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On a New Subclass of Meromorphic Univalent Functions with Negative Coefficients Defined by Liu-Srivastava Linear Operator

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Abstract

In The present paper, we introduce and discuss a new subclass of meromorphic univalent functions defined by Liu-Srivastava linear operator, we obtain various important properties, like, coefficient inequalities, extreme points, closure theorems, (n,δ)-neighhborhoods of a functions $f \in A^*$ and partial sums. We also consider integral transforms of functions in the class $A^*(g,\gamma,k,\lambda)$ and obtain some results.

Keywords: Meromorphic Univalent Functions, Liu-Srivastava Linear Operator, HadamardProduct, Extreme Points, Closure Theorems, Neighborhoods, Partial Sums, Integral Transform.
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1. Introduction

Let A* denote the class of functions of the form:

$$f(z)=\frac{1}{z}-\sum_{n=1}^{\infty}a_n z^n, ...(a_n \geq 0, n \in N=\{1,2,...\}) \tag{1}$$

Which are analytic andmeromorphic univalent in the punctured unit disk $U^*=\{z \in \mathbb{C}: 0<|z|<1\}=U\{0\}$.

Let $A^*(\gamma)$ and $A_k^*(\gamma)$, $(0 \leq \gamma < 1)$. denote the subclass of A^* that are meromorphicallystalikefunctions of order γ and meromorphically convex functions of order γ respectively. Analytically, $f \in A^*(\gamma)$ if and only if, f is of the form (1) and satisfy

$$-\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma, z \in U,$$

Similarly, $f \in A_k^*(\gamma)$, if and only if, f is of the form (1) and satisfies

$$-\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \gamma, \qquad z \in U,$$

and similar other classes of meromorphically univalent functions have been extensively studied by (for example) Altintas et al. [2],Aouf [3] ,Mogra et al. [13], Uralegadi et al.[18,19, 20] and others (see [7,14,15]).

Let $f, g \in A^*$, where f is given by (1) and g is defined by

$$g(z) = \frac{1}{z} - \sum_{n=1}^{\infty} b_n z^n \tag{2}$$

Then the Hadamad product (or convolution) f*g of the functions f and g is defined by

$$(f * g)(z) = \frac{1}{z} - \sum_{n=1}^{\infty} a_n b_n z^n = (g * f)(z). \tag{3}$$

For complex parameters $\alpha_1, ..., \alpha_l$ *and* $\beta_1, ..., \beta_m$ $(\beta \neq 0, -1, ...; j = 1, 2, ..., m)$

The generalized hypergeometric function $\iota f_m(z)$ is defined by

$$\iota F_m(z) = \iota F_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ... (\alpha_l)_n z^n}{(\beta_1)_n ... (\beta_m)_n n!} \tag{4}$$

$(\iota \leq m+1; \iota, m \in N_0 = N \cup \{0\}; Z \in U),$

Where $(\theta)_n$ is the pochhammer symbol defined by

$$(\theta)_n = \frac{\Gamma(\theta+n)}{\Gamma(\theta)} = \begin{cases} 1, & \mathrm{n} = 0; \theta \in \mathrm{C} \setminus \{0\}, \\ \theta(\theta+1)(\theta+2)\ldots(\theta+\mathrm{n}-1), & \mathrm{n} \in \mathrm{N}; \theta \in \mathbb{C}. \end{cases} \tag{5}$$

Corresponding to a function $\iota F_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z)$ defined by

$$Q(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z) = z^{-1} \iota F_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z) \tag{6}$$

Liu and Srivastava[11] considered a linear operator

$L(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m) : A^* \to A^*,$

Defined by the following Hadamard product (or convolution):

$$L(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m) f(z) = Q(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z) * f(z)$$

$$= z^{-1} - \sum_{n=1}^{\infty} \left| \frac{(\alpha_1)_{n+1} ... (\alpha_l)_{n+1}}{(\beta_1)_{n+1} ... (\beta_m)_{n+1}} \right| \frac{a_n z^n}{(n+1)!'} \tag{7}$$

Where, $\alpha_i > 0, (i = 1, 2, ..., l), \beta_j > 0, (j = 1, 2, ..., m), l \leq m+1; m \in N_0 = N \cup \{0\}$. For notional simplicity, we use a shorter notations $L[\alpha_1]$ for

