*Injective Modules Relative To a Preradical

By

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Abstract

The concept of ρ-injective modules (where ρ is a preradical) is introduced in this work as a generalization of injective modules. The definition of ρ-injectivity unifies several definitions on generalizations of injectivity such as nearly injective modules and special injective modules. Many characterizations and properties of ρ-injectivity are given. We study the endomorphisms rings of ρ-injective modules. The results of this work unify and extend many results in the literature.

Keywords: Injective modules; nearly-injective modules; preradical; endomorphisms ring.

1. Introduction:

Throughout this work, R stands a commutative ring with identity element 1 and a module means a unitary left R-modules. The class of all R-module will be denoted by R-Mod and the symbol ρ means a preradical on R-Mod (A preradical ρ is defined to be a subfunctor of the identity functor of R-Mod). For an R-module M, the notations $\mathrm{J}(M)$, $\mathrm{L}(M)$, $\mathrm{E}(M)$ and $S = \mathrm{End}_R(M)$ will respectively stand for the Jacobson radical of M, the prime radical of M, the injective envelope of M and the endomorphism ring of M. The notation $\mathrm{Hom}_R(N, M)$ denoted to the set of all R-homomorphism from R-module N into R-module M. An R-module M is called injective, if for every R-monomorphism $f: A \to B$ (where A and B are R-modules) and every R-monomorphism $g: A \to M$, there exists an R-homomorphism $h: B \to M$ such that $g = h \circ f$ [1].

Injective modules have been studied extensively, and several generalizations for these modules are given, for example, quasi-injective modules [2], P-injective Modules [3], and S-injective module [4].

In 2000, nearly-injective modules were discussed in [5] as generalization of injective modules. An R-module M is said to be nearly injective if for each R-monomorphism $f: A \to B$ (where A and B are two R-modules), each R-homomorphism $g: A \to M$, there exists an R-homomorphism $h: B \to M$ such that $(h\, o\, f)(a) - g(a) \in \mathrm{J}(M)$,for all $a \in A$ [5].

Also, in [6] M. S. Abbas and Sh. N. Abd-Alridha introduced the concept of special injective modules as a generalization of injectivity. An R-module M is said to be special injective if for each R-monomorphism $f: A \to B$ (where A and B are two R-modules), each R-homomorphism $g: A \to M,$ there exists an R-homomorphism $h: B \to M$ such that $(h\, o\, f)(a) - g(a) \in \mathrm{L}(M)$,for all $a \in A$ [6].

A ring R is called Von Neumann regular (in short, regular) if for each $a \in R$, there exsits $b \in R$ such that $a = aba$. For a submodule N of an R-module M and $a \in M$, $[N:_R a] = \{r \in R \mid ra \in N\}$. For an R-module M and $a \in M$. A submodule N of an R-module M is called essential and denoted by $N \leq^e M$ if every non zero submodule of M has nonzero intersection with N.

* The results of this paper will be part of a MSc thesis of the second author, under the supervision of the first author at the University of Al-Qadisiyah.

2. Injective Modules Relative to a Preradical

In this section, we will introduce a new generalization of injective module namely, injective module relative to a preradical. We will study some properties and characterizations of these modules.

We start by the following definition:-

Definition 2.1. Let ρ be a preradical on R-Mod and let M, N and K be R-modules. A module M is said to be N-injective relative to the preradical ρ (shortly, ρ-N-injective) if for each R-monomorphism $f: K \to N$ and each R-homomorphism $g: K \to M$ there is an R-homomorphism $h: N \to M$ such that $(hof)(x) - g(x) \in \rho(M)$, for each x in K.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "g"'] & N \arrow[dl, dashed, "h"] \\ & M & \end{tikzcd}$$

An R-module M is said to be injective relative to the preradical ρ (shortly, ρ-injective) if M is ρ-N-injective for all R-modules N. A ring R is said to be ρ-injective ring, if R is a ρ-injective R-module.

Examples and Remarks 2.2.
(1) It is clear that injective modules and N-injective modules are ρ-N-injective for every R-module N.
(2) There are many types of preradical functors, for examples: the Jacobson radical functor (J), the socle functor (soc), the prime radical functor (L) and the torsion functor (T) [7]. Each one of these functors gives a special case of ρ-injective modules, for example a left R-module M is said to be (soc)-injective if M is ρ-injective, where $\rho = \text{soc}$.
(3) The concept of nearly-injective module (which is introduced in [5]) is a special case of ρ-injective R-modules by taking $\rho = \text{J}$, where J is the Jacobson radical functor.
(4) Special injective modules (which are introduced in [6]) are special case of ρ-injectivity by taking $\rho = \text{L}$, where L is the prime radical functor.
(5) Let M be a module such that $\rho(M) = 0$, thus M is injective if and only if M is ρ-injective.
(6) It is clear that if $\rho(M) = M$, then M is a ρ-injective module, in particular:
(a) Every module M which has no maximal submodule (i.e, $\text{J}(M) = M$) is J-injective.
(b) Every semisimple module M (i.e., $\text{soc}(M) = M$) is (soc)-injective. Thus ρ-injective modules may not be injective, for example: let $M = \mathbb{Z}_p$ as $\mathbb{Z}$-module, where p is a prime number. Since M is semisimple, thus $\text{soc}(M) = M$ and hence M is (soc)-injective but M is not injective.
(7) Let M_1 be an R-module. If $M_1$is a ρ-N-injective R-module and M_1 is isomorphic to M_2, then M_2 is a ρ-N-injective.
(8) Form (7) above we have that ρ-injectivity is an algebraic property.
(9) Every submodule of semisimple R-module is ρ-injective, where ρ is the socle functor.

Lemma 2.3. Let N and M be R-modules. Then the following statements are equivalent:
(1) M is ρ-N-injective;
(2) for any diagram,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "g"'] & N \arrow[dl, dashed, "h"] \\ & M & \end{tikzcd}$$

where A is a submodule of an R-module N, $g: A \to M$ is any R-homomorphism and i is the inclusion mapping, there exists an R-homomorphism $h: N \to M$ such that $(h \circ i)(a) - g(a) \in \rho(M)$, for all a in A.
Proof: The proof is obvious. ☐

In the following proposition we show that the set of all essential submodules of N is a test set for ρ-N-injectivity.

Proposition 2.4. Let N be an R-module. Then an R-module M is ρ-N-injective if and only if for each essential submodule A of N and each R-homomorphism $f: A \to M$, there is an R-homomorphism $g: N \to M$ such that $(g \circ i)(a) - f(a) \in \rho(M)$ for each a in A.

Proof: ($\Longrightarrow$) This is obvious.

($\Longleftarrow$) Let A be any essential submodule of N and $f: A \to M$ be any R-homomorphism. Consider the diagram (1).

