European Journal of Scientific Research
ISSN 1450-216X Vol.64 No.2 (2011), pp. 340-352

http://www.europeanjournalofscientificresearch.com

On a Class of Univalent Functions with Negative Coefficients Defined by Generalized Ruscheweyh Derivatives II

Waggas Galib Atshan
Department of Mathematics, College of Computer Science and Mathematics
University of Al-Qadisiya, Diwaniya-Iraq
E-mail: waggashnd@yahoo.com

Ahmed Sallal Joudah
Department of Mathematics, College of Computer Science and Mathematics
University of Al-Qadisiya, Diwaniya-Iraq
E-mail: ahmedhiq@yahoo.com

Abstract

In this paper , we have introduced a class $T^{\vartheta,\mu,v}(n,\tau,\alpha,\beta)$ of analytic and univalent functions as defined by making use of the generalized Ruscheweyh derivatives in the unit disk U .Here we obtain coefficient inequality, subordination property. We also introduce the subclass $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ consisting of functions with negative and fixed finitely many coefficients. We discuss some interesting properties of the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$

Keywords: Univalent Function, Generalized Ruscheweyh Derivatives, Extreme points, Subordination property, Weighted Mean, Arithmetic mean.
AMS Subject Classification Codes: Primary 30C45, Secondary 30C50, 26A33.

1. Introduction

Let W denote the class of functions analytic in the unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ and let $T(n)$ denote the subclass of W consisting of functions of the form:

$$f(z)=z-\sum_{k=n+1}^{\infty}a_k z^k,\left(a_k\geq 0,n\in\mathbb{N}=\{1,2,3,\ldots\}\right), \tag{1.}$$

which are analytic and univalent in the unit disk U . Then the function $f\in T(n)$ is said to be in the class $S(n,\rho)$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\rho\ ,\left(z\in U,0\leq\rho<1\right). \tag{2.}$$

function $f\in S(n,\rho)$ is called starlike function of order ρ

A function $f\in T(n)$ is said to be in the class $C(n,\rho)$ if and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\rho\ ,\left(z\in U,0\leq\rho<1\right). \tag{3.}$$

A function $f \in C(n,\rho)$ is called convex function of order ρ.

It is observed that $f \in C(n,\rho)$ if and only if

$zf' \in S(n,\rho), \forall n \in \mathbb{N}$ [2]. (4).

A function $f \in T(n)$ is said to be in the class $K(n,\rho)$ if there is a convex function g such that

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > \rho\ ,\left(\forall z \in U, 0 \le \rho < 1\right). \qquad (5).$$

A function $f \in K(n,\rho)$ is called close-to-convex of order ρ. We shall need the fractional derivative operator ([13], [14]) in this paper.

Let $a,b,c \in \mathbb{C}$ with $c \neq 0,-1,-2,\ldots$ The Gaussain hypergeometric function $_2F_1$ is defined by

$$_2F_1 \equiv {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \qquad (6).$$

Where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}). \end{cases}$$

Definition1. Let $0 \le \vartheta < 1$ and $\mu, v \in \mathbb{R}$ .Then, in terms of familiar (Gauss's) hypergeometric function $_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\vartheta,\mu,v}$ of a function f is defined by:

$$J_{0,z}^{\vartheta,\mu,v} f(z) = \begin{cases} \dfrac{1}{\Gamma(1-\vartheta)} \dfrac{d}{dz}\{z^{\vartheta-\mu} \displaystyle\int_0^z (z-\varepsilon)^{-\vartheta} f(\varepsilon)\, {}_2F_1\left(\mu-\vartheta, 1-v; 1-\vartheta; 1-\frac{\varepsilon}{z}\right) d\varepsilon\,\} & (0 \le \vartheta < 1) \\ \dfrac{d^n}{dz^n} J_{0,z}^{\vartheta-n,\mu,v} f(z) & (n \le \vartheta < n+1, n \in \mathbb{N}), \end{cases} \qquad (7).$$

Where the function f is analytic in a simply-connected region of the z-plane containing the origin, with the order

$$f(z) = O\left(|z|^{\epsilon}\right), \qquad (z \to 0), \qquad (8).$$

for $\epsilon > \max\{0, \mu - v\} - 1$ and the multiplicity of $(z-\varepsilon)^{-\vartheta}$ is removed by required $\log(z-\varepsilon)$ to be real when $(z-\varepsilon) > 0$.

