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On A New Class of Univalent Functions with Negative Coefficients

Waggas Galib Atshan
Department of Mathematics, College of Computer Science and Mathematics
University of Al-Qadisiya, Diwaniya - Iraq
E-mail: waggashnd@yahoo.com

Haeder Younis Ghawi
Department of Mathematics, College of Computer Science and Mathematics
University of Al-Qadisiya, Diwaniya - Iraq
E-mail: hadirkawi@yahoo.com

Abstract

In the present paper, we have introduced a subclass $S(\gamma, \alpha, \mu)$ of univalent functions with negative coefficients in the unit disc. We derive basic properties like coefficient inequality, distortion and covering theorem, radius of convexity, extreme points, Hadamard product, closure theorems and convolution operator for functions belonging to this class.

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1. Introduction

Let *A* denote the class of function of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

Which are analytic and univalent in $U = \{z : |z| < 1\}$. If a function f is given by (1) and g is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{2}$$

is in A, then convolution or Hadamard product of f(z) and g(z) is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, z \in U \tag{3}$$

Let S denote the subclass of A consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ \ a_n \geq 0 \tag{4}$$

We aim to study the subclass $S(\gamma, \alpha, \mu)$ consisting of functions $f \in S$ and satisfying:

$$\left|\frac{\gamma(f'(z)-\frac{f(z)}{z})}{\alpha f'(z)+(1-\gamma)\frac{f(z)}{z}}\right|<\mu,\quad z\in U \tag{5}$$

For $0\le\gamma<1, 0\le\alpha<1, 0<\mu<1$. Another classes studied by Atshan and Kulkarni [2] and Darus, [3] consisting of functions of the form (4).

2. Coefficient Inequality

The following theorem gives a necessary and sufficient condition for function to be in the class $S(\gamma,\alpha,\mu)$.

Theorem 1: Let the function *f* be defined by (4) . Then $f\in S(\gamma,\alpha,\mu)$ if and only if

$$\sum_{n=2}^{\infty}[\gamma(n-1)+\mu(n\alpha+1-\gamma)]a_n\le\mu(\alpha+(1-\gamma)), \tag{6}$$

where $0\le\gamma<1, 0\le\alpha<1, 0<\mu<1$. The result (6) is sharp for the function

$$f(z)=z-\frac{\mu(\alpha+(1-\gamma)}{\gamma(n-1)+\mu(n\alpha+1-\gamma)}z^n,\quad n\ge 2.$$

Proof: Assume that the inequality (6) hold true and $|z|=1$. Then we obtain

$$\left|\gamma(f'(z)-\frac{f(z)}{z})\right|-\mu\left|\alpha f'(z)+(1-\gamma)\frac{f(z)}{z}\right|$$

$$=\left|-\gamma\sum_{n=2}^{\infty}(n-1)a_nz^{n-1}\right|-\mu\left|\alpha+(1-\gamma)-\sum(n\alpha+1-\gamma)a_nz^{n-1}\right|$$

$$\le\sum_{n=2}^{\infty}[\gamma(n-1)+\mu(n\alpha+1-\gamma)]a_n-\mu(\alpha+(1-\gamma))\le 0.$$

Hence, by maximum modules principle, $f\in S(\gamma,\alpha,\mu)$.

Now, assume that $f\in S(\gamma,\alpha,\mu)$ so that

$$\left|\frac{\gamma\left(f'(z)-\frac{f(z)}{z}\right)}{\alpha f'(z)+(1-\gamma)\frac{f(z)}{z}}\right|<\mu,\quad z\in U$$

$$\left|\gamma\left(f'(z)-\frac{f(z)}{z}\right)\right|<\mu\left|\alpha f'(z)+(1-\gamma)\frac{f(z)}{z}\right|;$$

We get

$$\left|-\gamma\sum_{n=2}^{\infty}(n-1)a_nz^{n-1}\right|<\mu\left|\alpha+(1-\gamma)-\sum_{n=2}^{\infty}(n\alpha+1-\gamma)a_nz^{n-1}\right|.$$

Thus

$$\sum_{n=2}^{\infty}[\gamma(n-1)+\mu(n\alpha+1-\gamma)]a_n\le\mu(\alpha+(1-\gamma))$$

and this proof is complete.