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & N \\ & M & \end{tikzcd} \qquad \text{(diagram (1))}$$

Let A^c be any complement submodule of A in N. By [8, p.16], we have that $A \oplus A^c \leq^e N$. Define $g: A \oplus A^c \to M$ by $g(a + a_1) = f(a)$, for all $a \in A$ and $a_1 \in A^c$. It is easy to prove that g is a well-defined R-homomorphism. Therefore, we have the diagram (2).

$$\begin{tikzcd} 0 \arrow[r] & A \oplus A^c \arrow[r, "i"] \arrow[d, "g"'] & N \arrow[dl, dashed, "h"] \\ & M & \end{tikzcd}$$

By hypothesis, there exists an R-homomorphism $h: N \to M$ such that $(h \circ i)(x) - g(x) \epsilon\, \rho(M)$ for all x in $A \oplus A^c$. For the diagram (1), we get that $(h \circ i)(a) - f(a) = (h \circ i)(a) - g(a) \in \rho(M)$ for all a in A. Therefore, M is a ρ-N-injective R-module, by Lemma 2.3. $\square$

Now, we will study the direct product and the direct sum of ρ-N-injective modules.

Proposition 2.5. Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of R-modules. Then :

(1) if $\prod_{\lambda \in \Lambda} M_\lambda$ is a ρ-N-injective (where N is an R-module), then each M_λ is ρ-N-injective.

(2) if $\rho(\prod_{\lambda \in \Lambda} M_\lambda) = \prod_{\lambda \in \Lambda}(\rho(M_\lambda))$, then the converse of (1) is true.

Proof: **(1)** Put $M = \prod_{\lambda \in \Lambda} M_\lambda$ and let $i_\lambda: M_\lambda \to M$ and $p_\lambda: M \to M_\lambda$ be the injections and projections associated with this direct product respectively. Suppose that M is ρ-N-injective. To prove that M_λ is ρ-N-injective for each $\lambda \in \Lambda$. Consider the following diagram where A is a submodule of N and α_λ is an R-homomorphism.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\alpha_\lambda"'] & N \arrow[dl, dashed, "g_\lambda"'] \arrow[ddl, dashed, "h"] \\ & M_\lambda \arrow[d, shift right, "i_\lambda"'] & \\ & M \arrow[u, shift right, "p_\lambda"'] & \end{tikzcd}$$

Since M is a ρ-N-injective module, thus there exists an R-homomorphism $h: N \to M$ such that $(h \circ i)(a) - (i_\lambda \circ \alpha_\lambda)(a) \in \rho(M)$ for all a in A. Put $g_\lambda = p_\lambda \circ h : N \to M_\lambda$. For every a in A, we have that $(g_\lambda \circ i)(a) - \alpha_\lambda(x) = g_\lambda(a) - \alpha_\lambda(a) = (p_\lambda \circ h)(a) - \alpha_\lambda(a) = (p_\lambda \circ h)(a) - ((p_\lambda \circ i_\lambda) \circ \alpha_\lambda)(a) = p_\lambda(h(a) - (i_\lambda \circ \alpha_\lambda)(a)) \in \rho(M_\lambda)$.

Thus $(g_\lambda \circ i)(a) - \alpha_\lambda(a) \in \rho(M_\lambda)$, for each $\lambda \in \Lambda$ and for every $a \in A$ and hence M_λ is ρ-N-injective, for each $\lambda \in \Lambda$.

(2) Suppose that $\rho(\prod_{\lambda \in \Lambda} M_\lambda) = \prod_{\lambda \in \Lambda}(\rho(M_\lambda))$ and consider the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\alpha"'] & N \arrow[dl, dashed, "g"'] \arrow[ddl, dashed, "g_\lambda"] \\ & M \arrow[d, "p_\lambda"'] & \\ & M_\lambda & \end{tikzcd}$$

For each $\lambda \in \Lambda$, let $p_\lambda: M \to M_\lambda$ be the projection R-homomorphism. Since each M_λ is ρ-N-injective, thus there exists an R-homomorphism $g_\lambda: N \to M_\lambda$, for each $\lambda \in \Lambda$ such that $(g_\lambda \circ i)(a) - (p_\lambda \circ \alpha)(a) \in \rho(M_\lambda)$, for every a in A. Define $g: N \to M$ by $g(x) = \{g_\lambda(x)\}_{\lambda \in \Lambda}$, for every $x \in N$. It is clear that g is an R-homomorphism. For every a in A, we have that

$(g \circ i)(a) - \alpha(a) = \{g_\lambda(i(a))\}_{\lambda \in \Lambda} -$

$\{(p_\lambda \circ \alpha)(a)\}_{\lambda\in\Lambda} = \{(g_\lambda \circ i)(a) - (p_\lambda \circ \alpha)(a)\}_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda}(\rho(M_\lambda))$. Since $\prod_{\lambda\in\Lambda}(\rho(M_\lambda)) = \rho(\prod_{\lambda\in\Lambda} M_\lambda)$ (by hypothesis) it follows that $(g \circ i)(a) - \alpha(a) \in \rho(M)$, for every a in A. Therefore, M is a ρ-N-injective module. □

Corollary 2.6. Let R be a ring such that $R/J(R)$ is a semisimple R-module, let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of R-modules and let N be any R-module. Then $\prod_{\lambda\in\Lambda} M_\lambda$ is (soc)-N-injective if and only if M_λ is (soc)-N-injective, for each $\lambda \in \Lambda$.

Proof: Since $R/J(R)$ is a semisimple R-module, $\mathrm{soc}(\prod_{\lambda\in\Lambda} M_\lambda) = \prod_{\lambda\in\Lambda} \mathrm{soc}(M_\lambda)$ [7, Exercise (11), p.239]. Therefore, the result follows from Proposition 2.5. □

Corollary 2.7. Let R be a ring and let I be a finitely generated ideal of R. Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of R-modules and let N be R-module. Then $\prod_{\lambda\in\Lambda} M_\lambda$ is ρ_I-N-injective if and only if M_λ is ρ_I-N-injective.

Proof: Since I is a finitely generated ideal of R it follows from [9, Exercise 3(1), p.174] that $I(\prod_{\lambda\in\Lambda} M_\lambda) = \prod_{\lambda\in\Lambda}(IM_\lambda)$ and hence $\rho_I(\prod_{\lambda\in\Lambda} M_\lambda) = \prod_{\lambda\in\Lambda}(\rho_I(M_\lambda))$. Therefore, the result follows from Proposition 2.5. □

For any family $\{M_\lambda\}_{\lambda\in\Lambda}$ of R-modules, if $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is an N-injective R-module, then each M_λ is an N-injective and the converse is true, if Λ is finite by [3, Proposition(1.11), p. 6].

The following proposition shows that this result is true in case of ρ-N-injectivity.

Proposition 2.8. Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of R-modules, let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ and let N be any R-module.

(1) If M is ρ-N-injective, then each M_λ is ρ-N-injective.

(2) If Λ is a finite set, then the converse of (1) is true.

Proof: Suppose that M is a ρ-N-injective module. To prove that each M_λ is ρ-N-injective.