The fractional derivative of order ϑ of a function f is defined by

$$D_z^{\vartheta}\{f(z)\} = \frac{1}{\Gamma(1-\vartheta)} \frac{d}{dz} \int_0^z \frac{f(\varepsilon)}{(z-\varepsilon)^{\vartheta}} d\varepsilon \qquad , 0 \le \vartheta < 1, \qquad (9).$$

Where f is chosen as in (7), and the multiplicity of $(z-\varepsilon)^{-\vartheta}$ is removed by required $\log(z-\varepsilon)$ to be real when $(z-\varepsilon) > 0$. y

By comparing (7) and (9), we find

$$J_{0,z}^{\vartheta,\vartheta,v} f(z) = D_z^{\vartheta}\{f(z)\} \qquad ,\left(0 \le \vartheta < 1\right). \qquad (10).$$

In terms of gamma function, we have

$$J_{0,z}^{\vartheta,\mu,v} z^k = \frac{\Gamma(k+1)\Gamma(1-\mu+v+k)}{\Gamma(1-\mu+k)\Gamma(1-\vartheta+v+k)} z^{k-\mu} \quad , \quad .$$

$$\left(0 \le \vartheta < 1, \mu, v \in \mathbb{R}\, and\, k > \max\{0, \mu - v\} - 1\right). \qquad (11).$$

Definition2. Let $f \in T(n)$ be given by (1). Then the class $T^{\vartheta,\mu,v}(n,\tau,\alpha,\beta)$ is defined by

$$T^{\vartheta,\mu,v}(n,\tau,\alpha,\beta)=\{f\in T(n):\left|\frac{\frac{z\left(\Im_1^{\vartheta,\mu}f(z)\right)''}{\left(\Im_1^{\vartheta,\mu}f(z)\right)'}}{2\tau\left(1-\alpha+\frac{z\left(\Im_1^{\vartheta,\mu}f(z)\right)''}{\left(\Im_1^{\vartheta,\mu}f(z)\right)'}\right)-\frac{z\left(\Im_1^{\vartheta,\mu}f(z)\right)''}{\left(\Im_1^{\vartheta,\mu}f(z)\right)'}}\right|<\beta \quad .$$

$$,\left(z\in U,0<\beta\leq 1,\frac{1}{2}\leq\tau\leq 1,0\leq\alpha<\frac{1}{2\tau}\, and\, \vartheta>-1\right)\}, \quad (12).$$

Where $\Im_1^{\vartheta,\mu}f(z)$ is a generalized Ruscheweyh derivative defined by Goyal and Goyal [3, p.442] as

$$\Im_1^{\vartheta,\mu}f(z)=\frac{\Gamma(\mu-\vartheta+v+2)}{\Gamma(v+2)\Gamma(\mu+1)}zJ_{0,z}^{\vartheta,\mu,v}\left(z^{\mu-1}f(z)\right)$$

$$=z-\sum_{k=n+1}^{\infty}a_kC_1^{\vartheta,\mu}(k)z^k, \quad (13).$$

where

$$C_1^{\vartheta,\mu}(k)=\frac{\Gamma(k+\mu)\Gamma(v+2+\mu-\vartheta)\Gamma(k+v+1)}{\Gamma(k)\Gamma(k+v+1+\mu-\vartheta)\Gamma(v+2)\Gamma(1+\mu)}\, . \quad (14).$$

For $\mu=\vartheta=\gamma, v=1$ the generalized Ruscheweyh derivatives reduce to ordinary Ruscheweyh derivatives of f of order γ [9]:

$$D^{\gamma}f(z)=\frac{z}{\Gamma(\gamma+1)}D^{\gamma}\left(z^{\gamma-1}f(z)\right)=z-\sum_{k=n+1}^{\infty}a_kC_k(\gamma)z^k, \quad (15).$$

where

$$C_k(\gamma)=\frac{(\gamma+1)(\gamma+2)\ldots(\gamma+k-1)}{(k-1)!}. \quad (16).$$

The class $T^{\vartheta,\mu,v}(n,\tau,\alpha,\beta)$ contains well-known classes of analytic functions, for example,

(i) If $\mu=\vartheta=0, v=1, n=1$ we get the class $T^{0,0.1}(1,\tau,\alpha,\beta)$ studied by Aqlan and Kulkarni [1].

(ii) If $\mu=\vartheta=0, v=1, \rho=\beta, \alpha=0, \tau=\frac{1}{2}$ we get the class of convex functions

of order p, $(C(n,\rho))$.

The same properties have been found for other classes in [4],[11] and [12].

2. Coefficient Inequality

The following theorem gives a necessary and sufficient condition for function to be in the class $T^{\vartheta,\mu,v}(n,\tau,\alpha,\beta)$

Theorem 1.