Corollary 1: If the function $f(z)\in S(\gamma,\alpha,\mu)$, then

$$a_n\le\frac{\mu(\alpha+(1-\gamma)}{\gamma(n-1)+\mu(n\alpha+1-\gamma)},\quad \text{for } n\ge 2$$

3. Distortion and Covering Theorem

Now, we prove the growth and distortion theorems for the functions in the class $S(\gamma,\alpha,\mu)$.

Theorem 2: If the function $f \in S(\gamma,\alpha,\mu)$, then

$$|z|-\frac{\mu(\alpha+(1-\gamma))}{\gamma+\mu(2\alpha+1-\gamma)}|z|^2 \le |f(z)| \le |z|+\frac{\mu(\alpha+(1-\gamma))}{\gamma+\mu(2\alpha+1-\gamma)}|z|^2 .$$

The result is sharp and attained for

$$f(z)= z-\frac{\mu(\alpha+(1-\gamma))}{\gamma+\mu(2\alpha+1-\gamma)}z^2$$

Proof:

$$|f(z)|=\left|z-\sum_{n=2}^{\infty}a_n z^n\right| \le |z|+\sum_{n=2}^{\infty}a_n|z|^n$$

$$\le |z|+|z|^2\sum_{n=2}^{\infty}a_n$$

By Theorem 1, we get

$$\sum_{n=2}^{\infty}a_n \le \frac{\mu(\alpha+(1-\gamma))}{\gamma+\mu(2\alpha+1-\gamma)}.$$

Thus

$$|f(z)|\le |z|+\frac{\mu(\alpha+(1-\gamma))}{\gamma+\mu(2\alpha+1-\gamma)}|z|^2.$$

Also

$$|f(z)|\ge |z|-\sum_{n=2}^{\infty}a_n|z|^n \ge |z|-|z|^2\sum_{n=2}^{\infty}a_n$$

$$\ge |z|-\frac{\mu(\alpha+(1-\gamma))}{\gamma+\mu(2\alpha+1-\gamma)}|z|^2.$$

Theorem 3: If $f \in S(\gamma,\alpha,\mu)$, then

$$1-\frac{\mu(\alpha+(1-\gamma))}{\gamma+\mu(2\alpha+1-\gamma)}|z| \le |f'(z)| \le 1+\frac{\mu(\alpha+(1-\gamma))}{\gamma+\mu(2\alpha+1-\gamma)}|z| ,$$

with equality for

$$f(z)= z-\frac{\mu(\alpha+(1-\gamma))}{\gamma+\mu(2\alpha+1-\gamma)}z^2 .$$

Proof: Notice that

$$[\gamma+\mu(2\alpha+1-\gamma)]\sum_{n=2}^{\infty}na_n \le \sum_{n=2}^{\infty}n[\gamma(n-1)+\mu(n\alpha+1-\gamma)]a_n \le \mu(\alpha+(1-\gamma)), \tag{8}$$

from Theorem 1. Thus

$$|f'(z)|=\left|1-\sum_{n=2}^{\infty}na_n z^{n-1}\right|\le 1+\sum_{n=2}^{\infty}na_n|z|^{n-1}$$

$$\le 1+|z|\sum_{n=2}^{\infty}na_n$$

$$\le 1+\frac{\mu(\alpha+(1-\gamma))}{\gamma+\mu(2\alpha+1-\gamma)}|z| . \tag{9}$$

On the other hand

$$|f'(z)|=\left|1-\sum_{n=2}^{\infty}na_n z^{n-1}\right| \ge 1-\sum_{n=2}^{\infty}na_n|z|^{n-1}$$

$$\geq 1-|z|\sum_{n=2}^{\infty} na_n$$

$$\geq 1-\frac{\mu(\alpha+(1-\gamma))}{\gamma+\mu(2\alpha+1-\gamma)}|z| \ . \tag{10}$$

Combining (9) and (10), we get the result .

Theorem 4: The disc $|z|<1$ is mapped onto a domain that contains the disc

$$|w|< \frac{(\gamma+\mu(2\alpha+1-\gamma)-\mu(\alpha+(1-\gamma))}{(\gamma+\mu(2\alpha+1-\gamma))}$$

The result is sharp with extremal function

$$f(z)= z-\frac{\mu(\alpha+(1-\gamma))}{\gamma+\mu(2\alpha+1-\gamma)}z^2 .$$

Proof: The result follows upon allowing $|z|\to 1$ in Theorem 2.