$L(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m)$ *and*

$$\Gamma_n(\alpha_1) = \left| \frac{(\alpha_1)_{n+1} ... (\alpha_l)_{n+1}}{(\beta_1)_{n+1} ... (\beta_m)_{n+1}} \right| \frac{1}{(n+1)!'} \tag{8}$$

Unless otherwise stated in the sequel. We note that the linear operator $H_m^l[\alpha_1]$ was earlier defined for multivalent functions by Dziok and Srivastava [8] and was investigated by Liu and Srivastava [11]. Motivated by Ravichandaran et al.[17] and Atshan et al.[6], making use of the operator $L[\alpha_1]$, now defined a new subclass $A^*(g, \gamma, k, \lambda)$ *of* A^*.

Definition 1: For $0 \le \gamma < 1, \mathrm{k} \ge 0$ and $0 \le \lambda < \frac{1}{3}$, we let $A^*(g,\gamma,k,\lambda)$ by the subclass of A^* consisting of functions of the form (1) and satisfying the analytic criterion

$$-\operatorname{Re}\left\{\frac{z^2(L[\alpha_1]f*g(z))''}{L[\alpha_1]f*g(z)}+\lambda\frac{z^3(L[\alpha_1]f*g(z))'''}{L[\alpha_1]f*g(z)}+\gamma\right\} > k\left|\frac{z^2(L[\alpha_1]f*g(z))''}{L[\alpha_1]f*g(z)}+\lambda\frac{z^3(L[\alpha_1]f*g(z))'''}{L[\alpha_1]f*g(z)}+1\right|. \tag{9}$$

Also by suitably choosing $g(z)$ involved in the class, the class $A^*(g,\gamma,k,\lambda)$ reduces to various new subclasses. These considerations canfruitfully be worked out and we skip the details in the regard

The main object of this paper is to study some usual properties of the geometric function theory such as the coefficient inequalities, extreme points, closure theorems, neighborhoods of a function $f \in A^*$, partial sums and integral transforms of functions in the class $A^*(g,\gamma,k,\lambda)$.

2. Cofficient Inequalities

In the following theorem, we obtain necessary and sufficient condition for a function f to be in the class $f \in A^*(g,\gamma,k,\lambda)$. In this connection, we need and state the following lemmas:

Lemma 1 [4]: If γ is a real number and $w=-(u+iv)$ is a complex number, then $\operatorname{Re}(w) \ge \gamma$ *if and only if* $|w+(1-\gamma)|-|w-(1+\gamma)| \ge 0.$

Lemma 2 [6]: If $w = u + iv$ is a complex number and γ,k are real numbers, then $-\operatorname{Re}(w) \ge k|w+1|+\gamma$ *if and only if* $-\operatorname{Re}(w(1+ke^{i\theta})+ke^{i\theta}) \ge \gamma, -\pi \le \theta \le \pi.$

Theorem 1: Let $f \in A^*$ be given by (1). Then $A^*(g,\gamma,k,\lambda)$ if and only if

$$\sum_{n=1}^{\infty}(n(n-1)(1+k)(1+\lambda(n-2))+(k+\gamma))\Gamma_n(\alpha_1)a_nb_n \le 3(k(1-2\lambda)-2\lambda)+(2+\gamma). \tag{10}$$

Proof: By Definition 1, we obtain

$$-\operatorname{Re}\left\{\frac{z^2(L[\alpha_1]f*g(z))''}{L[\alpha_1]f*g(z)}+\lambda\frac{z^3(L[\alpha_1]f*g(z))'''}{L[\alpha_1]f*g(z)}+\gamma\right\} > k\left|\frac{z^2(L[\alpha_1]f*g(z))''}{L[\alpha_1]f*g(z)}+\lambda\frac{z^3(L[\alpha_1]f*g(z))'''}{L[\alpha_1]f*g(z)}+1\right|$$

Then by Lemma 2, we have

$$-\operatorname{Re}\left\{\left(\frac{z^2(L[\alpha_1]f*g(z))''}{L[\alpha_1]f*g(z)}+\lambda\frac{z^3(L[\alpha_1]f*g(z))'''}{L[\alpha_1]f*g(z)}\right)(1+ke^{i\theta})+ke^{i\theta}\right\} \ge \gamma, -\pi \le \theta \le \pi. \tag{11}$$