(1) Let $i_\lambda: M_\lambda \to M$ and $p_\lambda: M \to M_\lambda$ be the injections and projections associated with this direct product respectively. Consider the following diagram, where A is a submodule of N and α_λ is an R-homomorphism.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\alpha_\lambda"'] & N \arrow[dl, dashed, "g_\lambda"'] \arrow[ddl, dashed, "h"] \\
& M_\lambda \arrow[d, shift right, "i_\lambda"'] & \\
& M \arrow[u, shift right, "p_\lambda"'] &
\end{tikzcd}$$

Since M is ρ-N-injective, there exists an R-homomorphism $h: N \to M$ such that $(h \circ i)(a) - (i_\lambda \circ \alpha_\lambda)(a) \in \rho(M)$, for all a in A. For each $\lambda \in \Lambda$, put $g_\lambda = p_\lambda \circ h: N \to M_\lambda$. For every a in A, we have that $(g_\lambda \circ i)(a) - \alpha_\lambda(a) = g_\lambda(a) - \alpha_\lambda(a) = (p_\lambda \circ h)(a) - \alpha_\lambda(a) = (p_\lambda \circ h)(a) - ((p_\lambda \circ i_\lambda) \circ \alpha_\lambda)(a) = (p_\lambda \circ h)(a) - (p_\lambda(i_\lambda \circ \alpha_\lambda)(a)) = p_\lambda(h(a) - (i_\lambda \circ \alpha_\lambda)(a)) \in \rho(M_\lambda)$ (because ρ is a preradical). Thus $g_\lambda(a) - \alpha_\lambda(a) \in \rho(M_\lambda)$, for each $\lambda \in \Lambda$ and for every $a \in A$. Therefore, M_λ is ρ-N-injective, for each $\lambda \in \Lambda$.

(2) Suppose that Λ is a finite set. Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of ρ-N-injective modules. Since Λ is finite it follows from [7, p.82] that $\bigoplus_{\lambda\in\Lambda} M_\lambda = \prod_{\lambda\in\Lambda} M_\lambda$. Since $\rho(\bigoplus_{\lambda\in\Lambda} M_\lambda) = \bigoplus_{\lambda\in\Lambda} \rho(M_\lambda)$ (by [10, Proposition 2, p.76]) it follows that $\rho(\prod_{\lambda\in\Lambda} M_\lambda) = \prod_{\lambda\in\Lambda}\rho(M_\lambda)$. By Proposition 2.5 (2), $\prod_{\lambda\in\Lambda} M_\lambda$ is ρ-N-injective and hence $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is ρ-N-injective. □

The following corollary is immediate from Proposition 2.8(1).

Corollary 2.9. Let M be a ρ-N-injective R-module and let K be a direct summand of M. Then K is a ρ-N-injective R-module. □

Corollary 2.10. Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of R-modules and let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ .Then

(i) (1) If ρ is a preradical and $M/\rho(M)$ is ρ-N-injective, then each $M_\lambda/\rho(M_\lambda)$ is ρ-N-injective.

(2) If ρ is a radical and $M/\rho(M)$ is ρ-N-injective, then each $M_\lambda/\rho(M_\lambda)$ is N-injective.

(ii) (1) If ρ is a preradical, then $M_\lambda/\rho(M_\lambda)$ is ρ-N-injective and Λ is a finite set, then $M/\rho(M)$ is ρ-N-injective.

(2) If ρ is a radical, each $M_\lambda/\rho(M_\lambda)$ is ρ-N-injective and Λ is a finite set, then $M/\rho(M)$ is N-injective.

Proof: (i)(1) Suppose that ρ is a preradical and $M/\rho(M)$ is a ρ-N-injective R-module. Since $M/\rho(M) = \bigoplus_{\lambda\in\Lambda}\left(M_\lambda/\rho(M_\lambda)\right)$ and $M/\rho(M)$ is ρ-N-injective (by hypothesis) it follows that $\bigoplus_{\lambda\in\Lambda}\left(M_\lambda/\rho(M_\lambda)\right)$ is ρ-N-injective. By Proposition 2.8(1) , $M_\lambda/\rho(M_\lambda)$ is ρ-N-injective, for all $\lambda\in\Lambda$.

(i)(2) Suppose that ρ is a radical and $M/\rho(M)$ is a ρ-N-injective module. By (i)(1), $M_\lambda/\rho(M_\lambda)$ is ρ-N-injective, for all $\lambda\in\Lambda$. Since ρ is a radical, $\rho\left(M_\lambda/\rho(M_\lambda)\right) = 0$ and hence $M_\lambda/\rho(M_\lambda)$ is N-injective, for all $\lambda\in\Lambda$.

(ii)(1) Suppose that ρ is a preradical, each $M_\lambda/\rho(M_\lambda)$ is ρ-N-injective and Λ is a finite set. By Proposition 2.8(2), $\bigoplus_{\lambda\in\Lambda}\left(M_\lambda/\rho(M_\lambda)\right)$ is ρ-N-injective. Since $\bigoplus_{\lambda\in\Lambda}\left(M_\lambda/\rho(M_\lambda)\right) = \bigoplus_{\lambda\in\Lambda} M_\lambda/\bigoplus_{\lambda\in\Lambda}\rho(M_\lambda) = M/\rho(\bigoplus_{\lambda\in\Lambda} M_\lambda) = M/\rho(M)$ it follows that $M/\rho(M)$ is ρ-N-injective.

(ii(2)) Suppose that ρ is a radical, each $M_\lambda/\rho(M_\lambda)$ is ρ-N-injective and Λ is a finite set. By (ii(1)), $M/\rho(M)$ is ρ-N-injective. Since ρ is a radical, $\rho\left(M_\lambda/\rho(M_\lambda)\right) = 0$ and hence $M_\lambda/\rho(M_\lambda)$ is N-injective. ☐

Examples 2.11.

(1) The converse of Proposition 2.8(1) is not true in general. For example, let Λ be an infinite countable index set and let $T_\lambda = Q$ for all $\lambda\in\Lambda$ (where Q is the field of rational numbers). Let $R = \prod_{\lambda\in\Lambda} T_\lambda$ be the ring product of the family $\{T_\lambda | \lambda\in\Lambda\}$. It is easy to prove that R is a regular ring. For $k\in\Lambda$, let e_k be the element of R whose kth-component is 1 and whose remaining components are 0.

Let $A = \bigoplus_{\lambda\in\Lambda} Re_\lambda$, it is clear that A is a submodule of an R-module R. By [7, p.140], A is a direct sum of injective R-modules, but A is not injective R-module. Since every injective R-module is ρ-injective, thus A is a direct sum of ρ-injective R-modules. Let ρ be any J-preradical. Assume that A is ρ-injective. Since R is a regular ring, thus $\mathrm{J}(A) = 0$ (by [7, p.272]). Since ρ is a J-preradical, thus $\rho(A) = 0$ and hence A is injective and this is a contradiction. Thus A is not ρ-injective. Therefore, A is a direct sum of ρ-injective modules, but it is not ρ-injective.