Let $f\in T(n)$ Then $f\in T^{\vartheta,\mu,v}(n,\tau,\alpha,\beta)$ if and only if

$$\sum_{k=n+1}^{\infty}k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)a_k\leq 2\beta\tau(1-\alpha), \quad (17).$$

Where $0<\beta\leq 1,\frac{1}{2}\leq\tau\leq 1,0\leq\alpha<\frac{1}{2\tau}, \vartheta>-1, n\in\mathbb{N}\, and\, C_1^{\vartheta,\mu}(k)$ given by (14). The result (17) is sharp for the function

$$f(z)=z-\frac{2\beta\tau(1-\alpha)}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}z^k \quad ,k\geq n+1..$$

proof . For $|z|=1$ we have

$$\left|z(\Im_1^{\vartheta,\mu}f(z))''\right|-\beta\left|2\tau\left[(1-\alpha)\left(\Im_1^{\vartheta,\mu}f(z)\right)'+z\left(\Im_1^{\vartheta,\mu}f(z)\right)''\right]-z\left(\Im_1^{\vartheta,\mu}f(z)\right)''\right|$$

$$\left|-\sum_{k=n+1}^{\infty}k(k-1)C_1^{\vartheta,\mu}(k)a_k z^{k-1}\right|$$

$$-\beta\left|2\tau(1-\alpha)-\sum_{k=n+1}^{\infty}\left[2k\tau(k-\alpha)-(k-1)\right]C_1^{\vartheta,\mu}(k)a_k z^{k-1}\right|$$

$$\leq\sum_{k=n+1}^{\infty}k(k-1)C_1^{\vartheta,\mu}(k)a_k-2\tau\beta(1-\alpha)$$

$$+\beta\sum_{k=n+1}^{\infty}k\left[2k\tau(k-\alpha)-(k-1)\right]C_1^{\vartheta,\mu}(k)a_k$$

$$=\sum_{k=n+1}^{\infty}k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-a)]C_1^{\vartheta,\mu}(k)a_k-2\beta\tau(1-a)\leq 0$$

By hypothesis.

Thus by maximum modulus Theorem $f\in T^{\vartheta,\mu,v}(n,\tau,\alpha,\beta)$ Conversely, assume that

$$\left|\frac{\dfrac{z\left(\Im_1^{\vartheta,\mu}f(z)\right)''}{\left(\Im_1^{\vartheta,\mu}f(z)\right)'}}{2\tau\left(1-\alpha+\dfrac{z\left(\Im_1^{\vartheta,\mu}f(z)\right)''}{\left(\Im_1^{\vartheta,\mu}f(z)\right)'}\right)-\dfrac{z\left(\Im_1^{\vartheta,\mu}f(z)\right)''}{\left(\Im_1^{\vartheta,\mu}f(z)\right)'}}\right|=\left|\frac{z\left(\Im_1^{\vartheta,\mu}f(z)\right)''}{2\tau(1-\alpha)\left(\Im_1^{\vartheta,\mu}f(z)\right)'+2\tau\left(\Im_1^{\vartheta,\mu}f(z)\right)''-z\left(\Im_1^{\vartheta,\mu}f(z)\right)''}\right|$$

$$=\left|\frac{-\sum_{k=n+1}^{\infty}k(k-1)C_1^{\vartheta,\mu}(k)a_k z^{k-1}}{2\tau(1-a)\left(1-\sum_{k=n+1}^{\infty}k\,C_1^{\vartheta,\mu}(k)a_k z^{k-1}\right)+\sum_{k=n+1}^{\infty}k(k-1)[1-2\tau]C_1^{\vartheta,\mu}(k)a_k z^{k-1}}\right|<\beta$$

Since $|Re(z)|\leq|z|$ or all z ,we have

$$\mathrm{Re}\left\{\frac{\sum_{k=n+1}^{\infty}k(k-1)C_1^{\vartheta,\mu}(k)a_k z^{k-1}}{2\tau(1-a)\left(1-\sum_{k=n+1}^{\infty}k\,C_1^{\vartheta,\mu}(k)a_k z^{k-1}\right)+\sum_{k=n+1}^{\infty}k(k-1)[1-2\tau]C_1^{\vartheta,\mu}(k)a_k z^{k-1}}\right\}<\beta \tag{18}$$

We can choose value of z n the real axis so that $\left(\Im_1^{\vartheta,\mu}f(z)\right)'$ is real.

Let $z\to 1^-$ through real values, so we can write (18) as

$$\sum_{k=n+1}^{\infty}k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)a_k\leq 2\beta\tau(1-\alpha).$$

Finally , sharpness follows if we take

$$f(z)=z-\frac{2\beta\tau(1-\alpha)}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}z^k, k=n+1,n+2,\ldots . \tag{19}$$

The proof is complete.

