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On a New Subclass of Univalent Functions with Positive Coefficients Defined by Ruscheweyh Derivative

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Abstract

In this paper, we have discussed a subclass $N(\alpha,\delta,\beta,\lambda)$ of analytic and univalent functions with positive coefficients defined by Ruscheweyh derivative in unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. We obtain basic properties like coefficient inequality, distortion theorem, extreme points, radii of starlikeness, convexity and close-to-convexity and neighborhoods.

Keywords: Univalent function, Ruscheweyh derivative, radius of Starlikeness, Distortion theorem, Extreme points, neighborhoods.

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1- Introduction:

Let μ denote the class of functions given by

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \tag{1.1}$$

which are analytic and univalent in open unit disk $U=\{z\in\mathbb{C}:\ |z|<1\}$.

If a function f is given by (1.1) and g is defined by

$$g(z)=z+\sum_{n=2}^{\infty}b_n z^n \quad , \tag{1.2}$$

is in the class μ, then the convolution (or Hadamard product) of f and g is defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n,\ z\in U. \tag{1.3}$$

Let N be a subclass of μ consisting of functions of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\ (\mathrm{a_n}\geq 0). \tag{1.4}$$

We denote by $S^*(\alpha), K(\alpha)$ consisting of all functions which are respectively starlike and convex of order α in U with $0\leq\alpha<1$, thus

$$S^*(\alpha)=\left\{f\in S : Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha : 0\leq\alpha<1, z\in U\right\}$$

$$k(\alpha)=\left\{f\in S : Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha : 0\leq\alpha<1, z\in U\right\}.$$

The Ruscheweyh derivative [5], of $f\in N$ denoted by $D^\lambda f(z)$ of order λ is defined by

$$D^\lambda f(z)=z+\sum_{\mathrm{n}=2}^{\infty}a_nT_n(\lambda)z^n,(a_n\geq 0)$$

Where

$$T_n(\lambda)=\frac{(\lambda+1)(\lambda+2)...(\lambda+n-1)}{(n-1)!},\lambda>-1, z\in U.$$

Definition (1): A function $f\in N$ is said to be in the class $N(\alpha,\delta,\beta,\lambda)$ if the following inequality is satisfied:

$$\left|\frac{\frac{z(D^\lambda f(z))''}{(D^\lambda f(z))'}+(1-\alpha)}{(1-\alpha)\frac{z(D^\lambda f(z))''}{(D^\lambda f(z))'}+(2-\delta)}\right|<\beta, \tag{1.5}$$

for $|z|<1,\ \frac{1}{2}\leq\beta<1,\ \frac{1}{2}<\alpha<1$ *and* $\frac{1}{2}\leq\delta\leq 1.$

Some anthors studied univalent functions for other classes, like, [1], [2], [3].

2. Coefficient Estimates

Theorem (1): Let the function f be defined by (1.4). Then $f\in N(\alpha,\delta,\beta,\lambda)$ if and only if

$$\sum_{n=2}^{\infty}n[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)a_n\leq\beta(2-\delta)+\alpha-1, \tag{2.1}$$

where $\frac{1}{2}\leq\beta<1, \frac{1}{2}<\alpha<1$ *and* $\frac{1}{2}\leq\delta\leq 1.$ The result (2.1) is sharp for the function

$$f(z)=z+\frac{\beta(2-\delta)+\alpha-1}{n[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)}z^n, n\geq 2.$$

Proof: Assume that the inequality (2.1) holds true and let $|z|=1$, then, we have

$$\left|z(D^\lambda f(z))'' + (1-\alpha)(D^\lambda f(z))'\right| - \beta\left|(1-\alpha)z(D^\lambda f(z))'' + (2-\delta)(D^\lambda f(z))'\right|$$

$$= \left|(1-\alpha) + \sum_{n=2}^{\infty} n(n-\alpha)T_n(\lambda)a_n z^{n-1}\right| - \beta\left|(2-\delta) + \sum_{n=2}^{\infty} n[(n-1)(1-\alpha) + (2-\delta)]T_n(\lambda)a_n z^{n-1}\right|$$