(2) Let $M = Q\oplus\mathbb{Z}$. Thus M is not ρ-injective $\mathbb{Z}$-module, where ρ is a J-preradical. In fact, if M is ρ-injective, then by Proposition 2.8(1) we have $\mathbb{Z}$ is ρ-injective $\mathbb{Z}$-module and hence $\mathbb{Z}$ is an injective $\mathbb{Z}$-module (because $\rho(\mathbb{Z}) = \mathrm{J}(\mathbb{Z}) = 0$) and this is a contradiction. Thus M is not ρ-injective $\mathbb{Z}$-module.

In following, we will introduce further characterizations of ρ-injective modules.

Recall that a submodule N of an R-module M is said to be a direct summand of M if there exists a submodule K of M such that $M = N\oplus K$, (i.e., $M = N + K$ and $N\cap K = 0$) [7]. This is equivalent to saying that, for every commutative diagram with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\ 0 \arrow[r] & N \arrow[r, "\beta"] & M \end{tikzcd}$$

(where A and B are two R-modules), there exists an R-homomorphism $h: B\to N$ such that $f = h\circ\alpha$ [11]. It is well-known that an R-module M is injective if and only if M is a direct summand of every extension of it self [1, Theorem (2.1.5)].

For analogous result for ρ-injective R-modules, we introduce the following concept as a generalization of direct summands.

Definition 2.12. A submodule N of an R-module M is said to be ρ-direct summand of M if for every commutative diagram with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\ 0 \arrow[r] & N \arrow[r, "\beta"] & M \end{tikzcd}$$

(where A and B are two R-modules), there exists an R-homomorphism $h: B \to N$ such that $(h \circ \alpha)(a) - f(a) \epsilon\, \rho(N)$, for all a in A.

Proposition 2.13. Let N be a submodule of an R-module M. Then the following statements are equivalent:-

(1) N is ρ-direct summand of M;

(2) for each diagram with exact row,

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, "I_N"'] & M \arrow[dl, dashed, "h"] \\ & N & \end{tikzcd}$$

where I_N is the identity homomorphism of N, there exists an R-homomorphism $h: M \to N$ such that $(h \circ \alpha)(a) - a \in \rho(N)$,for all $a \in N$.

Proof: $(\mathbf{1}) \Rightarrow (\mathbf{2})$ Suppose that N is a ρ-direct summand of M and consider the following diagram with exact row.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, "I_N"'] & M \\ & N & \end{tikzcd}$$

Thus we have the following commutative diagram with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, "I_N"'] & M \arrow[d, "I_M"] \arrow[dl, dashed, "h"'] \\ 0 \arrow[r] & N \arrow[r, "\alpha"] & M \end{tikzcd}$$

By hypothesis, there exists a homomorphism $h: M \to N$ such that $(h \circ \alpha)(a) - I_N(a) \epsilon\, \rho(N)$, for all a in A and hence $(h \circ \alpha)(a) - a \in \rho(N)$, for all a in N.

$(\mathbf{2}) \Rightarrow (\mathbf{1})$ Consider the following commutative diagram with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ 0 \arrow[r] & N \arrow[r, "\beta"] & M \end{tikzcd}$$

Thus we have the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "h_1"'] \\ 0 \arrow[r] & N \arrow[r, "\beta"] \arrow[d, "I_N"'] & M \arrow[dl, dashed, "h"] \\ & N & \end{tikzcd}$$

By hypothesis, there exists a homomorphism $h: M \to N$ such that $(h \circ \beta)(a) - a \in \rho(N)$, for all $a \in N$. Put $h_1 = h \circ g: B \longrightarrow N$. It is clear that h_1 is a homomorphism. Let $a \in A$, thus $(h_1 \circ \alpha)(a) - f(a) = \big((h \circ g) \circ \alpha\big)(a) - f(a) = \big(h \circ (g \circ \alpha)\big)(a) - f(a) = \big(h \circ (\beta \circ f)\big)(a) - f(a) = (h \circ \beta)\big(f(a)\big) - f(a) \in \rho(N)$. Hence $(h_1 \circ \alpha)(a) - f(a) \in \rho(N)$, for all a in A and this implies that N is a ρ-direct summand of M. ☐

In the following theorem we will give a characterization of ρ-injective modules, by using ρ-direct summands.

Theorem 2.14. For an R-module M, the following statements are equivalent:

(1) M is ρ-injective.

(2) M is a ρ-direct summand of every extension of itself.

(3) M is a ρ-direct summand of every injective extension of itself.

(4) M is a ρ-direct summand of at least, one injective extension of itself.

(5) M is a ρ-direct summand of $\mathrm{E}(M)$, where $\mathrm{E}(M)$ is the injective hull of M.

Proof:- $(\mathbf{1}) \Longrightarrow (\mathbf{2})$ Suppose that M is a ρ-injective R-module and let M_1 be any extension R-module of M. We will prove that

M is ρ-direct summand of M_1. Consider the following diagram with exact row.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, "I_M"'] & M_1 \arrow[dl, dashed, "f"] \\ & M & \end{tikzcd}$$

Since M is ρ-injective, there exists an R-homomorphism $f\colon M_1 \to M$ such that $(f \circ \alpha)(a) - a \in \rho(M)$, for all $a \in M$. Thus Proposition 2.13. implies that M is a ρ-direct summand of M_1.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$ and $(\mathbf{3}) \Rightarrow (\mathbf{4})$ are clear.

$(\mathbf{4}) \Rightarrow (\mathbf{1})$ Suppose that M is a ρ-direct summand of at least, one injective extension R-module of M, say E. To prove that M is a ρ-injective module. Consider the diagram (1) with exact row, where A and B are R-modules and $f\colon A \to M$ is an R-homomorphism.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \\ & M & \end{tikzcd}$$

(diagram (1))

Since E is an extension of M, there is an R-monomorphism, say $\beta\colon M \longrightarrow E$. Thus we have the diagram (2)

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[ddl, dashed, "g"] \\ & M \arrow[d, "\beta"'] & \\ & E & \end{tikzcd}$$

(diagram (2))

Since E is an injective R-module, there exists an R-homomorphism $g\colon B \to E$ such that $(g \circ \alpha)(a) = (\beta \circ f)(a)$ for all a in A. Thus we have the commutative diagram (3) with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\ 0 \arrow[r] & M \arrow[r, "\beta"] & E \end{tikzcd}$$

(diagram (3))

Since M is a ρ-direct summand of E (by hypothesis), thus there exists a homomorphism $h\colon B \to M$ such that $(h \circ \alpha)(a) - f(a) \in \rho(M)$, for all $a \in A$. Thus, for the diagram (1), we get a homomorphism $h\colon B \to M$ such that $(h \circ \alpha)(a) - f(a) \in \rho(M)$, for all a in A. Therefore, M is ρ-injective.

$(\mathbf{3}) \Rightarrow (\mathbf{5})$ This is clear.

$(\mathbf{5}) \Rightarrow (\mathbf{1})$ Suppose that M is a ρ-direct summand of $\mathrm{E}(M)$. Since $\mathrm{E}(M)$ is an injective extension of M, thus M is a ρ-direct summand of at least, one injective extension of itself. ☐

In the following corollary we will give an inner characterization of ρ-injective modules, for the term inner see [7].