Corollary 1. Let $f\in T^{\vartheta,\mu,v}(n,\tau,\alpha,\beta)$ Then

$$a_k\le\frac{2\beta\tau(1-\alpha)}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)} \quad ,k=n+1,n+2,\ldots . \tag{20}$$

The equality in (20) is attained for the function f iven by (19).

3. Subordination Property

Definition3. Let f nd g e analytic in U then g is said to be subordinate to f written $g\prec f$ $g(z)\prec f(z)$ if there exists a Shwarz function w, which is analytic in U ith $w(0)=0$ and $|w(z)|<1$ $(z\in U)$ such that $g(z)=f(w(z))$ $(z\in U)$ indeed it is known that

$$g(z)\prec f(z)\ (z\in U)\Rightarrow g(0)=f(0) and\ g(U)\subset f(U)..$$

particular, if the function f is univalent in U we have the following equivalence ([5], [6]):

$$g(z)\prec f(z)\ (z\in U)\Leftrightarrow g(0)=f(0) and\ g(U)\subset f(U)..$$

Theorem 2.

For $n=1$ let $f\in T^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ nd g e an arbitrary element of $T(1)$ such that $g\prec f$ defined in Definition 3, and if

$$g_k=\frac{1}{k!}\left[\frac{d^k(f(w(z)))}{dz^k}\right]_{z=0} \tag{21}$$

also if

$$\frac{\sum_{k=2}^{\infty}k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)|g_k|}{|g_1|}\le 2\beta\tau(1-\alpha). \tag{22}$$

Then $g\in T^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ Proof. Since $g\prec f$ by definition of subordination there is analytic function $w(z)$ uch that $|w(z)|\le|z|$ nd $g(z)=f(w(z))$ But g s the composition of two analytic functions in the unit disk , therefore we can expend this function in terms of Taylor series at origin as below

$$g(z)=\sum_{k=0}^{\infty}g_kz^k\ ,.$$

Where g_k s defined in (21). Hence

$$g_0=\frac{f(w(0))}{0!}=0\ ,g_1=\frac{w'(0)f'(0)}{1!}=w'(0).$$

Therefore , we can write

$$g(z)=g_1z-\sum_{k=2}^{\infty}g_kz^k$$

and

$$\Im_1^{\vartheta,\mu}g(z)=g_1z-\sum_{k=2}^{\infty}C_1^{\vartheta,\mu}(k)g_kz^k\ ,$$

we must prove $g\in T^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ in other words , we show that

$$\left|\frac{z\left(\Im_1^{\vartheta,\mu}g(z)\right)''}{2\tau(1-\alpha)\left(\Im_1^{\vartheta,\mu}g(z)\right)'+2\tau\left(\Im_1^{\vartheta,\mu}g(z)\right)''-z\left(\Im_1^{\vartheta,\mu}g(z)\right)''}\right|<\beta .$$

or

$$\left|\frac{-\sum_{k=2}^{\infty}\mathrm{k}(\mathrm{k}-1)C_1^{\vartheta,\mu}(k)\,g_k z^{k-1}}{2\tau(1-\alpha)\left(g_1-\sum_{k=2}^{\infty}\mathrm{k}\,C_1^{\vartheta,\mu}(k)\,g_k z^{k-1}\right)+\sum_{k=2}^{\infty}\mathrm{k}(\mathrm{k}-1)[1-2\tau]C_1^{\vartheta,\mu}(k)\,g_k z^{k-1}}\right|<\beta .$$

Since $|Re(z)|\le|z|$ or all z e have

$$Re\left\{\frac{\sum_{k=2}^{\infty}\mathrm{k}(\mathrm{k}-1)C_1^{\vartheta,\mu}(k)\,g_k z^{k-1}}{2\tau(1-\alpha)\left(g_1-\sum_{k=2}^{\infty}\mathrm{k}\,C_1^{\vartheta,\mu}(k)\,g_k z^{k-1}\right)+\sum_{k=2}^{\infty}\mathrm{k}(\mathrm{k}-1)[1-2\tau]C_1^{\vartheta,\mu}(k)\,g_k z^{k-1}}\right\}<\beta . \tag{23}$$

We can choose value of z n the real axis so that $\left(\Im_1^{\vartheta,\mu}g(z)\right)'$ s real.

Let $z\to 1^-$ through real values, so we can write (23) as

$$\sum_{k=2}^{\infty}k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)\,g_k\le g_1 2\beta\tau(1-\alpha).$$

The proof is complete.