$$\le \sum_{n=2}^{\infty} n[n(1-\beta) + \alpha(\beta(n-1)-1) + \beta(\delta-1)]T_n(\lambda)a_n - [\beta(2-\delta) + \alpha - 1] \le 0,$$

by hypothesis. Hence, by maximum modulus principle, $f \in N(\alpha, \delta, \beta, \lambda)$.
Conversely, suppose that f defined by (1.4) is in the class $N(\alpha, \delta, \beta, \lambda)$ .Hence

$$\left|\frac{z(D^\lambda f(z))'' + (1-\alpha)(D^\lambda f(z))'}{(1-\alpha)z(D^\lambda f(z))'' + (2-\delta)(D^\lambda f(z))'}\right| = \left|\frac{(1-\alpha) + \sum_{n=2}^{\infty} n(n-\alpha)T_n(\lambda)a_n z^{n-1}}{(2-\delta) + \sum_{n=2}^{\infty} n[(n-1)(1-\alpha) + (2-\delta)]T_n(\lambda)a_n z^{n-1}}\right| < \beta.$$

Since $Re(z) < |z|$ for all z, we have

$$\operatorname{Re}\left\{\frac{(1-\alpha) + \sum_{n=2}^{\infty} n(n-\alpha)T_n(\lambda)a_n z^{n-1}}{(2-\delta) + \sum_{n=2}^{\infty} n[(n-1)(1-\alpha) + (2-\delta)]T_n(\lambda)a_n z^{n-1}}\right\} < \beta, \tag{2.2}$$

we can choose the value of z on the real axis. Let $z \to 1^-$ through real values, we obtain the inequality (2.1).
Finally, sharpness follows if, we take

$$f(z) = z + \frac{\beta(2-\delta) + \alpha - 1}{n[n(1-\beta) + \alpha(\beta(n-1)-1) + \beta(\delta-1)]T_n(\lambda)} z^n, n \ge 2 \tag{2.3}$$

Corollary (1): Let $f \in N(\alpha, \delta, \beta, \lambda).$ Then

$$a_n \le \frac{\beta(2-\delta) + \alpha - 1}{n[n(1-\beta) + \alpha(\beta(n-1)-1) + \beta(\delta-1)]T_n(\lambda)}, n = 2,3,\dots . \tag{2.4}$$

3- Growth and Distortion Theorems

In the following theorems, we obtain the growth and distortion theorems for function $f \in N(\alpha, \delta, \beta, \lambda)$.
Theorem (2): Let the function $f(z)$ defined by (1.4) be in the class $N(\alpha, \delta, \beta, \lambda)$. Then

$$r - \frac{\beta(2-\delta) + \alpha - 1}{2[2(1-\beta) + \alpha(\beta-1) + \beta(\delta-1)](\lambda+1)} r^2 \le |f(z)| \le r + \frac{\beta(2-\delta) + \alpha - 1}{2[2(1-\beta) + \alpha(\beta-1) + \beta(\delta-1)](\lambda+1)} r^2,$$
$$|z| = r < 1 . \tag{3.1}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z + \frac{\beta(2-\delta) + \alpha - 1}{2[2(1-\beta) + \alpha(\beta-1) + \beta(\delta-1)](\lambda+1)} z^2.$$

Proof: Let $f(z) \in N(\alpha, \delta, \beta, \lambda).$ Then by Theorem (1), we have

$$\sum_{n=2}^{\infty} a_n \le \frac{\beta(2-\delta) + \alpha - 1}{2[2(1-\beta) + \alpha(\beta-1) + \beta(\delta-1)](\lambda+1)}.$$

Hence

$$\left|f(z)\right| \le \left|z\right| + \sum_{n=2}^{\infty} a_n \left|z^n\right| = r + r^2 \sum_{n=2}^{\infty} a_n \le r + \frac{\beta(2-\delta)+\alpha-1}{2[2(1-\beta)+\alpha(\beta-1)+\beta(\delta-1)](\lambda+1)} \mathrm{r}^2 . \tag{3.2}$$

Similarly, we obtain

$$\left|f(z)\right| \ge \left|z\right| - \sum_{n=2}^{\infty} a_n \left|z^n\right| = r - r^2 \sum_{n=2}^{\infty} a_n \ge r - \frac{\beta(2-\delta)+\alpha-1}{2[2(1-\beta)+\alpha(\beta-1)+\beta(\delta-1)](\lambda+1)} r^2. \tag{3.3}$$

From bounds (3.2) and (3.3), we get (3.1).