4. Radius of Convexity

In the next theorem , we obtain the radius of convexity for the class $S(\gamma,\alpha,\mu)$.

Theorem 5: Let $f\in S(\gamma,\alpha,\mu)$. Then f is p-valently convex in $|z|<\Re$ of order $\delta, 0\leq\delta<1$, where

$$R=\inf_n\left\{\frac{(1-\delta)(\gamma(n-1)+\mu(n\alpha+1-\gamma))}{n(n-\delta)\mu(\alpha+(1-\gamma))}\right\}^{\frac{1}{n-1}} , \ n\geq 2 . \tag{11}$$

Proof: f is p-valently convex of order $\delta, 0\leq\delta<1$ if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\delta .$$

Thus it is enough to show that

$$\left|\frac{zf''(z)}{f'(z)}\right|=\left|\frac{-\sum_{n=2}^{\infty}n(n-1)a_nz^{n-1}}{1-\sum_{n=2}^{\infty}na_nz^{n-1}}\right|\leq\frac{\sum_{n=2}^{\infty}n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}na_n|z|^{n-1}} .$$

Thus

$$\left|\frac{zf''(z)}{f'(z)}\right|\leq 1-\delta \ \text{ if } \sum_{n=2}^{\infty}\frac{n(n-\delta)a_n|z|^{n-1}}{1-\delta}\leq 1. \tag{12}$$

Hence by Theorem 1, (12) will be true if

$$\frac{n(n-\delta)|z|^{n-1}}{1-\delta}\leq\frac{(\gamma(n-1)+\mu(n\alpha+1-\gamma))}{\mu(\alpha+(1-\gamma))},$$

or if

$$|z|\leq\left\{\frac{(1-\delta)(\gamma(n-1)+\mu(n\alpha+1-\gamma))}{n(n-\delta)\mu(\alpha+(1-\gamma))}\right\}^{\frac{1}{n-1}}, \ n\geq 2 . \tag{13}$$

The theorem follows easily from (13).

5. Extreme Points

In the following theorem, we obtain extreme points for the class $S(\gamma,\alpha,\mu)$.

Theorem 6: Let $f_1(z)=z$ and $f_n(z)=z-\frac{\mu(\alpha+(1-\gamma))}{\gamma(n-1)+\mu(2\alpha+1-\gamma)}z^2$,for n=2,3,… . Then $f\in S(\gamma,\alpha,\mu)$ if and only if it can be expressed in the form $f(z)=\sum_{n=1}^{\infty}\theta_n f_n(z)$ where $\theta_n\geq 0$, and $\sum_{n=1}^{\infty}\theta_n=1$

Proof: Suppose that $f(z)=\sum_{n=1}^{\infty}\theta_n f_n(z)$, hence we get

$$f(z)=z-\sum_{n=2}^{\infty}\frac{\mu(\alpha+(1-\gamma))\theta_n}{\gamma(n-1)+\mu(2\alpha+1-\gamma)}z^n.$$

Now, $f\in S(\gamma,\alpha,\mu)$, since

$$\sum_{n=2}^{\infty}\frac{\gamma(n-1)+\mu(n\alpha+1-\gamma)}{\mu(\alpha+(1-\gamma))}.\frac{\mu(\alpha+(1-\gamma))\theta_n}{\gamma(n-1)+\mu(n\alpha+1-\gamma)}=\sum_{n=2}^{\infty}\theta_n=1-\theta_1\leq 1.$$

Conversely, assume $f\in S(\gamma,\alpha,\mu)$. Then we show that f can be written in the form $f(z)=\sum_{n=1}^{\infty}\theta_n f_n(z)$.

Now, $f\in S(\gamma,\alpha,\mu)$ implies from Theorem 1

$$a_n\leq\frac{\mu(\alpha+(1-\gamma))}{\gamma(n-1)+\mu(n\alpha+1-\gamma)}$$

Setting $\theta_n=\frac{\gamma(n-1)+\mu(n\alpha+(1-\gamma))}{\mu(\alpha+1-\gamma)}a_n$, n=2,3,…and $\theta_1=1-\sum_{n=2}^{\infty}\theta_n$, we obtain $f(z)=\sum_{n=1}^{\infty}\theta_n f_n(z)$.