For convenience, we let

$$A(z) = -[z^2(L[\alpha_1]f*g(z))''+\lambda z^3(L[\alpha_1]f*g(z))'''](1+ke^{i\theta})-ke^{i\theta}L[\alpha_1]f*g(z).$$

$$B(z) = L[\alpha_1]f*g(z).$$

That is, the equation (11) is equivalent to

$$\operatorname{Re}\left(\frac{A(z)}{B(z)}\right) \ge \gamma.$$

In view of Lemma 1, we only need to prove that

$$|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)| \ge 0.$$

Therefore

$$\left|A(z)+(1-\gamma)B(z)\right|=$$

$$|-\left(3ke^{i\theta}(1-2\lambda)-6\lambda+(1+\gamma)\right)\frac{1}{z}+\sum_{n=1}^{\infty}\left(n(n-1)\left(1+ke^{i\theta}\right)\left(1+\lambda(n-2)\right)+ke^{i\theta}-(1-\gamma)\right)\Gamma_n(\alpha_1)a_nb_nz^n|$$

$$\geq\left(3k(1-2\lambda)-6\lambda+(1+\gamma)\right)\frac{1}{|z|}-\sum_{n=1}^{\infty}\left(n(n-1)(1+k)\left(1+\lambda(n-2)\right)+k-(1-\gamma)\right)\Gamma_n(\alpha_1)a_nb_n|z|^n.$$

And

$$|A(z)-(1+\gamma)B(z)|=|(3(2\lambda-1)(1+k)-\gamma)\frac{1}{z}+\sum_{n=1}^{\infty}(n(n-1)\left(1+ke^{i\theta}\right)\left(1+\lambda(n-2)\right)+ke^{i\theta}+(1+\gamma))\Gamma_n(\alpha_1)a_nb_nz^n|$$

$$\leq(3(2\lambda-1)(1+k)-\gamma)\frac{1}{|z|}+\sum_{n=1}^{\infty}(n(n-1)(1+k)\left(1+\lambda(n-2)\right)+k+(1+\gamma))\Gamma_n(\alpha_1)a_nb_n|z|^n.$$

It is now easy to show that

$$\left|A(z)+(1-\gamma)B(z)\right|-\left|A(z)+(1+\gamma)B(z)\right|$$

$$\geq 2(3k(1-2\lambda)+(2+\gamma)-6\lambda)\frac{1}{|z|}-2\sum_{n=1}^{\infty}\left(n(n-1)(1+k)\left(1+\lambda(n-2)\right)+(k+\gamma)\right)\Gamma_n(\alpha_1)a_nb_n|z|^n\geq 0,$$

Which is equivalent to

$$\sum_{n=1}^{\infty}\left(n(n-1)(1+k)\left(1+\lambda(n-2)\right)+(k+\gamma)\right)\Gamma_n(\alpha_1)a_nb_n\leq 3(k(1-2\lambda)-2\lambda)+(2+\gamma).$$

Conversely, suppose that (10) hold true. Then, we must show

$$-Re\left\{\frac{\left[z^2\left(L[\alpha_1]f*g(z)\right)''+\lambda z^3\left(L[\alpha_1]f*g(z)\right)'''\right]\left(1+ke^{i\theta}\right)+ke^{i\theta}L[\alpha_1]f*g(z)}{L[\alpha_1]f*g(z)}\right\}\geq\gamma.$$