4. Some Properties of a Subclass

$\boldsymbol{T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)}$ e introduce the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ he subclass of $T^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ where

$$T^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)=\left\{f\in T(1):\left|\frac{\dfrac{z\left(\Im_1^{\vartheta,\mu}f(z)\right)''}{\left(\Im_1^{\vartheta,\mu}f(z)\right)'}}{2\tau\left(1-\alpha+\dfrac{z\left(\Im_1^{\vartheta,\mu}f(z)\right)''}{\left(\Im_1^{\vartheta,\mu}f(z)\right)'}\right)-\dfrac{z\left(\Im_1^{\vartheta,\mu}f(z)\right)''}{\left(\Im_1^{\vartheta,\mu}f(z)\right)'}}\right|<\beta\right\},$$

consisting of functions with negative and fixed finitely many coefficients of the form:

$$f(z)=z-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i-\sum_{k=m+1}^{\infty}a_k z^k , \tag{24}$$

Where $m=2,3,\ldots,$ $a_k\ge 0$ or $k=m+1,m+2,\ldots,$ $0\le c_i\le 1$ for $i=2,3,\ldots,m$ and

$$o\le\sum_{i=2}^{m}c_i\le 1.$$

The different cases were studied earlier by many authors e.g. [7], [8], [10].

We need the following lemma which has been proved in general case in Theorem 1.

Lemma 1. Let

$$f(z)=z-\sum_{k=2}^{\infty}a_k z^k\in T(1) .$$

Then $f\in T^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ f and only if

$$\sum_{k=2}^{\infty}\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)}{2\beta\tau(1-\alpha)}C_1^{\vartheta,\mu}(k)a_k\le 1 .$$

The following theorem gives a necessary and sufficient condition for a function to be in $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$

Theorem 3.

Let f be defined by (24). Then $f\in T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ f and only if

$$\sum_{k=m+1}^{\infty}\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)}{2\beta\tau(1-\alpha)}C_1^{\vartheta,\mu}(k)a_k\le 1-\sum_{i=2}^{m}c_i \quad . \tag{25}$$

Proof. By letting

$$a_i=\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)} \ ,.$$

Since $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)\subset T^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ o $f\in T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ f and only if

$$\sum_{i=2}^{m}\frac{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)}{2\beta\tau(1-\alpha)}a_i\,C_1^{\vartheta,\mu}(i)+\sum_{k=m+1}^{\infty}\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)}{2\beta\tau(1-\alpha)}a_k\,C_1^{\vartheta,\mu}(k)\le 1$$

$$\sum_{k=m+1}^{\infty}\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)}{2\beta\tau(1-\alpha)}a_k\,C_1^{\vartheta,\mu}(k)\le 1-\sum_{i=2}^{m}c_i$$

d this complete the proof.

Corollary 2.

Let f efined by (24) be in the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ then for $k\ge m+1$ we have

$$a_k\le\frac{2\beta\tau(1-\alpha)\left(1-\sum_{i=2}^{m}c_i\right)}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)} \quad .$$

is result is sharp due to the function f defined by

$$f(z)=z-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i-\frac{2\beta\tau(1-\alpha)\left(1-\sum_{i=2}^{m}c_i\right)}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}z^k.$$

Theorem 4.

Let

$$f_j(z)=z-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i-\sum_{k=m+1}^{\infty}a_{k,j}z^k \ , \tag{26}$$

for $j=1,2,\ldots,\ell$ be in the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ Then the function

$$F(z)=\sum_{j=1}^{\ell}\eta_j f_j(z).$$

is also in $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ where

$$\sum_{j=1}^{\ell}\eta_j=1\ ,0\le c_i\le 1\ ,0\le\sum_{i=2}^{m}c_i\le 1. \quad .$$

Proof. By Theorem 3 for every $j=1,2,\ldots,\ell$, we have

$$\sum_{k=m+1}^{\infty}\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)}a_{k,j}\le 1-\sum_{i=2}^{m}c_i \quad . \quad .$$

But

$$F(z)=\sum_{j=1}^{\ell}\eta_j f_j(z)=z-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i-\sum_{k=m+1}^{\infty}\left(\sum_{j=1}^{\ell}\eta_j a_{k,j}\right)z^k .$$

So,

$$\sum_{k=m+1}^{\infty}\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)}\left(\sum_{j=1}^{\ell}\eta_j a_{k,j}\right).$$

$$\sum_{j=1}^{\ell}\sum_{k=m+1}^{\infty}\left(\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C}{2\beta\tau(1-\alpha)}\right.$$

$$\leq\sum_{j=1}^{\ell}\left(1-\sum_{i=2}^{m}c_i\right)n_j=1-\sum_{i=2}^{m}c_i$$

and the proof is complete.

