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On a New Subclass of Multivalent Functions with Positive Coefficients

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Abstract

In this paper, we define a new subclass $H_p^{\beta}(\gamma, A, B, \nu)$ of multivalent functions in the open unit disk U. We obtain some interesting properties, like, coefficient estimates and closure theorems. Also we discuss integral representation, convolution properties and integral mean related with fractional integral.

Keywords: Multivalent Function, Convolution, Integral mean, Fractional integral.
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1. Introduction

Let A_p^{β} be the class of functions of the form:-

$$f(z)^{\beta} = z^{\beta+p-1} + \sum_{n=p+1}^{\infty} \beta a_n z^{\beta+n-1}, (\beta > 1, p \in N = \{1, 2, \ldots\}) \tag{1}$$

Which are analytic and multivalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$

Let H_p^{β} be denote the subclass of A_p^{β} containing of functions of the form**:**

$$f(z)^{\beta} = z^{\beta+p-1} + \sum_{n=p+1}^{\infty} \beta a_n z^{\beta+n-1}, (a_n \geq 0, \beta > 1. p \in N) \tag{2}$$

Which are analytic and multivalent in the open unit disk U.

Definition 1: Let $f \in H_p^{\beta}$ be given by (2). Then, the class $H_p^{\beta}(\gamma, A, B, \nu)$ is defined by

$$H_p^{\beta}(\gamma, A, B, \nu) = \left\{ f \in H_p^{\beta} : \left| \frac{f'''(z)^{\beta} - (\beta + p - 3)\frac{f''(z)^{\beta}}{z}}{\gamma f'''(z)^{\beta} + (A + B + 2\gamma)\frac{f''(z)^{\beta}}{z}} \right| < \nu, 0 \leq \gamma < 1, \beta > 1, 0 < A \leq 1, 0 \leq B < 1 \, and \, 0 < \nu < 1 \right\} \tag{3}$$

Such type of study was carried out by several different authors for another classes, like, Atshan and Wanas [3], Atshan and Kulkarni [2], Najafzadeh et al.[9],Aouf [1], Khairnar and Rajas [6] ,Rajabikafshgar and Latha [10] and Bulut [4].

2. Coefficient Estimates

The following theorem gives a necessary and sufficient condition for a function f to be in the class $H_p^{\beta}(\gamma, A, B, \nu)$

Theorem 1: Let $f \in H_p^{\beta}$. Then $H_p^{\beta}(\gamma, A, B, \nu)$ if and only if

$$\sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(n-p+\nu(A+B)+\gamma\nu(\beta+n-1))a_n \le \nu(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1)), \quad (4)$$

Where $0 \le \gamma < 1, \beta > 1, 0 < A \le 1, 0 \le B < 1$ and p $\in N$.

The result is sharp for the function

$$f(z)^{\beta} = z^{\beta+p-1} + \frac{\nu(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))}{\beta(\beta+n+1)(\beta+n+2)(n-p+\nu(A+B)+\gamma\nu(\beta+n-1))} z^{\beta+n-1}$$

Proof: Suppose that the inequality (4) holds true and $|z| = 1$.Then, we have

$$\left| f'''(z)^{\beta} - (\beta+p-3)\frac{f''(z)^{\beta}}{z} \right| - \nu \left| \gamma f'''(z)^{\beta} + (A+B+2\gamma)\frac{f''(z)^{\beta}}{z} \right|$$

$$= \left| \sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(n-p)a_n z^{\beta+n-4} \right|$$

$$-\nu \left| (\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))z^{\beta+p-4} + \sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(A+B+\gamma(\beta+n-1))a_n z^{\beta+n-4} \right|$$

$$\le \sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(n-p)a_n |z|^{\beta+n-4}$$

$$-\nu(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))|z|^{\beta+p-4}$$

$$+ \sum_{n=p+1}^{\infty} \nu\beta(\beta+n-1)(\beta+n-2)(A+B+\gamma(\beta+n-1))a_n |z|^{\beta+n-4}$$

$$= \sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(n-p+\nu(A+B)+\gamma\nu(\beta+n-1))a_n$$

$$-\nu(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1)) \le 0,$$

By hypothesis

Hence, by maximum modulus principle, $f \in H_p^{\beta}(\gamma, A, B, \nu)$.