Upon choosing the values of z on the positive real axis, where $0\leq z=r<1$, the above inequality reduces to

$$Re\left\{\frac{\left(3\left((1-2\lambda)ke^{i\theta}-2\lambda\right)+(2+\gamma)\right)\frac{1}{z^2}-\sum_{n=1}^{\infty}(n(n-1)\left(1+ke^{i\theta}\right)\left(1+\lambda(n-2)\right)+ke^{i\theta}+\gamma)\Gamma_n(\alpha_1)a_nb_nz^{n-1}}{\frac{1}{z^2}-\sum_{n=1}^{\infty}\Gamma_n(\alpha_1)a_nb_nz^{n-1}}\right\}$$

$$\geq 0.$$

Since $\operatorname{Re}(-e^{i\theta}) \geq -\left|e^{i\theta}\right| = -1$, the above inequality reduce to

$$Re\left\{\frac{\left(3\big((1-2\lambda)k-2\lambda\big)+(2+\gamma)\right)\frac{1}{r^2}-\sum_{n=1}^{\infty}\big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+k+\gamma\big)\Gamma_n(\alpha_1)a_nb_nr^{n-1}}{\frac{1}{r^2}-\sum_{n=1}^{\infty}\Gamma_n(\alpha_1)a_nb_nz^{n-1}}\right\}$$

$\geq 0.$

Letting $r \to 1^-$ and by the mean value theorem, we have desired inequality (10).

Corollary 1: If $f \in A^*(g,\gamma,k,\lambda)$, then

$$a_n \leq \frac{3(k(1-2\lambda)-2\lambda)+(2+\gamma)}{\big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\big)\Gamma_n(\alpha_1)b_n}. \tag{12}$$

In the next theorem, we obtain the extreme points for the class $A^*(g,\gamma,k,\lambda)$.

Theorem2: Let

$$f_0(z) = \frac{1}{z} \text{ and}$$

$$f_n(z) = \frac{1}{z} - \frac{3(k(1-2\lambda)-2\lambda)+(2+\gamma)}{\big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\big)\Gamma_n(\alpha_1)b_n}z^n,$$

Where $(n \geq 1, n \in N, 0 \leq \gamma < 1, k \geq 0, 0 \leq \lambda < \frac{1}{3}$ and $\Gamma_n(\alpha_1)$ is given by (8))

Then f is in the class $A^*(g,\gamma,k,\lambda)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \sigma_n f_n(z),$$

Where

$$\left(\sigma_n \geq 0 \text{ and } \sum_{n=0}^{\infty} \sigma_n = 1 \text{ or } 1 = \sigma_0 + \sum_{n=1}^{\infty} \sigma_n\right)$$

Proof: Let

$$(z) = \sum_{n=0}^{\infty} \sigma_n f_n(z)$$

$$= \frac{1}{z} - \sum_{n=1}^{\infty} \frac{\big(3(k(1-2\lambda)-2\lambda)+(2+\gamma)\big)\sigma_n}{\big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\big)\Gamma_n(\alpha_1)b_n}z^n.$$

Then

$$\sum_{n=1}^{\infty}\left(\frac{\big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\big)\Gamma_n(\alpha_1)b_n}{3(k(1-2\lambda)-2\lambda)+(2+\gamma)}\right) *$$

$$\sigma_n \frac{3(k(1-2\lambda)-2\lambda)+(2+\gamma)}{\big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\big)\Gamma_n(\alpha_1)b_n}$$

$$= \sum_{n=1}^{\infty} \sigma_n = 1 - \sigma_0 \leq 1.$$

Using Theorem 1, we easily get $f \in A^*(g,\gamma,k,\lambda)$.

Conversely, let $f \in A^*(g,\gamma,k,\lambda)$ is of the form (1). Then

$$a_n \leq \frac{3(k(1-2\lambda)-2\lambda)+(2+\gamma)}{\big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\big)\Gamma_n(\alpha_1)b_n}, (n \in N, n \geq 1).$$

Setting

$$\sigma_n = \frac{\left(n(n-1)(1+k)\left(1+\lambda(n-2)\right)+(k+\gamma)\right)\Gamma_n(\alpha_1)b_n}{3(k(1-2\lambda)-2\lambda)+(2+\gamma)} a_n \text{ , for } n = 1,2,3,\ldots$$

And

$$\sigma_0 = 1 - \sum_{n=1}^{\infty} \sigma_n \text{ . Then}$$

$$f(z) = \sum_{n=0}^{\infty} \sigma_n f_n(z) = \sigma_0 f_0 + \sum_{n=1}^{\infty} \sigma_n f_n(z).$$

Now, we shall prove that the class $A^*(g,\gamma,k,\lambda)$ is closed under arithmetic mean and convex linear combinations.