Corollary 2.15. An R-module M is ρ-injective if and only if M is a ρ-direct summand of an R-module $\mathrm{Hom}_{\mathbb{Z}}(R, B)$, with B is a divisible Abelian group.

Proof: $(\Rightarrow)$ Suppose that M is ρ-injective. By [7, p.91], there is a $\mathbb{Z}$-monomorphism $f\colon M \to B$, where B is a divisible Abelian group. Thus Lemma (5.5.2) in [7] implies that $\mathrm{Hom}_{\mathbb{Z}}(R, B)$ is an injective R-module.

Define $\theta\colon M \to \mathrm{Hom}_{\mathbb{Z}}(R, B)$ by $\theta(m)(r) = f(rm)$, for all $m \in M$ and for all $r \in R$. It is easy to see that θ is an R-monomorphism and hence $\mathrm{Hom}_{\mathbb{Z}}(R, B)$ is an extension R-module of M. Since M is a ρ-injective R-module, thus Theorem 2.14. implies that M is a ρ-direct summand of an R-module $\mathrm{Hom}_{\mathbb{Z}}(R, B)$.

$(\Leftarrow)$ Suppose that M is a ρ-direct summand of an R-module $\mathrm{Hom}_{\mathbb{Z}}(R, B)$ with B is a divisible Abelian group. By [7, Lemma (5.5.2)], we have that $\mathrm{Hom}_{\mathbb{Z}}(R, B)$ is an injective R-module. Thus M is a ρ-direct summand of an injective extension R-module. Therefore, M is a ρ-injective R-module, by Theorem 2.14. ☐

An R-monomorphism $\alpha\colon \mathrm{N} \to \mathrm{M}$ (where N and M are R-modules) is called split, if there exists an R-homomorphism $\beta\colon \mathrm{M} \to \mathrm{N}$ such that $\beta \circ \alpha = \mathrm{I_N}$ [7].

An R-module M is injective if and only if for every R-module N, each R-monomorphism $\alpha\colon M \to N$ is split [7].

For analogous result for ρ-injective modules, we introduce the following concept.

Definition 2.16. An R-monomorphism $\alpha: N \to M$ is said to be ρ-split, if there exists an R-homomorphism $\beta: M \to N$ such that $(\beta \circ \alpha)(a) - a \in \rho(N)$, for all a in N.

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, "I_N"'] & M \arrow[dl, dashed, "\beta"] \\
& N &
\end{tikzcd}$$

The following theorem gives and characterization of ρ-injectivity by using ρ-split monomorphisms.

Theorem 2.17. The following statements are equivalent for an R-module M:

(1) M is ρ-injective;

(2) for each R-module N, each R-monomorphism $\alpha: M \to N$ is a ρ-split;

(3) for each injective R-module N, each R-monomorphism $\alpha: M \to N$ is a ρ-split;

(4) each R-monomorphism $\alpha: M \to \mathrm{E}(M)$ is ρ-split.

Proof: $(\mathbf{1}) \Rightarrow (\mathbf{2})$ Suppose that M is a ρ-injective R-module. Let N be any R-module and let $\alpha: M \to N$ be any R-monomorphism. Consider the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, "I_M"'] & N \arrow[dl, dashed, "\beta"] \\
& M &
\end{tikzcd}$$

Since M is ρ-injective, there exists an R-homomorphism $\beta: N \to M$ such that $(\beta \circ \alpha)(a) - a \in \rho(M)$, for all $a \in M$. Hence α is a ρ-split.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$ and $(\mathbf{3}) \Rightarrow (\mathbf{4})$ are obvious.

$(\mathbf{4}) \Rightarrow (\mathbf{1})$ Suppose that each R-monomorphism $\alpha: M \to E(M)$ is a ρ-split. To prove that M is a ρ-injective. Consider the following diagram with exact row, where A and B are R-modules and $g: A \to M$ is any R-homomorphism.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"'] & B \\
& M &
\end{tikzcd}$$

Since $\mathrm{E}(M)$ is an extension of M, thus there is a monomorphism, say $\alpha: M \to \mathrm{E}(M)$ and hence we get the following diagram with exact row.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, dashed, "h_1"'] \arrow[ddl, dashed, "h"] \\
& M \arrow[d, shift right, "\alpha"'] & \\
& \mathrm{E}(M) \arrow[u, shift right, "\beta"'] &
\end{tikzcd}$$

Since $\mathrm{E}(M)$ is an injective module, there exists a homomorphism $h: B \to E(M)$ such that $(h \circ f)(a) = (\alpha \circ g)(a)$, for all $a \in A$. By hypothesis, we have $\alpha: M \to \mathrm{E}(M)$is a ρ-split and hence there exists a homomorphism $\beta: E(M) \to M$ such that $(\beta \circ \alpha)(a) - a \in \rho(M)$, for all $a \in M$. Put $h_1 = \beta \circ h$, it is clear that h_1 is an R-homomorphism. For each a in A, we have that $(h_1 \circ f)(a) - g(a) = \big((\beta \circ h) \circ f\big)(a) - g(a) = \big(\beta(h \circ f)\big)(a) - g(a) = \big(\beta(\alpha \circ g)\big)(a) - g(a) = (\beta \circ \alpha)\big(g(a)\big) - g(a) \in \rho(M)$. Thus $(h_1 \circ f)(a) - g(a) \in \rho(M)$, for all $a \in A$ and hence M is a ρ-injective module. ☐

The following proposition gives a characterization of ρ-injective modules by using the class of injective modules.

Proposition 2.18. The following statements are equivalent for an R-modules M:

(1) M is ρ-injective;

(2) M is ρ-B-injective, for every injective module B;

(3) for each diagram with B is an injective R-module and A is an essential submodule in B,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[dl, dashed, "g"] \\ & M & \end{tikzcd}$$

there exists a homomorphism $g: B \to M$ such that $(g \circ i)(a) - f(a) \in \rho(M)$, for all $a \in A$.

Proof: $(\mathbf{1}) \Rightarrow (\mathbf{2})$ and $(\mathbf{2}) \Rightarrow (\mathbf{3})$ are obvious.

$(\mathbf{3}) \Rightarrow (\mathbf{1})$ Consider the following diagram with B is any R-module and A is any essential submodule in B.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i_A"] \arrow[d, "f"'] & B \\ & M & \end{tikzcd}$$

By [1], there exists an injective R-module say E, such that B is an essential submodule in E. Thus we have the following diagram,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i_A"] \arrow[d, "f"'] & B \arrow[r, "i_B"] \arrow[dl, dashed, "g"'] & E \arrow[dll, dashed, "h"] \\ & M & & \end{tikzcd}$$

where i_A and i_B are inclusion R-homomorphisms. Since $A \leq^e B$ (by hypothesis) and $B \leq^e E$ it follows from [8] that $A \leq^e E$. By hypothesis, there exists an R-homomorphism $h: E \to M$ such that $(h \circ i_B \circ i_A)(a) - f(a) \in \rho(M)$, for all $a \in A$. Put $g = h \circ i_B$, thus $(g \circ i_A)(a) - f(a) \in \rho(M)$, for all $a \in A$. By Proposition 2.4., M is ρ-B-injective, for every R-module B and hence M is a ρ-injective R-module. □

In the following proposition, we will give another characterization of ρ-injectivity by using the class of free modules.