Conversely, suppose that $f \in H_p^{\beta}(\gamma, A, B, \nu)$.Then from (3), we have

$$\left| \frac{f'''(z)^{\beta} - (\beta+p-3)\frac{f''(z)^{\beta}}{z}}{\gamma f'''(z)^{\beta} + (A+B+2\gamma)\frac{f''(z)^{\beta}}{z}} \right|$$

$$= \left| \frac{\sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(n-p)a_n z^{\beta+n-4}}{(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))z^{\beta+p-4} + \sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(A+B+\gamma(\beta+n-1))a_n z^{\beta+n-4}} \right| < \nu$$

Since Re (z) $\leq |z|$ for all z ($z \in U$) we get

$$\text{Re} = \left| \frac{\sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(n-p)a_n z^{\beta+n-4}}{(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))z^{\beta+p-4} + \sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(A+B+\gamma(\beta+n-1))a_n z^{\beta+n-4}} \right| < \nu \quad (5)$$

We choose the value of z on the real axis so that $\frac{zf'''(z)}{f''(z)}$ is real.

Letting $z \to 1^-$, through real values, we obtain inequality (4).Finally, sharpness follows, if we take

$$f(z)^{\beta} = z^{\beta+p-1} + \frac{\nu(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))}{\beta(\beta+n-1)(\beta+n-2)(n-p+\nu(A+B)+\gamma\nu(\beta+n-1))} z^{\beta+n-1}, n = p+1, p+2,..$$

Corollary 1: Let $f \in H_p^{\beta}(\gamma, A, B, \nu)$. Then

$$a_n \leq \frac{\nu(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))}{\beta(\beta+n-1)(\beta+n-2)(n-p+\nu(A+B)+\gamma\nu(\beta+n-1))}, = p+1, p+2... \quad (6)$$

3. Closure Theorems

Theorem 2: Let the functions f_i defined by

$$f_i(z)^{\beta} = z^{\beta+p-1} + \sum_{n=p+1}^{\infty} \beta a_{n,i} z^{\beta+n-1}, (a_{n,i} \geq 0, p \in N, i = 1, 2, ...\ell) \quad (7)$$

Be in the class $H_p^{\beta}(\gamma, A, B, \nu)$ for every $i = 1,2,\ldots,\ell$. Then the function h_1 defined by

$$h_1(z)^{\beta} = z^{\beta+p-1} + \sum_{n=p+1}^{\infty} \beta w_n z^{\beta+n-1}, (w_n \geq 0, p \in N),$$

Also belongs to the class $H_p^{\beta}(\gamma, A, B, \nu)$, where

$$w_n = \frac{1}{\ell} \sum_{i=1}^{\ell} a_{n.i}, \quad n = p+1, p+2....$$

Proof: Since $f_i \in H_p^{\beta}(\gamma, A, B, \nu)$, we note that

$$\sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(n-p+\nu(A+B)+\gamma\nu(\beta+n-1))a_{n,i}$$
$$\leq \nu(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1)),$$

For every i =1,2,…,ℓ. Hence

$$= \sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(n-p+\nu(A+B)+\gamma\nu(\beta+n-1))w_n$$

$$= \sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(n-p+\nu(A+B)+\gamma\nu(\beta+n-1))(\frac{1}{\ell} \sum_{i=1}^{\ell} a_{n,i})$$

$$= \frac{1}{\ell} \sum_{i=1}^{\ell} \sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(n-p+\nu(A+B)+\gamma\nu(\beta+n-1))a_{n,i}$$

$\leq v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1)),$

Therefore by Theorem 1, we have $h_1 \in H_p^\beta(\gamma, A, B, v)$

This completes the proof of the theorem.