Theorem (3): Let the function $f(z)$ defined by (1.4) be in the class $N(\alpha,\delta,\beta,\lambda)$ .Then

$$1 - \frac{\beta(2-\delta)+\alpha-1}{[2(1-\beta)+\alpha(\beta-1)+\beta(\delta-1)](\lambda+1)} r \le \left|f'(z)\right| \le 1 + \frac{\beta(2-\delta)+\alpha-1}{[2(1-\beta)+\alpha(\beta-1)+\beta(\delta-1)](\lambda+1)} r. \tag{3.4}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z + \frac{\beta(2-\delta)+\alpha-1}{2[2(1-\beta)+\alpha(\beta-1)+\beta(\delta-1)](\lambda+1)} z^2.$$

Proof: Let $f(z) \in N(\alpha,\delta,\beta,\lambda)$. Then by Theorem (1), we have

$$\sum_{n=2}^{\infty} a_n \le \frac{\beta(2-\delta)+\alpha-1}{[2(1-\beta)+\alpha(\beta-1)+\beta(\delta-1)](\lambda+1)}.$$

Hence

$$\left|f'(z)\right| \le \left|1\right| + \sum_{n=2}^{\infty} n a_n \left|z\right|^{n-1} = 1 + r\sum_{n=2}^{\infty} a_n \le 1 + \frac{\beta(2-\delta)+\alpha-1}{[2(1-\beta)+\alpha(\beta-1)+\beta(\delta-1)](\lambda+1)} r. \tag{3.5}$$

Similarly, we obtain

$$\left|f'(z)\right| \ge \left|1\right| - \sum_{n=2}^{\infty} n a_n \left|z\right|^{n-1} = 1 - r\sum_{n=2}^{\infty} a_n \ge 1 - \frac{\beta(2-\delta)+\alpha-1}{[2(1-\beta)+\alpha(\beta-1)+\beta(\delta-1)](\lambda+1)} r. \tag{3.6}$$

From bounds (3.5) and (3.6), we get (3.4).

4. Extreme points:

In the following theorem, we obtain extreme points for the class $N(\alpha,\delta,\beta,\lambda)$.

Theorem (4): Let $f_1(z) = z$ and

$$f_n(z) = z + \frac{\beta(2-\delta)+\alpha-1}{n[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)} z^n, \textit{for } \mathrm{n} = 2,3,\ldots .$$

Then $f \in N(\alpha,\delta,\beta,\lambda)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z),$$

where

$$(\mu_n \ge 0 \ \textit{and} \ \sum_{n=1}^{\infty} \mu_n = 1 \ \textit{ or } \ 1 = \mu_1 + \sum_{n=2}^{\infty} \mu_n).$$

Proof: Let

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) = z + \sum_{n=2}^{\infty} \frac{\beta(2-\delta)+\alpha-1}{n[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)} \mu_n z^n.$$

Then

$$\sum_{n=2}^{\infty}\frac{n[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)}{\beta(2-\delta)+\alpha-1}\mu_n\frac{\beta(2-\delta)+\alpha-1}{n[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)}$$

$$=\sum_{n=2}^{\infty}\mu_n=1-\mu_1\le 1.$$

Using Theorem (1), we easily get $f\in N(\alpha,\delta,\beta,\lambda)$.
Conversely, let $f\in N(\alpha,\delta,\beta,\lambda)$ is of the form (1.4). Then

$$a_n\le\frac{\beta(2-\delta)+\alpha-1}{n[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)},n\ge 2.$$

Setting

$$\mu_n=\frac{n[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)}{\beta(2-\delta)+\alpha-1}a_n, for \quad n\ge 2$$

and

$$\mu_1=1-\sum_{n=2}^{\infty}\mu_n.$$

Then

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n=z+\sum_{n=2}^{\infty}\frac{\beta(2-\delta)+\alpha-1}{n[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)}\mu_nz^n$$

$$=\mu_1z+\sum_{n=2}^{\infty}\mu_nf_n(z).$$

Thus

$$f(z)=\sum_{n=2}^{\infty}\mu_nf_n(z)=\mu_1f_1(z)+\sum_{n=2}^{\infty}\mu_nf_n(z).$$

5. Radii of Starlikeness, Convexity and Close-to-convexity:

In the following theorems, we obtain the radii of starlikeness and convexity and close-to-convexity for the class $N(\alpha,\delta,\beta,\lambda)$.

