy\ equivalence$) , then

K $is\ an\ I_{z,m}.cw-complex \iff \bar{K}\ is\ an\ I_{z,m}.cw-complex$

Proof

Since $K \equiv_z \bar{K}$ then there are two cellular maps $f: K^z \to \bar{K}^z$ and $g: \bar{K}^z \to K^z$ such that $g \circ f \approx 1_{K^z}$ & $f \circ g \approx 1_{\bar{K}^z}$

Hence , f & g induce homomorphisms

$\alpha_n: i_{n,m}(\pi_{n+m}(K^{n-1})) \to i_{n,m}(\pi_{n+m}(\bar{K}^{n-1}))$ $\forall n \leq z$

$\beta_n: i_{n,m}(\pi_{n+m}(\bar{K}^{n-1})) \to i_{n,m}(\pi_{n+m}(K^{n-1}))$ $\forall n \leq z$

We may take the homotopy joining $(f \circ g)|_{\bar{K}^{z-1}}$ and $1|_{\bar{K}^{z-1}}$ to be cellular . Hence $(f \circ g)|_{\bar{K}^{n-1}}: \bar{K}^{n-1} \to \bar{K}^n$ induce an automorphism of $i_{n,m}(\pi_{n+m}(\bar{K}^{n-1}))$ for each $n = 2,3,\cdots,z$. Thus $\alpha_n \circ \beta_n$ is an automorphism , thus α_n maps $i_{n,m}(\pi_{n+m}(K^{n-1}))$ onto $i_{n,m}(\pi_{n+m}(\bar{K}^{n-1}))$, that is

$\alpha_n\left(i_{n,m}(\pi_{n+m}(K^{n-1}))\right) = i_{n,m}(\pi_{n+m}(\bar{K}^{n-1}))$

Now , if K is a $I_{z,m}.cw-complex$, it follows that

$i_{n,m}(\pi_{n+m}(K^{n-1})) = 0 \implies i_{n,m}(\pi_{n+m}(\bar{K}^{n-1})) = 0$ $\forall n = 2,3,\cdots,z$

Therefore $\bar{K}$ is a $I_{z,m}.cw-complex$.

Conversely , $(g \circ f)|_{K^{z-1}}: K^{n-1} \to K^n$ induce an automorphism $\beta_n \circ \alpha_n$ for each $n = 2,3,\cdots,z$. Thus β_n maps $i_{n,m}(\pi_{n+m}(\bar{K}^{n-1}))$ onto $i_{n,m}(\pi_{n+m}(K^{n-1}))$, that is

$\beta_n\left(i_{n,m}(\pi_{n+m}(\bar{K}^{n-1}))\right) = i_{n,m}(\pi_{n+m}(K^{n-1}))$

Now , if $\bar{K}$ is a $I_{z,m}.cw-complex$, it follows that

$i_{n,m}(\pi_{n+m}(\bar{K}^{n-1})) = 0 \implies i_{n,m}(\pi_{n+m}(K^{n-1})) = 0$ $\forall n = 2,3,\cdots,z$

Therefore K is an $I_{z,m}.cw-complex$ ∎

The following corollary is a direct consequence from Theorem 2.5 , Lemma 2.3 and Remark 1.2

Corollary 2.6

If $K \equiv_z \bar{K}$, then $\eta_{n,m}(K) = 0 \iff \eta_{n,m}(\bar{K}) = 0$ for each $n = 2,3,\cdots,z$.

Theorem 2.7

If K is an $I_{z,m}.cw-complex$, then for each r we have

$\eta^r_{n,m} \cong \pi^r_{n,m}$ $\forall n = 2,3,\cdots,z$,

and $j^r_{z+1,m}$ $is\ onto$

Proof

Let K be a complex .

First , consider the following sequence which is known to be exact , see [3] ;

$$\cdots \to \pi_{n+m}(K^{n-1}) \xrightarrow{i_{n,m}} \pi_{n+m}(K^n) \xrightarrow{j_{n,m}} \pi_{n+m}(K^n, K^{n-1}) \xrightarrow{\partial_{n,m}} \pi_{n+m-1}(K^{n-1}) \to \cdots$$

By using our symbols in Remark 1.2

Since K is an $I_{z,m}.cw-complex$ and $im.\, i_{n,m} = ker.\, j_{n,m}$

then

$\eta^0_{n,m} \cong \pi^0_{n,m} \quad \forall n = 2,3,\cdots,z$ and $j^0_{z+1,m}$ is onto