Theorem 3: Let the function f_i defined by (7) be in the class $H_p^\beta(\gamma, A, B, v)$

For every $i = 1,2,\ldots,\ell$. Then the function h_2 defined by

$$h_2(z)^\beta = \sum_{i=1}^{\ell} c_i f_i(z)^\beta$$

Is also in the class $H_p^\beta(\gamma, A, B, v)$,where

$$\sum_{i=1}^{\ell} c_i = 1, (c_i \geq 0)$$

Proof: By Theorem 1 for every i =1,2,..,ℓ, we have

$$\sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))a_{n,i}$$
$$\leq v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1)),$$

But

$$h_2\,(z)^\beta = \sum_{i=1}^{\ell} c_i f_i(z)^\beta = \sum_{i=1}^{\ell} c_i \left(z^{\beta+p-1} + \sum_{n=p+1}^{\infty} \beta\, a_{n,i}\, z^{\beta+n-1} \right)$$
$$= z^{\beta+p-1} + \sum_{n=p+1}^{\infty} \beta \left(\sum_{i=1}^{\ell} c_i a_{n,i} \right) z^{\beta+n-1}$$

Therefore

$$\sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))(\sum_{i=1}^{\ell} c_i\, a_{n,i}\ )$$
$$= \sum_{i=1}^{\ell} c_i \left(\sum_{n=p+1}^{\infty} \beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))\, a_{n,i} \right)$$
$$\leq \sum_{i=1}^{\ell} c_i \left(v(\beta+p-1)(\beta+p-2)\big(A+B+\gamma(\beta+p-1)\big) \right)$$
$$= v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1)),$$

And the proof is complete.

4. Integral Representation

In the following theorem, we obtain integral representation for the function $f'(z)^\beta$

Theorem 4: Let $f \in H_p^\beta(\gamma, A, B, v)$.Then

$$f'(z)^\beta = \int_0^z e^{\int_0^z \frac{(\beta+p-3)+v(A+B+2\gamma)\emptyset(t)}{t(1-V\gamma\emptyset(t))} dt}\, dt.$$

Proof: By putting $\frac{zf'''(z)}{f''(z)}$ =TH (z) in (3), we have

$$\left| \frac{TH(Z)-(\beta+P-3)}{\gamma TH(Z)+(A+B+2\gamma)} \right| \quad v\,,$$

Or equivalently

$$\frac{TH(Z)-(\beta+P-3)}{\gamma TH(Z)+(A+B+2\gamma)}=v\,\emptyset(z),$$

Where $|\emptyset(z)|\Re 1, z\in U$. So

$$\frac{f''(z)^{\beta}}{f'(z)^{\beta}}=\frac{(\beta+p-3)+v(A+B+2\gamma)\emptyset(Z)}{z(1-V\gamma\emptyset(Z))},$$

After integration, we obtain

$$\log\left(f'(z)^{\beta}\right)=\int_0^z\frac{(\beta+p-3)+v(A+B+2\gamma)\emptyset(t)}{t\left(1-V\gamma\emptyset(t)\right)}\,dt.$$

Thus

$$f'(z)^{\beta}=e^{\int_0^z\frac{(\beta+p-3)+v(A+B+2\gamma)\emptyset(t)}{t\left(1-V\gamma\emptyset(t)\right)}\,dt}$$

After integration, we have

$$f(z)^{\beta}=\int_0^z e^{\int_0^z\frac{(\beta+p-3)+v(A+B+2\gamma)\emptyset(t)}{t(1-V\gamma\emptyset(t))}\,dt}\,dt$$

And this gives the result.

5. Convolution Properties

Theorem 5: Let the function f_1 (j=1,2) defined by

$$f_j(z)^{\beta}=z^{\beta+p-1}+\sum_{n=p+1}^{\infty}\beta\,a_{n,j}\,z^{\beta+n-1}\quad,\left(a_{n,j}\geq 0,p\in N,j=1,2\right)\tag{8}$$

be in the class $H_p^{\beta}(\gamma,A,B,v)$.Then $f_1*f_2\in H_p^{\beta}(\gamma,A,\sigma,v)$, where

$$\sigma\leq\frac{Q\left(A+\gamma(\beta+p-1)\right)-Y(n-p+v(A+\gamma(\beta+n-1))}{Y\gamma-Q}.$$

Proof: We must find the largest σ such that

$$\sum_{n=p+1}^{\infty}\frac{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+\sigma)+\gamma v(\beta+n-1))}{v(\beta+p-1)(\beta+p-2)(A+\sigma+\gamma(\beta+p-1))}a_{n,1}a_{n,2}\leq 1.$$

Since $f_j\in H_p^{\beta}(\gamma,A,B,v)$ (j= 1,2), then

$$\sum_{n=p+1}^{\infty}\frac{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))}{v(\beta+p-1)(\beta+p-2)\left(A+B+\gamma(\beta+p-1)\right)}a_{n,j}\leq 1,\tag{9}$$