Theorem (5): Let $f\in N(\alpha,\delta,\beta,\lambda)$. Then f is starlike in the disk $|z|<R_1$, of order $\varphi, 0\le\varphi<1$, where

$$R_1=\inf_{n\ge 2}\left[\frac{(1-\varphi)n[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)}{(n-\varphi)[\beta(2-\delta)+\alpha-1]}\right]^{\frac{1}{n-1}},n\ge 2 \tag{5.1}$$

Proof: A function f is starlike of order $\varphi, 0\le\varphi<1,$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\varphi.$$

We must show that

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\varphi,\quad for\ |z|<R_1.$$

We have

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le\left|\frac{\sum_{n=2}^{\infty}(n-1)a_nz^{n-1}}{1+\sum_{n=2}^{\infty}a_nz^{n-1}}\right|\le\frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1+\sum_{n=2}^{\infty}a_n|z|^{n-1}}.$$

The last expression above is bounded by $(1-\varphi)$ if

$$\sum_{n=2}^{\infty}\frac{n-\varphi}{1-\varphi}a_n|z|^{n-1}\leq 1. \tag{5.2}$$

Hence, by Theorem (1), (5.2) will be true if

$$\frac{n-\varphi}{1-\varphi}|z|^{n-1}\leq\frac{n[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)}{\beta(2-\delta)+\alpha-1},$$

or equivalently

$$|z|\leq\left[\frac{(1-\varphi)n[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)}{(n-\varphi)[\beta(2-\delta)+\alpha-1]}\right]^{\frac{1}{n-1}},n\geq 2\ . \tag{5.3}$$

The theorem follows easily from (5.3).

Theorem (6): Let $f\in N(\alpha,\delta,\beta,\lambda)$. Then f is convex in the disk $|z|<R_2$, of order $\varphi, 0\leq\varphi<1$, where

$$R_2=\inf_{n\geq 2}\left[\frac{(1-\varphi)[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)}{(n-\varphi)[\beta(2-\delta)+\alpha-1]}\right]^{\frac{1}{n-1}},n\geq 2\ . \tag{5.4}$$

Proof: A function f is convex of order $\varphi, 0\leq\varphi<1$, if

$$\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]>\varphi.$$

Thus is enough to show that

$$\left|\frac{zf''(z)}{f'(z)}\right|\leq 1-\varphi, for|z|<R_2.$$

We have

$$\left|\frac{zf''(z)}{f'(z)}\right|=\left|\frac{\sum_{n=2}^{\infty}n(n-1)a_nz^{n-1}}{1+\sum_{n=2}^{\infty}na_nz^{n-1}}\right|\leq\frac{\sum_{n=2}^{\infty}n(n-1)a_n|z|^{n-1}}{1+\sum_{n=2}^{\infty}na_n|z|^{n-1}}.$$

The last expression above is bounded by $(1-\varphi)$ if

$$\sum_{n=2}^{\infty}\frac{n(n-\varphi)}{(1-\varphi)}a_n|z|^{n-1}\leq 1\ . \tag{5.5}$$

Hence, by Theorem (1), (5.5) will be true if

$$\frac{n(n-\varphi)}{(1-\varphi)}|z|^{n-1}\leq\frac{n[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)}{\beta(2-\delta)+\alpha-1},$$

or equivalently

$$|z|\leq\left[\frac{(1-\varphi)[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)}{(n-\varphi)[\beta(2-\delta)+\alpha-1]}\right]^{\frac{1}{n-1}},n\geq 2\ . \tag{5.6}$$

The theorem follows easily from (5.6).