. Second , consider the second exact couple ;

$$\cdots \to \eta^1_{z+1,m} \xrightarrow{j^1_{z+1,m}} \pi^1_{z+1,m} \xrightarrow{\partial^1_{z+1,m}} \eta^1_{z,m} \xrightarrow{i^1_{z,m}} \eta^1_{z,m} \xrightarrow{j^1_{z,m}} \pi^1_{z,m} \to \eta^1{}_{z-1,m} \to \cdots$$

Since $\eta^1{}_{n,m} = ker.j_{n,m} = 0$ $\forall n = 2,\cdots,z$.

It follows from exactness , that $\eta^1{}_{n,m} \cong \pi^1{}_{n,m} \quad \forall n = 2,\cdots,z$.

and $im.j^1_{z+1,m} = ker.\partial^1_{z+1,m} = \pi^1_{z+1,m}$.

Therefore $j^1_{z+1,m}$ is onto .

Final , consider the r-th exact couple ;

$$\cdots \to \eta^r_{z+1,m} \xrightarrow{j^r_{z+1,m}} \pi^r_{z+1,m} \xrightarrow{\partial^r_{z+1,m}} \eta^r_{z,m} \xrightarrow{i^r_{z,m}} \eta^r_{z,m} \xrightarrow{j^r_{z,m}} \pi^r_{z,m} \to \eta^r{}_{z-1,m} \to \cdots$$

Since $\eta^r_{n,m} = ker.j^{r-1}_{n,m} = 0$ $\forall n = 2,\cdots,z$.

It follows from exactness , that $\eta^r_{n,m} \cong \pi^r_{n,m} \quad \forall n = 2,\cdots,z$.

and $im.j^r_{z+1,m} = ker.\partial^r_{z+1,m} = \pi^r_{z+1,m}$.

Therefore $j^r_{z+1,m}$ is onto .

So on ■

The following corollary is a direct consequence from Theorem 2.4 and Theorem 2.7

Corollary 2.8

If K is an $I_{z,m}.cw-complex$,and $K \equiv_z \bar{K}$, then

$\eta^r_{n,m}(\bar{K}) \cong \pi^r_{n,m}(\bar{K})$ $\forall p = 2,\cdots,m$ and $\bar{j}^r_{z+1,m}$ is onto ,$\forall r$.

Lemma 2.9

Let K be a $(z-1)-connected$, then K is a $I_{z,m}.cw-complex$.

Proof

Suppose that K is an $(z-1)-connected$, that is $\pi_i(K) = 0$ $for\ each$ $i = 1,2,\cdots,z-1$. So there is a singleton complex L such that $K^z \equiv_{z-1} L$ $(i.e.\ K \equiv_z L = \{e^0\})$. (_Theorem 2_ in [1])

Hence , from **Lemma 2.2** we have L is an $I_{z,m}.cw-complex$, and from **Theorem 2.4** it follows that K is an $I_{z,m}.cw-complex$ ■

Remarks 2.10

(1) If $m < 0$, then K is a $(z-1)-connected \Leftrightarrow$ K is an $I_{z,m}.cw-complex$

.[2] If $m \geq 0$, then K is a $(z-1)-connected \Rightarrow$ K is an $I_{z,m}.cw-complex$

The converse in general is not true , (depend on values of $n\ \&\ m$) .

[1] If m=0 , coming back to $J_z-complex$ of J.H.C. Whitehead , [1] .

Example 2.11

Let $K = e^0 \cup e^3 \cup e^4$, where e^0 is a 0-cell , e^3 is a 3-cell whose closure is a 3-sphere , $S^3 = e^0 \cup e^3$, and e^4 is attached to S^3 by a map, $f: \partial e^4 \to S^3$, of degree $(2r+1)\ (r > 0)$.

The complex K is not 3-connected (since $\pi_3(K) = \mathbb{Z}_{2r+1}$), while K is an $I_{4,0}.cw-comlpex$, since $K^2 = K^1 = K^0 = e^0$, $whence$ $\pi_n(K^{n-1}) = 0$ $\forall n = 1,2,3$ and $i_4(\pi_3(K^3)) = 0$, for , let $g: S^4 \to \partial e^4$ be an essential map , then $f \circ g: S^4 \to S^3$ is essential and hence represents the nonzero element of $\pi_4(S^3)$, [6] .

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