Remark 1. Let f_1,f_2e in the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$Then the function $H(z)=\frac{1}{2}[f_1(z)+f_2(z)]$ is also in the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$.

R**emark 2.** The class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$s a convex set.

In the next theorem , we will prove the arithmetic mean property.

Theorem 5.

Let f_j , $(j=1,2,\ldots,\ell)$ defined by (26) be in the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$Then the function

$$Q(z)=z-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i-\sum_{k=m+1}^{\infty}b_k z^k\ ,\quad (b_k\geq 0).$$

also in the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ where

$$b_k=\frac{1}{\ell}\sum_{j=1}^{\ell}a_{k,j}\ .$$

Proof. We have

$$\sum_{k=m+1}^{\infty}\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)}{2\beta\tau(1-\alpha)}b_k C_1^{\vartheta,\mu}(k)$$

$$=\sum_{k=m+1}^{\infty}\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)}{2\beta\tau(1-\alpha)}\left(\frac{1}{\ell}\sum_{j=1}^{\ell}a_{k,j}\right)C_1^{\vartheta,\mu}(k)$$

$$=\frac{1}{\ell}\sum_{j=1}^{\ell}\left(\sum_{k=m+1}^{\infty}\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)}{2\beta\tau(1-\alpha)}a_{k,j}C_1^{\vartheta,\mu}(k)\right)$$

by (Theorem 3)

$$\leq\frac{1}{\ell}\sum_{j=1}^{\ell}\left(1-\sum_{i=2}^{m}c_i\right)=1-\sum_{i=2}^{m}c_i\ ,$$

and the proof is complete.

Definition 4.

Let f nd g belong to $T(n)$ Then the weighted mean $h_j(z)$ f f and g s given by

$$h_j(z)=\frac{1}{2}\left[(1-j)f(z)+(1+j)g(z)\right] \text{ here } -1\le j\le 1$$

Theorem 6.

Let f nd g be in the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ Then the weighted mean of f and g is also in the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ Proof. By using Definition 4 , we obtain

$$h_j(z)=\frac{1}{2}\left[(1-j)\left(z-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i-\sum_{k=m+1}^{\infty}a_kz^k\right)+(1+j)\left(z-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i-\sum_{k=m+1}^{\infty}b_kz^k\right)\right]$$

$$=\left(z-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i-\sum_{k=m+1}^{\infty}\frac{1}{2}\left[(1-j)a_k+(1+j)b_k\right]z^k\right)$$

Since f nd g re in the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ sing Theorem 4 ,we have

$$\sum_{k=m+1}^{\infty}\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)}\frac{1}{2}\left[(1-j)a_k+(1+j)b_k\right].$$

$$=\frac{1}{2}\sum_{k=m+1}^{\infty}\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)}(1-j)a_k$$

$$+\frac{1}{2}\sum_{k=m+1}^{\infty}\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)}(1-j)b_k$$

$$\le\frac{1}{2}(1-j)\left(1-\sum_{i=2}^{m}c_i\right)+\frac{1}{2}(1+j)\left(1-\sum_{i=2}^{m}c_i\right)=\sum_{i=2}^{m}c_i$$

and again by Theorem 4 , $h_j(z)\in T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$

Now , we obtain the extreme points of the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ ut we need the following theorem to prove.

Theorem 7.

Let

$$f_m(z)=z-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i \tag{27}$$

and for $k\ge m+1$.

$$f_m(z)=z-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i$$
$$-\frac{2\beta\tau(1-\alpha)c_i\left(1-\sum_{i=2}^{m}c_i\right)}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}z^k$$

Then the function Y s in the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ f and only if it can be expressed in the form

$$\Upsilon(z)=\sum_{k=m}^{\infty}\sigma_k f_k(z)\ , \qquad .$$

Where $\sigma_k\ge 0\ (k\ge m)$ and

$$\sum_{k=m}^{\infty}\sigma_k=1\ .$$

Proof. Let

$$\Upsilon(z)=\sum_{k=m}^{\infty}\sigma_k f_k(z) \quad .$$

Then

$$\Upsilon(z)=\sigma_m f_m(z)+\sum_{k=m+1}^{\infty}\sigma_k f_k(z)$$

$$=\sigma_m z-\sigma_m\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i+\sum_{k=m+1}^{\infty}\sigma_k z$$

$$-\sum_{k=m+1}^{\infty}\sigma_k\left(\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i\right)$$