$(j=1,2)$

By Cauchy – Schwarz inequality, we obtain

$$\sum_{n=p+1}^{\infty}\frac{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))}{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))}\sqrt{a_{n,1}a_{n,2}}\leq 1.\tag{10}$$

$$\frac{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+\sigma)+\gamma v(\beta+n-1))}{v(\beta+p-1)(\beta+p-2)(A+\sigma+\gamma(\beta+p-1))}a_{n,1}a_{n,2}$$
$$\leq\frac{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))}{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))}\sqrt{a_{n,1}a_{n,2}}.$$

This equivalently to

$$\sqrt{a_{n,1}a_{n,2}}\leq\frac{(A+\sigma+\gamma(\beta+p-1))(n-p+v(A+B)+\gamma v(\beta+n-1))}{(A+B+\gamma(\beta+p-1))(n-p+v(A+\sigma)+\gamma v(\beta+n-1))}$$

From (10), we have

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))}{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))}$$

Thus, it is sufficient to show that

$$\frac{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))}{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))} \leq \frac{(A+\sigma+\gamma(\beta+p-1))(n-p+v(A+B)+\gamma v(\beta+n-1))}{(A+B+\gamma(\beta+p-1))(n-p+v(A+\sigma)+\gamma v(\beta+n-1))},$$

Which implies to

$$\sigma \leq \frac{Q\big(A+\gamma(\beta+p-1)\big)-Y(n-p+v(A+\gamma(\beta+n-1))}{Y\gamma-Q},$$

Where $Y = v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))^2$

And $Q = \beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))^2$

Theorem 6: Let the functions f_j (j=1,2) defined by (8) be in the class $H_p^\beta(\gamma, A, B, v)$ Then the function k defined by

$$\mathrm{k(z)}^\beta = \mathrm{z}^{\beta+\mathrm{p}-1} + \sum_{\mathrm{n=p+1}}^{\infty} \beta\big(a_{\mathrm{n},1}^2 + a_{\mathrm{n},2}^2\big)\mathrm{z}^{\beta+\mathrm{n}-1} \tag{11}$$

Belong to the class $H_p^\beta(\gamma, A, \varepsilon, v)$, where

$$\varepsilon \leq \frac{\beta(\beta+n-1)(\beta+n-2)\big(A+\gamma(\beta+p-1)\big)G - 2v(\beta+p-1)(\beta+p-2)\big(n-p+vA+\gamma v(\beta+n-1)\big)H}{2v^2(\beta+p-1)(\beta+p-2)H - \beta(\beta+n-1)(\beta+n-2)G}$$

Proof: We must find the largest ε such that

$$\sum_{n=p+1}^{\infty} \frac{\beta(\beta+n-1)(\beta+n-2)\big(n-p+v(A+\varepsilon)+\gamma v(\beta+n-1)\big)}{v(\beta+p-1)(\beta+p-2)\big(A+\varepsilon+\gamma(\beta+p-1)\big)}(a_{\mathrm{n},1}^2 + a_{\mathrm{n},2}^2) \leq 1.$$

Since $f_j \in H_p^\beta(\gamma, A, B, v)$ (j=1,2), we obtain

$$\sum_{n=p+1}^{\infty} \left(\frac{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))}{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))}\right)^2 a_{\mathrm{n},1}^2$$

$$\leq \left(\sum_{n=p+1}^{\infty} \frac{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))}{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))} a_{\mathrm{n},1}\right)^2 \leq 1 \tag{12}$$

And

$$\sum_{n=p+1}^{\infty} \left(\frac{\beta(\beta+n-1)(\beta+n-2)\,(n-p+v(A+B)+\gamma v(\beta+n-1))}{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))}\right)^2 a_{\mathrm{n},2}^2$$

$$\leq \left(\sum_{n=p+1}^{\infty} \frac{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))}{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))} a_{\mathrm{n},2}\right)^2 \leq 1 \tag{13}$$

Combining the inequalities (12) and (13), gives

$$\sum_{n=p+1}^{\infty} \frac{1}{2}\left(\frac{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))}{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1)}\right)^2 (a_{\mathrm{n},1}^2 + a_{\mathrm{n},2}^2) \leq 1. \tag{14}$$