Proposition 2.19. An R-module M is ρ-injective if and only if M is ρ-F-injective, for every free R-module F.

Proof: $(\Longrightarrow)$ This is obvious.

$(\Longleftarrow)$ Suppose that M is ρ-F-injective, for every free R-module F. Consider the following diagram with exact row.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[r, "i"] \arrow[dl, dashed, "h"'] & F \arrow[dll, dashed, "h_1"] \\ & M & & \end{tikzcd}$$

Since B is a set, thus there exists a free R-module, say F, such that B is a basis of F [12, p.58]. By hypothesis, there exists an R-homomorphism h_1: F → M such that $\big(h_1 \circ (i \circ f)\big)(a) - g(a) \in \rho(M)$, for all $a \in A$. Put $h =: h_1 \circ i: B \to M$, it is clear that h is an R-homomorphism. For every $a \in A$, we have that $(h \circ f)(a) - g(a) = \big((h_1 \circ i) \circ f\big)(a) - g(a) \in \rho(M)$ and hence M is a ρ-injective R-module. □

3. Endomorphism Ring of ρ-Injective Modules

Let M be an R-module, $S = \mathrm{End}_R(M)$ and let $\Delta = \{f \in S \mid \ker(f) \leq^e M\}$. It is well-known that Δ is a two-sided ideal of S [13] and if an R-module M is injective, then the ring S/Δ is regular. Moreover, if $\Delta = 0$, then the ring S is a right self-injective ring [8].

For analogous results for ρ-injective modules we consider the following.

Let M and N be R-modules and $f: M \to N$ be an R-homomorphism. The set $f^{-1}(\rho(N)) = \{x \in M \mid f(x) \in \rho(N)\}$ is said to be the kernel of f relative to a preradical ρ and denoted by $\rho\ker(f)$.

Let M be an R-module and $S = \mathrm{End}_R(M)$. We will use the notation $\rho\Delta$ for the set $\{f \in S \mid \rho\ker(f) \leq^e M\}$.

Proposition 3.1. Let M be an R-module and $S = \mathrm{End}_R(M)$. Then $\rho\Delta$ is a two-sided ideal of S.

Proof. Since the zero function belong to Δ , thus $\rho\Delta$ is a non-empty set. Let $f, g \in \rho\Delta$, thus $\rho\ker(f) \leq^e M$ and $\rho\ker(g) \leq^e M$ and hence Lemma 5.1.5(b) in [7] implies that $\rho\ker(f) \cap \rho\ker(g) \leq^e M$. Since $\rho\ker(f) \cap \rho ker(g) \subseteq \rho\ker(f - g)$, thus $\rho\ker(f - g) \leq^e$ M (by [7**,** Lemma 5.1.5(a)]**)** and hence $f - g \in \rho\Delta$.

Let $f \in \rho\Delta$ and $h \in S$, thus $\rho\ker(f) \leq^e M$. Since $\rho\ker(f) \subseteq \rho\ker(h \circ f)$, thus $\rho\ker(h \circ f) \leq^e M$ (by [7**,** Lemma 5.1.5(a)]**)** and hence $h \circ f \in \rho\Delta$. Now we will prove that $f \circ h \in \rho\Delta$. Since $\rho\ker(f) \leq^e M$, thus Lemma 5.1.5(c) in [7] implies that $h^{-1}\big(\rho\ker(f)\big) \leq^e M$. But $h^{-1}\big(\rho\ker(f)\big) \subseteq \rho\ker(f \circ h)$, therefore $\rho\ker(f \circ h) \leq^e M$, by [7**,** Lemma 5.1.5(a)]**.** Thus $f \circ h \in \rho\Delta$ and hence $\rho\Delta$ is a two-sided ideal of S. □

Now, we are ready to state and prove the main result in this section.

Theorem 3.2. Let M be an R-module and $S = \mathrm{End}_R(M)$. If M is ρ-injective, then:
(1) $S/\rho\Delta$ is a regular ring;
(2) if $\rho\Delta = 0$, then S is a right self-injective ring.

Proof. Suppose that M is a ρ-injective R-module.

(1) Let $\lambda + \rho\Delta \in S/\rho\Delta$, thus $\lambda \in S$. Put $K = \ker(\lambda)$ and let L be a relative complement of K in M. Define $\alpha: \lambda(L) \to M$ by $\alpha\big(\lambda(x)\big) = x$, for all $x \in L$. It is easy to prove that α is a well-defined R-homomorphism.

Thus we have the following diagram, where i is the inclusion R-homomorphism.

$$\begin{tikzcd} 0 \arrow[r] & \lambda(L) \arrow[r, "i"] \arrow[d, "\alpha"'] & M \arrow[dl, dashed, "\beta"] \\ & M & \end{tikzcd}$$

Since M is ρ-injective (by hypothesis), there exists an R-homomorphism $\beta: M \to M$ such that $\beta(\lambda(x)) - \alpha\big(\lambda(x)\big) \in \rho(M)$ for each $x \in L$. That is for each $x \in L$, we have that $\beta(\lambda(x)) = \alpha\big(\lambda(x)\big) + m_x$, for some $m_x \in \rho(M)$. Let $u \in K \oplus L$, thus $u = x + y$ where $x \in K$ and $y \in L$ and hence $(\lambda - \lambda\beta\lambda)(u) = (\lambda - \lambda\beta\lambda)(x + y) = \lambda(x) - \lambda\beta\big(\lambda(x)\big) + \lambda(y) - \lambda\beta\big(\lambda(y)\big) = 0 - 0 - \lambda(y) - \lambda\big(\alpha\lambda(y) + m_y\big) = \lambda(y) - \lambda(y) - \lambda\big(m_y\big) \in \rho(M)$ (because ρ is a preradical) and hence $u \in \rho\ker(\lambda - \lambda\beta\lambda)$. Thus for each $u \in K \oplus L$, we have that $u \in \rho\ker(\lambda - \lambda\beta\lambda)$ and this implies that $K \oplus L \subseteq \rho\ker(\lambda - \lambda\beta\lambda)$. Since $K \oplus L \leq^e M$ [8], thus Lemma 5.1.5(a) in [7] implies that $\rho\ker(\lambda - \lambda\beta\lambda) \leq^e M$ and hence $\lambda - \lambda\beta\lambda \in \rho\Delta$. Thus $\lambda + \rho\Delta = (\lambda\beta\lambda) + \rho\Delta$ and hence $S/\rho\Delta$ is a regular ring.