$$-\sum_{k=m+1}^{\infty}\sigma_k\left(\frac{2\beta\tau(1-\alpha)\left(1-\sum_{i=2}^{m}c_i\right)}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}z^k\right)$$

$$=\left(\sigma_m+\sum_{k=m+1}^{\infty}\sigma_k\right)z-\left(\sigma_m+\sum_{k=m+1}^{\infty}\sigma_k\right)\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i$$

$$-\sum_{k=m+1}^{\infty}\frac{2\beta\tau(1-\alpha)\left(1-\sum_{i=2}^{m}c_i\right)}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}\sigma_k z^k$$

$$=z-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i$$

$$-\sum_{k=m+1}^{\infty}\frac{2\beta\tau(1-\alpha)\left(1-\sum_{i=2}^{m}c_i\right)\sigma_k}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}z^k$$

Finally , we have

$$\sum_{k=m+1}^{\infty}\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)(1-)2\beta\tau(1-\alpha)\sigma_k\,C_1^{\vartheta,\mu}(k)}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)2\beta\tau(1-\alpha)}$$

$$=\left(1-\sum_{i=2}^{m}c_i\right)\sum_{k=m+1}^{\infty}\sigma_k=\left(1-\sum_{i=2}^{m}c_i\right)(1-\sigma_m)\le 1-\sum_{i=2}^{m}c_i \quad .$$

Thus $\Upsilon\in T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ Conversely , assume $\Upsilon\in T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ so

$$\Upsilon(z)=z-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i-\sum_{k=m+1}^{\infty}a_k z^k \quad .$$

By putting

$$\sigma_k=\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)\left(1-\sum_{i=2}^{m}c_i\right)}a_k \;,(k\ge m+1),$$

We have $\sigma_k\ge 0$ nd if we set

$$\sigma_m=1-\sum_{k=m+1}^{\infty}\sigma_k \quad ,$$

We get ,

$$\Upsilon(z)=z-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i-\sum_{k=m+1}^{\infty}\frac{2\beta\tau(1-\alpha)\left(1-\sum_{i=2}^{m}c_i\right)\sigma_k}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}z^k$$

$$=\mathrm{f_m}(z)-\sum_{k=m+1}^{\infty}\left(z-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i-f_k(z)\right)\sigma_k$$

$$=\mathrm{f_m}(z)-\sum_{k=m+1}^{\infty}f_m(z)-f_k(z))\sigma_k$$

$$=\left(1-\sum_{k=m+1}^{\infty}\sigma_k\right)\mathrm{f_m}(z)+\sum_{k=m+1}^{\infty}\sigma_k\, f_k(z)=\sum_{k=m}^{\infty}\sigma_k\, f_k(z)$$

Corollary 3.

The extreme points of the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ re the functions $f_k \quad (k\geq m)$ efined by (27),(28).

Now , we obtain the radii of starlikeness and convexity for the elements of the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$

Theorem 8.

Let the function f efined by (24) be in the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ Then f s starlike of order $\eta\ (0\leq\eta<1)$ n $|z|<r$ where r s the largest value such that

$$\sum_{i=2}^{m}\frac{c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}r^{i-1}+\frac{\left(1-\sum_{i=2}^{m}c_i\right)}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}r^{k-1}\leq\frac{1}{2\beta\tau(1-\alpha)}\ .$$

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\eta\ . \tag{29}$$

Thus ,we have

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq\frac{\sum_{i=2}^{m}\frac{(i-1)2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}|z|^{i-1}+\sum_{k=m+1}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}|z|^{i-1}-\sum_{k=m+1}^{\infty}a_k|z|^{k-1}}$$

$$\leq\frac{\sum_{i=2}^{m}\frac{(i-1)2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}r^{i-1}+\sum_{k=m+1}^{\infty}(k-1)\frac{2\beta\tau(1-\alpha)\left(1-\sum_{i=2}^{m}c_i\right)}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}r^{k-1}}{1-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}r^{i-1}-\sum_{k=m+1}^{\infty}\frac{2\beta\tau(1-\alpha)\left(1-\sum_{i=2}^{m}c_i\right)}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}r^{k-1}}$$

Therefore (29) holds true if the last term of above relationship is less than $1-\eta$ r equivalently

$$\sum_{i=2}^{m}\frac{(i-\eta)2\beta\tau(1-\alpha)c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)(1-\eta)C_1^{\vartheta,\mu}(i)}r^{i-1}+\frac{(k-\eta)2\beta\tau(1-\alpha)\left(1-\sum_{i=2}^{m}c_i\right)}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)(1-\eta)C_1^{\vartheta,\mu}(k)}r^{k-1}\leq 1.$$

Finally ,we find

$$\sum_{i=2}^{m}\frac{c_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}r^{i-1}+\frac{\left(1-\sum_{i=2}^{m}c_i\right)}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}r^{k-1}\le\frac{1}{2\beta\tau(1-\alpha)}$$

and this completes the proof.