But k$\in$ $H_p^\beta(\gamma, A, \varepsilon, v)$ if and only if

$$\sum_{n=p+1}^{\infty} \frac{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+\varepsilon)+\gamma v(\beta+n-1))}{v(\beta+p-1)(\beta+p-2)(A+\varepsilon+\gamma(\beta+p-1))}(a_{\mathrm{n},1}^2 + a_{\mathrm{n},2}^2) \leq 1 \tag{15}$$

The inequality (15) will be satisfied if

$$\frac{\beta(\beta+n-1)(\beta+n-2)\,(n-p+v(A+\varepsilon)+\gamma v(\beta+n-1))}{v(\beta+p-1)(\beta+p-2)(A+\varepsilon+\gamma(\beta+p-1))}$$

$$\le \frac{\beta^2(\beta+n-1)^2(\beta+n-2)^2\big(n-p+v(A+B)+\gamma v(\beta+n-1)\big)^2}{2v^2(\beta+p-1)^2(\beta+p-2)^2\big(A+B+\gamma(\beta+p-1)\big)^2}, (n=p+1, p+2, \dots),$$

So that

$$\varepsilon \le \frac{\beta(\beta+n-1)(\beta+n-2)\big(A+\gamma(\beta+p-1)\big)G - 2v(\beta+p-1)(\beta+p-2)\big(n-p+vA+\gamma v(\beta+n-1)\big)H}{2v^2(\beta+p-1)(\beta+p-2)H - \beta(\beta+n-1)(\beta+n-2)G},$$

Where

$$G = \big(n-p+v(A+B)+\gamma v(\beta+n-1)\big)^2$$

And

$$H = \big(A+B+\gamma(\beta+p-1)\big)^2$$

6. Integral Mean Inequalities for the Fractional Integral

Definition 2 [5]: The fractional integral of order λ (λ > 0) is defined for a function f by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(z)}{(z-t)^{1-\lambda}}\, dt, \qquad (16)$$

Where the function f is an analytic in a simple-connected region of the complex zplane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring log (z-t) to be real, when (z-t) > 0. ■

Definition 3[8]: Let f,g be analytic in U. Then f is said to be subordinate to g, written $f \prec g$, if there exists a Shwarz function w(z), which is analytic in U, with w (0)=0 and $|w(z)|$ 1 (z ∈ U) such that f(z) = g(w(z)), (z ∈ U) .

In particular, if the function g is univalent in U, we have the following:

f(z) <g(z) (z ∈ U) if and only if f(0) = g(0) and f(U) ⊂ g(U).

In 1925, Littlewood [7] proved the following subordination theorem :

Theorem 7 [7]: If f and g are analytic in U with,f<g then for μ > 0 and z = r $e^{i\theta}$ (0 ℜ r ℜ1)

$$\int_0^{2\pi} |f(z)|^{\mu}\, d\theta \le \int_0^{2\pi} |g(z)|^{\mu}\, d\theta.$$

Theorem 8: Let $f \in H_p^{\beta}(\gamma, A, B, v)$ and suppose that f_n is defined by

$$f_n(z)^{\beta} = z^{\beta+p-1} + \frac{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))}{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))} z^{\beta+n-1}, (n \ge p+1). \qquad (17)$$

Also let

$$\sum_{m=p+1}^{\infty} (m-\eta)_{\eta+1}\, a_m \le \frac{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))\Gamma(n+1)\Gamma(p+\lambda+\eta+2)}{\beta(\beta+n-1)(\beta+n-2)\big(n-p+v(A+B)+\gamma v(\beta+n-1)\big)\Gamma(n+\lambda+\eta+1)\Gamma(p-\eta+1)} \qquad (18)$$