(2) Suppose that $\rho\Delta = 0$, thus by **(1)** above, we have that S is a regular ring. Let I be any right ideal of S and let $f: I \to S$ be any right S-homomorphism. Consider the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "f"'] & S \\ & S & \end{tikzcd}$$

Let IM be the R-submodule of M generated by $\{\lambda m | \lambda \in I, m \in M\}$. Thus, if $x \in IM$, then $x = \sum_{i=1}^{n} \lambda_i m_i$ for some $\lambda_1, \lambda_2, \cdots, \lambda_n \in I$ and some $m_1, m_2, \cdots, m_n \in M$ where $n \in \mathbb{Z}^+$. Define $\theta: IM \to M$ as follows, for each $x = \sum_{i=1}^{n} \lambda_i m_i \in IM$, put $\theta(x) = \theta(\sum_{i=1}^{n} \lambda_i m_i) = \sum_{i=1}^{n} f(\lambda_i)(m_i)$. Let $x, y \in IM$, thus $x = \sum_{i=1}^{n} \lambda_i m_i$ and $y = \sum_{j=1}^{t} \alpha_j m_j'$, for some $\lambda_i, \alpha_j \in I$ and $m_i, m_j' \in M$, with $i = 1, \cdots, n$ and $j = 1, \cdots, t$ where $n, t \in \mathbb{Z}^+$. Since S is a regular ring, thus Proposition 4.14 in [8] implies that each finitely generated right ideal of S is generated by an idempotent. Hence the right ideal of a ring S which is generated by $\lambda_1, \cdots, \lambda_n, \alpha_1, \cdots, \alpha_t$ written as eS, where $e = e^2 \in I$ and hence $\lambda_i, \alpha_j \in eS$ for all $i = 1, \cdots, n, j = 1, \cdots, t$ and this implies that

$\lambda_i = eh_i$ and $\alpha_j = eh'_j$ for some $h_i, h'_j \in S$ and for all $i = 1, \cdots, n,\ j = 1, \cdots, t$. Hence $e\lambda_i = e(eh_i) = e^2h_i = eh_i = \lambda_i$, for all $i = 1, \cdots, n$ and $e\alpha_j = e(eh'_j) = e^2h'_j = eh'_j = \alpha_j$ for all $j = 1, \cdots, t$. Thus, $f(\lambda_i) = f(e)\lambda_i$ and $f(\alpha_j) = f(e)\alpha_j$ for all $i = 1, \cdots, n$ and $j = 1, \cdots, t$. Therefore, $\theta(x) = \theta(\sum_{i=1}^{n} \lambda_i m_i) = \sum_{i=1}^{n} f(\lambda_i)(m_i) = \sum_{i=1}^{n} f(e)\lambda_i m_i = f(e)\sum_{i=1}^{n} \lambda_i m_i = f(e)x$ and similarly we have that $\theta(y) = f(e)y$. Clearly, θ is a well-defined R-homomorphism, since for all $x, y \in IM$, if $x = y$, then $f(e)x = f(e)y$. Since $\theta(x) = f(e)x$ and $\theta(y) = f(e)y$ (as above), thus $\theta(x) = \theta(y)$. Let $x, y \in IM$ and $r \in R$, thus $\theta(x + y) = f(e)\ (x + y) = f(e)x + f(e)y = \theta(x) + \theta(y)$ and $\theta(rx) = f(e)(rx) = r(f(e)(x)) = r\theta(x)$. Therefore, θ is a well-defined R-homomorphism. Thus we have the following diagram (where i is the inclusion R-homomorphism).

$$\begin{tikzcd}
0 \arrow[r] & IM \arrow[r, "i"] \arrow[d, "\theta"'] & M \arrow[dl, dashed, "\varphi"] \\
 & M &
\end{tikzcd}$$

Since M is a ρ-injective, there exists an R-homomorphism $\varphi: M \to M$ such that $\varphi(x) - \theta(x) \in \rho(M)$, for all $x \in IM$.
Let $m \in M$ and $\lambda \in I$. Thus $(\varphi\lambda)(m) = \varphi(\lambda m) = \theta(\lambda m) + l_m = f(\lambda)m + l_m$, for some $l_m \in \rho(M)$ and hence $(\varphi\lambda - f(\lambda))(m) \in \rho(M)$ and this implies that $m \in \rho\ker(\varphi\lambda - f(\lambda))$. Thus $M = \rho\ker(\varphi\lambda - f(\lambda))$, for each $\lambda \in I$. Therefore $\rho\ker(\varphi\lambda - f(\lambda)) \leq^e M$ and hence $\varphi\lambda - f(\lambda) \in \rho\Delta$, for all $\lambda \in I$. Since $\rho\Delta = 0$ (by hypothesis), thus $f(\lambda) = \varphi\lambda$, for all $\lambda \in I$ and hence S satisfied Baer's condition. Therefore, S is a right self-injective ring, by [8, Theorem 1.6.]. ☐

Proposition 3.3. Let M be an ρ-injective R-module and $S = \mathrm{End}_R(M)$. Then $I \cap K = IK + \rho\Delta \cap (I \cap K)$, for every two-sided ideals I and K of S.

Proof. Suppose that M is a ρ-injective R-module, thus Theorem 3.2. implies that $S/\rho\Delta$ is a regular. Let I and K be any two-sided ideals of S. Let $\alpha \in I \cap K$, thus $\alpha + \rho\Delta \in S/\rho\Delta$. Since $S/\rho\Delta$ is a regular ring, thus there exists an element $\beta + \rho\Delta \in S/\rho\Delta$ such that $\alpha + \rho\Delta = \alpha\beta\alpha + \rho\Delta$ and hence $\alpha - \alpha\beta\alpha \in \rho\Delta$. Since $\alpha - \alpha\beta\alpha \in I \cap K$, thus $\alpha - \alpha\beta\alpha \in \rho\Delta \cap (I \cap K)$. Put $\alpha_1 = \alpha - \alpha\beta\alpha$, thus $\alpha = \alpha\beta\alpha + \alpha_1 \in IK + \rho\Delta \cap (I \cap K)$ and hence $I \cap K \subseteq IK + \rho\Delta \cap (I \cap K)$. Since $IK \subseteq I$ and $IK \subseteq K$, thus $IK \subseteq I \cap K$. Since $\rho\Delta \cap (I \cap K) \subseteq (I \cap K)$, thus $IK + \rho\Delta \cap (I \cap K) \subseteq I \cap K$. Therefore, $I \cap K = IK + \rho\Delta \cap (I \cap K)$. ☐

By applying Proposition 3.3. we have the following result.

Corollary 3.4. Let M be a ρ-injective R-module, $S = \mathrm{End}_R(M)$ and let K be any two-sided ideal of S.Then $K = K^2 + (\rho\Delta \cap K)$

In [14], Osofsky showed that, for an R-module M, if $Z(M) = 0$, then the Jacobson radical of the ring $S = \mathrm{End}_R(M)$ is zero. Also, if M is an injective R-module with $Z(M) = 0$, then the ring $S = \mathrm{End}_R(M)$ is a right self-injective regular [8].

In the following, we will state and prove analogous results for ρ-injective modules. Firsty, we need the following lemma.