Making use of (4), we obtain the following corollary:

Corollary 4.

Let $f\in T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ Then f s convex of order η $(0\le\eta<1)$ n $|z|<r$ where r s the largest value for which

$$\sum_{i=2}^{m}\frac{ic_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}r^{i-1}+\frac{k\left(1-\sum_{i=2}^{m}c_i\right)}{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}r^{k-1}<\frac{1}{2\beta\tau(1-\alpha)}.$$

Theorem 9.

Let $f\in T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$ nd

$$d_i=\frac{2\beta\tau(1-\alpha)c_i^{2}}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}\ ,\quad (2\le i\le m).$$

Then the function

$$h(z)=z-\sum_{i=2}^{m}\frac{2\beta\tau(1-\alpha)d_i}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}z^i-\sum_{k=m+1}^{\infty}a_kz^k$$

also in the class $T_{c_m}^{\vartheta,\mu,v}(1,\tau,\alpha,\beta)$

Proof. It can be verified that $i\left((i-1)(1-\beta+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)>1$; $i=2,3,\ldots,m$ .herefore

$$0\le d_i=\frac{2\beta\tau(1-\alpha)c_i^{2}}{i\left((i-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(i)}<c_i\le 1\ .$$

$$0\le\sum_{i=2}^{m}d_i\le\sum_{i=2}^{m}c_i\le 1\ .$$

Thus

$$\sum_{k=m+1}^{\infty}\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)\left(1-\sum_{i=2}^{m}d_i\right)}a_k\le\sum_{k=m+1}^{\infty}\frac{k\left((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)\right)C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)\left(1-\sum_{i=2}^{m}c_i\right)}a_k\le 1.$$

and this completes the proof.

References

[1] E. S. Aqlan , *Some Problems Connected with Geometric Function Theory* , Ph.D. Thesis (2004) , Pune University , Pune.

[2] P. L. Duren, *"Univalent Functions"*, Grundelheren der Mathematischen Wissenchaften 259 ,Springer-Verlag , New York , Berlin, Heidelberg , Tokyo,1983.

[3] S. P. Goyal and Ritu Goyal , *On a class of multivalent functions defined by generalized Ruscheweyh derivatives involving a general fractional derivative operator* , Journal of Indian Acad. Math 27(2)(2005) , 439-456.

[4] S. Kanas and A. Wisniowska , *Conic regions and K-uniformly convexity II* Folia Sci. Tech. Reso. 178(1998),65-78.

[5] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions* , Michigan Math. J. ,28(1981),157-171.

[6] S. S. Miller and P. T. Mocanu, *Differential subordinations*: *Theory and Applications,* Series on Monographs and Text Books in pure and Applied Mathematics , Vol. 225,Marcel Dekker, New York and Basel,2000.

[7] M. Nunokawa, *A sufficient condition for univalence and starlikeness*, proc. Japan Acad. Ser. A Math. Sci. 65(1989),163-164.

[8] S. Owa, M. Nunokawa and H. M. Srivastava, *A certain class of multivalent functions* , Appl. Math. Lett. 10(2)(1997), 7-10.

[9] T. Rosy , K. G. Subramanian and G. Murugusundaramoorthy , *Neighborhoods and Partial sums of starlike functions based on Ruscheweyh derivatives* , J. Inequal. Pure and Appl. Math. 4(4)(2003), Art. 64,1-8.

[10] S. Shams and S. R. Kulkarni , *A class of univalent functions with negative and fixed finitely many coefficients* , Acta Cienica Indica, XXXIXM (3)(2003),587-594.

[11] S. Shams and S. R. Kulkarni, *Certain properties of the class of univalent functions defined by Ruscheweyh derivative*, Bull. Cal. Math. Soc. 97(2005),35-48.

[12] H. Silverman, *Univalent functions with negative coefficients* ,Proc. Amer. Math. Soc. 51(1975),109-116.

[13] H. M. Srivastava ,*Distortion inequalities for analytic and univalent functions associated with certain fractional calculus and other linear operators* (In Analytic and Geometric Inequalities and Applications eds. T. M. Rassias and H. M. Srivastava), Kluwar Academic Publishers, 478(1999), 349-374.

[14] H. M. Srivastava and R. K. Saxena, *Operators of fractional Integration and their applications* , Applied Mathematics and computation,118(2001),1-52.