6. Hadamard Product

Here, we prove the convolution result for functions belongs to the class $S(\gamma,\alpha,\mu)$.

Theorem 7: Let $f,g\in S(\gamma,\alpha,\mu)$. Then $f*g\in S(\gamma,\alpha,\ell)$for $f(z)=z-\sum_{n=2}^{\infty}a_n z^n$, $g(z)=z-\sum_{n=2}^{\infty}b_n z^n$, and $f*g(z)=z-\sum_{n=2}^{\infty}a_n b_n z^n$, where $\ell\geq\frac{\mu^2(\alpha+(1-\gamma))\gamma(n-1)}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]^2-\mu^2(\alpha+(1-\gamma))(n\alpha+1-\gamma)}$.

Proof: $f,g\in S(\gamma,\alpha,\mu)$ and so

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]}{\mu(\alpha+(1-\gamma))}a_n\leq 1 \qquad (14)$$

and

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]}{\mu(\alpha+(1-\gamma))}b_n\leq 1. \qquad (15)$$

We have to find the smallest number ℓ such that

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\ell(n\alpha+1-\gamma)]}{\ell(\alpha+(1-\gamma))}a_n b_n\leq 1. \qquad (16)$$

By Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]}{\mu(\alpha+(1-\gamma))}\sqrt{a_nb_n}\le 1 . \quad (17)$$

Therefore its enough to show that

$$\frac{[\gamma(n-1)+\ell(n\alpha+1-\gamma)]}{\ell(\alpha+(1-\gamma))}a_nb_n \le \frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]}{\mu(\alpha+(1-\gamma))}\sqrt{a_nb_n} .$$

That is

$$\sqrt{a_nb_n} \le \frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\ell}{[\gamma(n-1)+\ell(n\alpha+(1-\gamma)]\mu} . \quad (18)$$

From (17)

$$\sqrt{a_nb_n} \le \frac{\mu(\alpha+1-\gamma)}{[\gamma(n-1)+\mu(n\alpha+(1-\gamma))]} .$$

Thus it is enough to show that

$$\frac{\mu(\alpha+1-\gamma)}{[\gamma(n-1)+\mu(n\alpha+(1-\gamma))]} \le \frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\ell}{[\gamma(n-1)+\ell(n\alpha+1-\gamma)]\mu} ,$$

which simplifies to

$$\ell \ge \frac{\mu^2(\alpha+(1-\gamma))\gamma(n-1)}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]^2-\mu^2(\alpha+(1-\gamma))(n\alpha+1-\gamma)} .$$

7. Closure Theorem

We shall prove the following closure theorems for the class $S(\gamma,\alpha,\mu)$.

Theorem 8: Let $f_j \in S(\gamma,\alpha,\mu),\ j=1,2,...,\delta$. Then

$$g(z)=\sum_{j=1}^{\delta}c_j f_j(z) \in S(\gamma,\alpha,\mu)$$

for $f_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}z^n$,where $\sum_{j=1}^{\delta}c_j=1$.

Proof:

$$g(z)=\sum_{j=1}^{\delta}c_j f_j(z)$$

$$=z-\sum_{n=2}^{\infty}\sum_{j=1}^{\delta}c_j a_{n,j}z^n = z-\sum_{n=2}^{\infty}e_n z^n ,$$

where $e_n=\sum_{j=1}^{\delta}c_j a_{n,j}$ .Thus $g\in S(\gamma,\alpha,\mu)$ if

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]}{\mu(\alpha+(1-\gamma))}e_n\le 1 ,$$

that is, if

$$\sum_{n=2}^{\infty}\sum_{j=1}^{\delta}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]}{\mu(\alpha+(1-\gamma))}c_j a_{n,j} = \sum_{j=1}^{\delta}c_j\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]}{\mu(\alpha+(1-\gamma))}a_{n,j}\le\sum_{j=1}^{\delta}c_j=1$$

Since $\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]}{\mu(\alpha+(1-\gamma))}a_{n,j}\le 1$ as $f_j\in S(\gamma,\alpha,\mu)$