Lemma 3.5. Let M be an R-module and $S = \mathrm{End}_R(M)$. Then for each $\lambda \in S$ and for each $x \in M$ we have $[\rho(M): \lambda(x)]_R = [\rho ker(\lambda): x]_R$.
Proof. Let $\lambda \in S$ and $x \in M$. Thus if $r \in [\rho(M): \lambda(x)]$, then $\lambda(x)r \in \rho(M)$ and hence $\lambda(xr) \in \rho(M)$ and this implies that $xr \in \rho ker(\lambda)$ and so $r \in [\rho ker(\lambda): x]_R$. Therefore, $[\rho(M): \lambda(x)]_R \subseteq [\rho ker(\lambda): x]_R$ and by similar way we can prove $[\rho\ker(\lambda): x]_R \subseteq [\rho(M): \lambda(x)]_R$. Thus $[\rho(M): \lambda(x)]_R = [\rho\ker(\lambda): x]_R$. ☐

Let M be an R-module. It is easy to prove that the set $\{m \in M | \, [\rho(M): m]_R$ is an essential ideal in $R\}$ is a submodule of M. This submodule is said to be the ρ-singular submodule of M and denoted by $\rho Z(M)$.

The following proposition is an analogous result of the Osofsky's result [14].

Proposition 3.6. Let M be an R-module and $S = \mathrm{End}_R(M)$. If $\rho\mathrm{Z}(M) = 0$, then $\rho\Delta = 0$.
Proof. Suppose that $\rho\mathrm{Z}(M) = 0$ and let $\alpha \in \rho\Delta$, thus $\rho\ker(\alpha) \leq^e M$ and hence [8, Lemma 3, p. 46] implies that $[\rho\ker(\alpha): x]_R \leq^{\mathrm{e}} R$, for each $x \in M$. Since $[\rho(M): \alpha(x)]_R = [\rho\ker(\alpha): x]_R$ (by Lemma 3.5.), thus $[\rho(M): \alpha(x)]_R \leq^{\mathrm{e}} R$ and hence $\alpha(x) \in \rho Z(M)$. Since $\rho Z(M) = 0$ (by hypothesis), thus $\alpha(x) = 0$, for all x in M (i.e $\alpha = 0$) and hence $\rho\Delta = 0$. ☐

The following corollary (for ρ-injective modules) is analogous of the statement for injective modules [8].

Corollary 3.7. Let M be a ρ-injective R-module and $S = \mathrm{End}_R(M)$. If $\rho\mathrm{Z}(M) = 0$, then S is a right self-injective regular ring.
Proof. Suppose that M is a ρ-injective module with $\rho\mathrm{Z}(M) = 0$. Thus Proposition 3.6. implies that $\rho\Delta = 0$. Therefore, S is a right self-injective regular ring, by Theorem 3.2. ☐

Corollary 3.8. If R is a self ρ-injective ring and $\rho\mathrm{Z}(R) = 0$, then R is a right self-injective regular ring.
Proof. Since $R \cong \mathrm{End}_R(R)$, thus the result follows from Corollary 3.7. ☐

Let R be a ring and $\mathrm{x} \in \mathrm{R}$. Let $\mathrm{x_L}: \mathrm{R} \to \mathrm{R}$ be the mapping defined by $\mathrm{x_L}(\mathrm{r}) = \mathrm{rx}$, for all $\mathrm{r} \in \mathrm{R}$. Then x_L is an R-homomorphism and $\mathrm{End}_R(R) = \{x_L | \, x \in R\}$ [8].

Lemma 3.9. Let R be a ring and $S = \mathrm{End}_R(R)$. Define $\alpha: R/\rho\mathrm{Z}(R) \to S/\rho\Delta$ as follows: $\alpha\big(x + \rho\mathrm{Z}(R)\big) = x_L + \rho\Delta$ for each $x \in R$. Then α is an R-isomorphism.
Proof. It is easy. ☐

The following proposition is an analogous result of the statement for self-injective rings [15].

Proposition 3.10. If R is a self ρ-injective ring, then $R/\rho\mathrm{Z}(R)$ is a regular ring.
Proof. Let $\alpha: R/\rho\mathrm{Z}(R) \to S/\rho\Delta$ be the R-isomorphism as in Lemma 3.9., where $S = \mathrm{End}_R(R)$. Let $x + \rho\mathrm{Z}(R) \in R/\rho\mathrm{Z}(R)$, thus $\alpha\big(x + \rho\mathrm{Z}(R)\big) = x_L + \rho\Delta \in S/\rho\Delta$. Since R is a self ρ-injective ring, thus $S/\rho\Delta$ is a regular ring (by Theorem 3.2.) and this implies that there exists an element $y_L + \rho\Delta \in S/\rho\Delta$ such that $x_L + \rho\Delta = x_L y_L x_L + \rho\Delta = (xyx)_L + \rho\Delta$. Since α is an R-isomorphism, thus α^{-1} exists and $\alpha^{-1}(x_L + \rho\Delta) = \alpha^{-1}((xyx)_L + \rho\Delta)$. Hence $x + \rho\mathrm{Z}(R) = xyx + \rho\mathrm{Z}(R) = \big(x + \rho\mathrm{Z}(R)\big) \cdot \big(y + \rho\mathrm{Z}(R)\big) \cdot \big(x + \rho\mathrm{Z}(R)\big)$. Since $\alpha^{-1}(y_L + \rho\Delta) = y + \rho\mathrm{Z}(R) \in R/\rho\mathrm{Z}(R)$, thus we get an element $y + \rho\mathrm{Z}(R)$ in $R/\rho\mathrm{Z}(R)$ such that $x + \rho\mathrm{Z}(R) = \big(x + \rho\mathrm{Z}(R)\big) \cdot \big(y + \rho\mathrm{Z}(R)\big) \cdot \big(x + \rho\mathrm{Z}(R)\big)$. Therefore, $R/\rho\mathrm{Z}(R)$ is a regular ring. ☐

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*الموديولات الأغمارية نسبة الى جذر ابتدائي

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الخلاصة

مفهوم الموديولات الاغمارية نسبة الى جذر ابتدائي ρ (الموديولات الاغمارية$-\rho$) طرحت في هذا العمل كتعميم للموديولات الاغمارية. تعريف الموديولات الاغمارية نسبة الى جذر ابتدائي ρ يوحد عدة تعريفات عن تعميمات الموديولات الاغمارية مثل الموديولات الاغمارية تقريبا والموديولات الاغمارية الخاصة. العديد من التشخيصات وخواص الموديولات الاغمارية نسبة الى جذر ابتدائي ρ قد اعطيت. درسنا حلقات التماثلات الموديولية الذاتية للموديولات الاغمارية نسبة الى جذر ابتدائي ρ. نتائج هذا العمل توحد وتوسع العديد من النتائج الموجودة في المصادر.

الكلمات المفتاحية: الموديولات الاغمارية، الموديولات الاغمارية تقريبا، الجذر الابتدائي، حلقات التماثلات الموديولية الذاتية.

* نتائج هذا البحث ستكون جزء من رسالة الماجستير للباحث الثاني وتحت اشراف الباحث الاول في جامعة القادسية.