Let the function

$$f_i(z)(i = 1,2,\ldots,m) \text{ be defined by } f_i(z) = \frac{1}{z} - \sum_{n=1}^{\infty} a_{n,i} z^n, (a_{n,i} \geq 0 \,, n \in N, n \geq 1)\,. \quad (13)$$

Theorem 3: Let the function $f_i(z)$ defined by (13) be in the class $A^*(g,\gamma,k,\lambda)$
For every i=1,2,…,m. Then the function h(z) defined by

$$h(z) = \frac{1}{z} - \sum_{n=1}^{\infty} c_n z^n \,, (c_n \geq 0 \,, n \in N \,, n \geq 1\,)$$

Also belongs to the class $A^*(g,\gamma,k,\lambda)$, where

$$c_n = \frac{1}{m}\sum_{i=1}^{m} a_{n,i}\,.$$

Proof: Since $f_i(z) \in A^*(g,\gamma,k,\lambda)$, therefore from Theorem 1, we obtain

$$\sum_{n=1}^{\infty}\left(n(n-1)(1+k)\left(1+\lambda(n-2)\right)+(k+\gamma)\right)\Gamma_n(\alpha_1)a_{n,i}b_n \leq 3(k(1-2\lambda)-2\lambda)+(2+\gamma). \quad (14)$$

Hence

$$\sum_{n=1}^{\infty}\left(n(n-1)(1+k)\left(1+\lambda(n-2)\right)+(k+\gamma)\right)\Gamma_n(\alpha_1)c_n b_n$$
$$= \sum_{n=1}^{\infty}\left(n(n-1)(1+k)\left(1+\lambda(n-2)\right)+(k+\gamma)\right)\Gamma_n(\alpha_1)b_n\left[\frac{1}{m}\sum_{i=1}^{m} a_{n,i}\right]$$
$$\leq 3(k(1-2\lambda)-2\lambda)+(2+\gamma)$$

(By (14)) which shows that $h(z) \in A^*(g,\gamma,k,\lambda)$.

Theorem 4: Letthe function $f_i(z)$ deined by (13) be in the class $A^*(g,\gamma,k,\lambda)$ for every i=1,2,…,m. Then the function h(z) defined by

$$h(z) = \sum_{i=1}^{m} d_i f_i(z) \text{ and } \sum_{i=1}^{m} d_i = 1, (\,d_i \geq 0).$$

In the class $A^*(g,\gamma,k,\lambda)$

Proof: By definition of h(z), we have

$$h(z) = \left[\sum_{i=1}^{m} d_i\right]\frac{1}{z} - \sum_{n=1}^{\infty}\left[\sum_{i=1}^{m} d_i a_{n,i}\right] z^n.$$

Since $f_i(z)$ are in the class $A^*(g,\gamma,k,\lambda)$ for every i=1,2,…,m, we obtain

$$\sum_{n=1}^{\infty}\Big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\Big)\Gamma_n(\alpha_1)a_{n,i}b_n$$
$$\le 3(k(1-2\lambda)-2\lambda)+(2+\gamma)$$

For every i=1,2,…,m. Hence we can see that

$$\sum_{n=1}^{\infty}\Big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\Big)\Gamma_n(\alpha_1)b_n\left[\sum_{i=1}^{\infty}d_i a_{n,i}\right]$$
$$=\sum_{i=1}^{m}d_i\left[\sum_{n=1}^{\infty}\Big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\Big)\Gamma_n(\alpha_1)a_{n,i}b_n\right]$$
$$\le\big(3(k(1-2\lambda)-2\lambda)+(2+\gamma)\big)\sum_{i=1}^{m}d_i=3(k(1-2\lambda)-2\lambda)+(2+\gamma)\ .$$

Thus $h(z)\in A^*(g,\gamma,k,\lambda)$.

Theorem 5: The class $A^*(g,\gamma,k,\lambda)$ is closed under convex linear combination.