Theorem 9: Let $f,g\in S(\gamma,\alpha,\mu)$. Then

$$h(z)=z-\sum_{n=2}^{\infty}(a_n^2+b_n^2)z^n$$

belongs to $S(\gamma,\alpha,\ell)$, where

$$\ell \geq \frac{\mu^2(\alpha+(1-\gamma))2\gamma(n-1)}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]^2-2\mu\ (n\alpha+1-\gamma)}$$

Proof: Since $f,g\in S(\gamma,\alpha,\mu)$, so Theorem 1 yields

$$\sum_{n=2}^{\infty}\left[\frac{\gamma(n-1)+\mu(n\alpha+1-\gamma)}{\mu(\alpha+(1-\gamma))}a_n\right]^2 \leq 1 \text{ and}$$

$$\sum_{n=2}^{\infty}\left[\frac{\gamma(n-1)+\mu(n\alpha+1-\gamma)}{\mu(\alpha+(1-\gamma))}b_n\right]^2 \leq 1.$$

We obtain from the last two inequalities

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{\gamma(n-1)+\mu(n\alpha+1-\gamma)}{\mu(\alpha+(1-\gamma))}\right]^2(a_n^2+b_n^2)\leq 1\ . \tag{19}$$

But $h\in S(\gamma,\alpha,\ell)$ if and only if

$$\sum_{n=2}^{\infty}\frac{\gamma(n-1)+\ell(n\alpha+1-\gamma)}{\ell(\alpha+(1-\gamma))}(a_n^2+b_n^2)\leq 1, \tag{20}$$

where 0< ℓ <1, however, (19) implies (20) if

$$\frac{[\gamma(n-1)+\ell(n\alpha+1-\gamma)]}{\ell(\alpha+(1-\gamma)}\leq \frac{1}{2}\left[\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]}{\mu(\alpha+(1-\gamma))}\right]^2$$

Simplifying, we get

$$\ell \geq \frac{\mu^2(\alpha+(1-\gamma))2\gamma(n-1)}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]^2-2\mu\ (n\alpha+1-\gamma)}.$$

8. Convolution Operator

Definition 1[4]: The Gaussian hypergeometric function denoted by

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}.\frac{z^n}{n!},|z|<1,$$

where c>b>0, c>a+b and

$$(x)_n=\begin{cases}x(x+1)(x+2)...(x+n-1), \ for & n=1,2,3,...\\ 1 & n=0\end{cases}$$

Definition 2[4]: For every $f\in S$, we define Convolution operator $W_{a,b,c}(f)(z)$ as below:

$$W_{a,b,c}(f)(z)={}_2F_1\ (a,b;c;z)*f(z)=z-\sum_{n=2}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}a_n z^n,$$

where ${}_2F_1(a,b;c;z)$ is Gaussian hypergeometric function (see [1] and [4]) introduced in Definition 1.

Theorem 10: Let *f* is given by (4) be in $S(\gamma,\alpha,\mu)$. Then the convolution operator $W_{a,b,c}(f)(z)$ is in $S(\gamma,\alpha,\ell)$ for $|z|\leq r(\mu,\ell)$, where

$$r(\mu,\ell)=\inf_{n}\left\{\frac{\ell[\gamma(n-1)+\mu(n\alpha+1-\gamma)]}{\mu[\gamma(n-1)+\ell(n\alpha+1-\gamma)]\frac{(a)_n(b)_n}{c_n n!}}\right\}^{\frac{1}{n-1}}.$$

The result is sharp for the function

$$f_n(z)=z-\frac{\mu(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]}z^n,\ n=2,3,\dots.$$

Proof: Since $f\in S(\gamma,\alpha,\mu)$, we have

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]}{\mu(\alpha+(1-\gamma))}a_n\le 1.$$

It is sufficient to show that

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\frac{(a)_n(b)_n}{(c)_n n!}}{\mu(\alpha+(1-\gamma))}a_n|z|^{n-1}\le 1.$$

Note that (21) is satisfied if

$$\frac{[\gamma(n-1)+\ell(n\alpha+1-\gamma)]\frac{(a)_n(b)_n}{(c)_n n!}}{\ell(\alpha+(1-\gamma))}a_n|z|^{n-1}\le\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]}{\mu(\alpha+(1-\gamma))}a_n.$$

Solving for $|z|$,we get the result .

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