For 0 ≤ η ≤ m, λ > 0, where $(m-\eta)_{\eta+1}$ denote the Pochhammer symbol defined by

$$(m-\eta)_{\eta+1} = (m-\eta)(m-\eta+1)\, m \qquad (19)$$

If there exists an analytic w defined by

$$(w(z)^{\beta})^{n-p} = \frac{\beta(\beta+n-1)(\beta+n-2)\big(n-p+v(A+B)+\gamma v(\beta+n-1)\big)\Gamma(n+\lambda+\eta+1)}{v(\beta+p-1)(\beta+p-2)\big(A+B+\gamma(\beta+p-1)\big)\Gamma(n+1)} \times$$

$$\sum_{m=p+1}^{\infty} (m-\eta)_{\eta+1}\, H(m) a_m\, z^{m-p}, \qquad (20)$$

Where m ≥ η and

$$H(m) = \frac{\Gamma(m-\eta)}{\Gamma(m+\lambda+\eta+1)} \text{ ■ ■ }, (\lambda > 0, m \ge p+1) \qquad (21)$$

Then, for z = r$e^{i\theta}$ and 0 r 1

$$\int_0^{2\pi} \left| D_z^{-\lambda-\eta} f(z)^{\beta} \right|^{\mu} \, d\theta \le \int_0^{2\pi} \left| D_z^{-\lambda-\eta} f_n(z)^{\beta} \right|^{\mu} \, d\theta \ (\lambda > 0, \mu > 0). \tag{22}$$

Proof: Let

$$f(\mathrm{z})^{\beta} = \mathrm{z}^{\beta+\mathrm{p}-1} + \sum_{\mathrm{m}=\mathrm{p}+1}^{\infty} a_{\mathrm{m}} \, \mathrm{z}^{\beta+\mathrm{m}-1}$$

For $\eta \ge 0$ and Definition 2, we obtain

$$D_z^{-\lambda-\eta} f(z)^{\beta} = \frac{\Gamma(p+1) z^{\beta+p+\lambda+\eta-1}}{\Gamma(p+\lambda+\eta+1)} \left(1 + \sum_{m=p+1}^{\infty} \frac{\Gamma(m+1)\Gamma(p+\lambda+\eta+1)}{\Gamma(p+1)\Gamma(m+\lambda+\eta+1)} \mathrm{a_m} \, \mathrm{z}^{\mathrm{m-p}} \right)$$

$$= \frac{\Gamma(p+1) z^{\beta+p+\lambda+\eta-1}}{\Gamma(p+\lambda+\eta+1)} \left(1 + \sum_{m=p+1}^{\infty} \frac{\Gamma(p+\lambda+\eta+1)}{\Gamma(p+1)} (m-\eta)_{\eta+1} \, H(m) a_m \, z^{m-p} \right)$$

Where

$$H(m) = \frac{\Gamma(m-\eta)}{\Gamma(m+\lambda+\eta+1)}, (\lambda > 0, m \ge p+1).$$

Since H is a decreasing function of m, we have

$$0 \quad \mathrm{H(m)} \le \mathrm{H(p+1)} = \frac{\Gamma(p-\eta+1)}{\Gamma(p+\lambda+\eta+2)}.$$

Similarly, from (17) and Definition 2, we obtain

$$D_z^{-\lambda-\eta} f_n(z)^{\beta} = \frac{\Gamma(p+1) z^{\beta+p+\lambda+\eta-1}}{\Gamma(p+\lambda+\eta+1)} \times$$

$$\left(1 + \frac{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))\Gamma(n+1)\Gamma(p+\lambda+\eta+1)}{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))\Gamma(p+1)\Gamma(n+\lambda+\eta+1)} z^{n-p} \right)$$

For $\mu > 0$ and $z = r e^{i\theta}$ (0 r 1), we must show that

$$\int_0^{2\pi} \left| 1 + \sum_{m=p+1}^{\infty} \frac{\Gamma(p+\lambda+\eta+1)}{\Gamma(p+1)} (m-\eta)_{\eta+1} \, H(m) a_m \, z^{m-p} \right|^{\mu} d\theta.$$

$$\le \int_0^{2\pi} \left| 1 + \frac{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))\Gamma(n+1)\Gamma(p+\lambda+\eta+1)}{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))\Gamma(p+1)\Gamma(n+\lambda+\eta+1)} z^{n-p} \right|^{\mu} d\theta.$$

By applying Littlewood' s subordination theorem, it would suffice to show that

$$1 + \sum_{m=p+1}^{\infty} \frac{\Gamma(p+\lambda+\eta+1)}{\Gamma(p+1)} (m-\eta)_{\eta+1} \, H(m) a_m \, z^{m-p}$$

$$\prec 1 + \frac{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))\Gamma(n+1)\Gamma(p+\lambda+\eta+1)}{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))\Gamma(p+1)\Gamma(n+\lambda+\eta+1)} z^{n-p}$$