Proof: Let the function $f_i(z)$ (i=1,2) defined by (13) be in the class $A^*(g,\gamma,k,\lambda)$.
We show the function

$$h(z)=\sigma f_1(z)+(1-\sigma)f_2(z)\,,\qquad (0\le\sigma\le 1)$$

Is also in the class $A^*(g,\gamma,k,\lambda)$. Since for $0\le\sigma\le 1$,

$$h(z)=\frac{1}{z}+\sum_{n=1}^{\infty}[\sigma a_{n,1}+(1-\sigma)a_{n,2}]z^n\ .$$

Therefor by Theorem 1, we have

$$\sum_{n=1}^{\infty}\Big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\Big)\Gamma_n(\alpha_1)b_n\big[\sigma a_{n,1}+(1-\sigma)a_{n,2}\big]$$
$$=\sigma\sum_{n=1}^{\infty}\Big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\Big)\Gamma_n(\alpha_1)b_n a_{n,1}$$
$$+(1-\sigma)\sum_{n=1}^{\infty}\Big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\Big)\Gamma_n(\alpha_1)b_n a_{n,2}$$
$$\le 3(k(1-2\lambda)-2\lambda)+(2+\gamma)$$

Hence by Theorem 1, we obtain $h(z)\in A^*(g,\gamma,k,\lambda)$ and this completes the proof.

The concept of neighborhood of analytic functions was first introduced by Goodman [9] and Ruscheweyh [16] investigated concept for the elements of severalfamous subclasses of analytic functions and Altintas and Owa [1] considered for a certain family of analytic functions with negative coefficients, also Liu and Srivastava [12] and Atshan [5] extended this concept for a certain subclass of meromorphically univalent and multivalent functions.

Now, we defined the (n,δ)-neighborhood of a function $f\in A^*$ by:

$$N_{n,\delta}(f)=\left\{h\in A^*\ :\mathrm{h(z)}=\frac{1}{z}-\sum_{n=1}^{\infty}c_n z^n\ and\ \sum_{n=1}^{\infty}n|a_n-c_n|\le\delta,0\le\delta<1\right\}. \qquad (15)$$

For the identity function $e(z)=z$, we have

$$N_{n,\delta}(e)=\left\{h\in A^*:h(z)=\frac{1}{z}-\sum_{n=1}^{\infty}c_n z^n\ \ and\ \ \sum_{n=1}^{\infty}n|c_n|\le\delta\right\}.$$

Definition 2: A function $f\in A^*$ is said to be in the class $A^*(g,\gamma,k,\lambda)$ if there exists a function $h\in A^*(g,\gamma,k,\lambda)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<1-\eta, \qquad (z\in U\,,0\le\eta<1).$$

Theorem 6: If $h\in A^{*}(g,\gamma,k,\lambda)$ and

$$\eta=1-\frac{(k+\gamma)\Gamma_1(\alpha_1)a_1(3(k(1-2\lambda)-2\lambda)+(2+\gamma)}{(k+\gamma)\Gamma_1(\alpha_1)a_1-3(k(1-2\lambda)-2\lambda)+(2+\gamma)}. \tag{16}$$

Thus $N_{n,\delta}(h)\subset A^{*\eta}(g,\gamma,k,\lambda)$.

Proof: Let $f\in N_{n,\delta}(h)$. We want to find from (15) that

$$\sum_{n=1}^{\infty}n|a_n-c_n|\le\delta,$$

Which readily implies the following coefficient inequality

$$\sum_{n=1}^{\infty}|a_n-c_n|\le\delta,\ \ (n\in N).$$

Next, Since $h\in A^{*}(g,\gamma,k,\lambda)$, we have from Theorem 1

$$\sum_{n=1}^{\infty}c_n\le\frac{3(k(1-2\lambda)-2\lambda)+(2+\gamma)}{(k+\gamma)\Gamma_1(\alpha_1)a_1}. \tag{17}$$

So that

$$\left|\frac{f(z)}{g(z)}-1\right|\le\frac{\sum_{n=1}^{\infty}|a_n-c_n|}{1-\sum_{n=1}^{\infty}c_n}\le\frac{(k+\gamma)\Gamma_1(\alpha_1)a_1(3(k(1-2\lambda)-2\lambda)+(2+\gamma)}{(k+\gamma)\Gamma_1(\alpha_1)a_1-3(k(1-2\lambda)-2\lambda)+(2+\gamma)}$$
$$=1-\eta\,.$$

Thus by Definition2, $f\in A^{*\eta}(g,\gamma,k,\lambda)$ for η given by (16).