Theorem (7): Let $f\in N(\alpha,\delta,\beta,\lambda)$. Then f is close-to-convex in the disk $|z|<R_3$, of order $\varphi, 0\leq\varphi<1$, where

$$R_3 = \inf_{n\geq 2}\left[\frac{(1-\varphi)[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)}{\beta(2-\delta)+\alpha-1}\right]^{\frac{1}{n-1}} . \tag{5.7}$$

Proof: A function f is close-to-convex function of order $\varphi, 0 \leq \varphi < 1$, if

$$\mathrm{Re}\{f'(z)\} > \varphi.$$

Thus it is enough to show that

$$|f'(z)-1| \leq 1-\varphi, \ for\ |z| < R_3.$$

We have

$$|f'(z)-1| = \left|\sum_{n=2}^{\infty} na_n z^{n-1}\right| \leq \sum_{n=2}^{\infty} na_n |z|^{n-1}.$$

Thus

$$|f'(z)-1| \leq 1-\varphi \ \ if \ \ \sum_{n=2}^{\infty}\frac{na_n|z|^{n-1}}{1-\varphi} \leq 1. \tag{5.8}$$

Hence, by Theorem (1), (5.8) will be true if

$$\frac{n|z|^{n-1}}{1-\varphi} \leq \frac{n[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)}{\beta(2-\delta)+\alpha-1},$$

or equivalently

$$|z| \leq \left[\frac{(1-\varphi)[n(1-\beta)+\alpha(\beta(n-1)-1)+\beta(\delta-1)]T_n(\lambda)}{\beta(2-\delta)+\alpha-1}\right]^{\frac{1}{n-1}}, n \geq 2. \tag{5.9}$$

The theorem follows easily from (5.9).

6. Neighborhoods:

Following the earlier works on neighborhoods of analytic functions by Goodman [4] and Ruscheeyh [6], we begin by introducing here the t –neighborhood of function $f \in N(\alpha,\delta,\beta,\lambda)$ of the form (1.4) by means of the definition below:

$$N_t(f) = \left\{ g \in N : g(z) = z + \sum_{n=2}^{\infty} b_n z^n \ \ and \ \ \sum_{n=2}^{\infty} n|a_n - b_n| \leq t, \ \ 0 \leq t < 1 \right\}. \tag{6.1}$$

Particularly for the identity function $e(z) = z$, we have

$$N_t(e) = \left\{ g \in N : g(z) = z + \sum_{n=2}^{\infty} b_n z^n \ \ and \ \ \sum_{n=2}^{\infty} n|b_n| \leq t, 0 \leq t < 1 \right\}. \tag{6.2}$$

Definition (2): A function $f \in N$ is said to be in the class $N(\alpha,\delta,\beta,\lambda)$ if there exists function $g \in N$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\rho, (z \in U, 0 < \rho < 1).$$

Theorem (8): If $g \in N(\alpha,\delta,\beta,\lambda)$ and

$$\rho = 1 - \frac{t(\lambda+1)[2(1-\beta)+\alpha(\beta-1)+\beta(\delta-1)]}{2(\lambda+1)[2(1-\beta)+\alpha(\beta-1)+\beta(\delta-1)]+\beta(2-\delta)+\alpha-1} . \tag{6.3}$$

Then $N_t(g) \subset N(\alpha,\delta,\beta,\lambda)$.

Proof: Let $f \in N_t(g)$ .Then we find from (6.1) that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \le t,$$

which implies the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \le \frac{t}{2}, (n \ge 2).$$

Since $g \in N(\alpha, \delta, \beta, \lambda)$ then by using theorem (1)

$$\sum_{n=2}^{\infty} b_n \le \frac{\beta(2-\delta)+\alpha-1}{2[2(1-\beta)+\alpha(\beta-1)+\beta(\delta-1)](\lambda+1)}. \tag{6.4}$$

So that

$$\left|\frac{f(z)}{g(z)}-1\right| \le \frac{\sum_{n=2}^{\infty}|a_n - b_n|}{1+\sum_{n=2}^{\infty} b_n} \le \frac{t(\lambda+1)[2(1-\beta)+\alpha(\beta-1)+\beta(\delta-1)]}{2(\lambda+1)[2(1-\beta)+\alpha(\beta-1)+\beta(\delta-1)]+\beta(2-\delta)+\alpha-1}$$

$$= 1-\rho$$

Hence by Definition (2) $f \in N \ \ for\ \rho$ given by (6.3).

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