By setting

$$1 + \sum_{m=p+1}^{\infty} \frac{\Gamma(p+\lambda+\eta+1)}{\Gamma(p+1)} (m-\eta)_{\eta+1} \, H(m) a_m \, z^{m-p}$$

$$= 1 + \frac{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))\Gamma(n+1)\Gamma(p+\lambda+\eta+1)}{\beta(\beta+n-1)(\beta+n-2)(n-p+v(A+B)+\gamma v(\beta+n-1))\Gamma(p+1)\Gamma(n+\lambda+\eta+1)} (w(z))^{n-p},$$

We find that

$$(w(z))^{n-p} = \frac{\beta(\beta+n-1)(\beta+n-2)\big(n-p+v(A+B)+\gamma v(\beta+n-1)\big)\Gamma(n+\lambda+\eta+1)}{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))\Gamma(n+1)} \times$$

$$\sum_{m=p+1}^{\infty} (m-\eta)_{\eta+1} \, H(m) a_m \, z^{m-p},$$

Which readily yields $w(0) = 0$ .For such a function w, we obtain

$$|w(z)|^{n-p} = \frac{\beta(\beta+n-1)(\beta+n-2)\big(n-p+v(A+B)+\gamma v(\beta+n-1)\big)\Gamma(n+\lambda+\eta+1)}{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))\Gamma(n+1)} \times$$

$$\sum_{m=p+1}^{\infty} (m-\eta)_{\eta+1}\, H(m) a_m\, |z|^{m-p}$$

$$\leq \frac{\beta(\beta+n-1)(\beta+n-2)\big(n-p+v(A+B)+\gamma v(\beta+n-1)\big)\Gamma(n+\lambda+\eta+1)}{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))\Gamma(n+1)} \times$$

$$H(p+1)|z| \sum_{m=p+1}^{\infty} (m-\eta)_{\eta+1}\, a_m$$

$$= |z| \frac{\beta(\beta+n-1)(\beta+n-2)\big(n-p+v(A+B)+\gamma v(\beta+n-1)\big)\Gamma(n+\lambda+\eta+1)\Gamma(p-\eta+1)}{v(\beta+p-1)(\beta+p-2)\big(A+B+\gamma(\beta+p-1)\big)\Gamma(n+1)\Gamma(p+\lambda+\eta+2)}$$

$$\times \sum_{m=p+1}^{\infty} (m-\eta)_{\eta+1}\, a_m \leq |z| \cdot 1.$$

This completes the proof of the theorem.

By taking η = 0 in the Theorem 8, we have the following corollary:-

Corollary 2: Let $f \in H_p^{\beta}(\gamma, A, B, v)$ and suppose that f_n is defined by (17).
Also let

$$\sum_{m=p+1}^{\infty} m\, a_m \leq \frac{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))\Gamma(n+1)\Gamma(n+1)\Gamma(p+\lambda+2)}{\beta(\beta+n-1)(\beta+n-2)\big(n-p+v(A+B)+\gamma v(\beta+n-1)\big)\Gamma(n+\lambda+1)\Gamma(p)},$$

(n ≥ p+1).

If there exists an analytic function w defined by

$$(w(z))^{n-p} = \frac{\beta(\beta+n-1)(\beta+n-2)\big(n-p+v(A+B)+\gamma v(\beta+n-1)\big)\Gamma(n+\lambda+1)}{v(\beta+p-1)(\beta+p-2)(A+B+\gamma(\beta+p-1))\Gamma(n+1)}$$

$$\times \sum_{m=p+1}^{\infty} m\, H(m)\, a_m\, z^{m-p},$$

Where

$$H(m) = \frac{\Gamma(m)}{\Gamma(m+\lambda+1)}, (\lambda > 0, m \geq p+1)$$

Then, for $z = re^{i\theta}$ and 0 r 1

$$\int_0^{2\pi} |D_z^{-\lambda} f(z)^{\beta}|^{\mu}\, d\theta \leq \int_0^{2\pi} |D_z^{-\lambda} f_n(z)^{\beta}|^{\mu}\, d\theta, (\lambda > 0, \mu > 0).$$

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