Now, we introduce the partial sums and the same property has been found for other class in [10].

Theorem 7: Let $f\in A^{*}$ begiven by (1) and define the partial sums $S_1(z)$ *and* $S_\iota(z)$ *by*:

$$S_1(z)=\frac{1}{z}\quad and\quad S_\iota(z)=\frac{1}{z}-\sum_{n=1}^{\iota-1}a_nz^n\,,$$

Suppose also that

$$\sum_{n=1}^{\infty}d_na_n\le1\,,$$

$$\left(d_n=\frac{\big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\big)\Gamma_n(\alpha_1)b_n}{3(k(1-2\lambda)-2\lambda)+(2+\gamma)}\right). \tag{18}$$

Then, We have

$$Re\left\{\frac{f(z)}{S_\iota(z)}\right\}>1-\frac{1}{d_\iota} \tag{19}$$

And

$$Re\left\{\frac{f(z)}{S_\iota(z)}\right\}>\frac{d_\iota}{1+d_\iota}. \tag{20}$$

Each of the bounds in (19) and (20) is the best possible for *nϵN*

Proof: For the coefficients d_n given by (18), it is not difficult to verify that

$$d_{n+1}>d_n>1\ ,\ \ n=1,2,....$$

Therefore, by using hypothesis (18), we have

$$\sum_{n=1}^{\iota-1} a_n + d_\iota \sum_{n=\iota}^{\infty} a_n \le \sum_{n=1}^{\infty} d_n a_n \le 1\,. \tag{21}$$

By setting

$$g_1(z) = d_\iota \left(\frac{f(z)}{S_\iota(z)} - \left(1 - \frac{1}{d_\iota}\right)\right) = 1 - \frac{d_\iota \sum_{n=\iota}^{\infty} a_n z^{n+1}}{1 - \sum_{n=1}^{\iota-1} a_n z^{n+1}}, \tag{22}$$

And applying (21), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \le \frac{d_\iota \sum_{n=\iota}^{\infty} a_n}{2 - 2\sum_{n=1}^{\iota-1} a_n - d_\iota \sum_{n=\iota}^{\infty} a_n} \le 1, \tag{23}$$

Which readily yields the assertion (19).if we take

$$f(z) = \frac{1}{z} - \frac{z^\iota}{d_\iota}, \tag{24}$$

Then

$$\frac{f(z)}{S_\iota(z)} = 1 - \frac{z^\iota}{d_\iota} \to 1 - \frac{1}{d_\iota} (z \to 1^-),$$

Which shows that the bound in (19) is the best possible for each $n \in N$.

Similarly, if we put

$$g_2(z) = (1 + d_\iota)\left[\frac{S_\iota(z)}{f(z)} - \frac{d_\iota}{1 + d_\iota}\right] = 1 + \frac{(1 + d_\iota)(\sum_{n=\iota}^{\infty} a_n z^{n+1})}{1 - \sum_{n=1}^{\infty} a_n z^{n+1}}, \tag{25}$$

And make use of (21), we have

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \le \frac{(1 + d_\iota)(\sum_{n=\iota}^{\infty} a_n)}{2 - 2\sum_{n=1}^{\infty} a_n + (1 + d_\iota)\sum_{n=\iota}^{\infty} a_n}, \tag{26}$$

Which leads us to the assertion (20) . The bound in (20) is sharp for each $n \in N$
With function given by (24).The proof of the theorem is complete.

In the below, we consider integral transforms of functions in the class $A^*(g,\gamma,k,\lambda)$,
Some of these integral transforms was studied byAtshan on the other class in [5].

Theorem 8: Let the function f given by (1) be in the class $A^*(g,\gamma,k,\lambda)$. Then the integral operator

$$F(z) = c \int_0^1 u^c f(uz)du\,, \quad (0 < u \le 1\,, \;\; 0 < c < \infty) \tag{27}$$

is in the class $A^*(g,\gamma,k,\lambda)$, where

$$\Omega = \frac{cw(k + n(n-1)(1+k)\big(1 + \lambda(n-2)\big) - y(c+n+1)(2 + 3(k(1-2\lambda) - 2\lambda)}{y(c+n+1) - cw}.$$

The result is sharp for the function

$$f(z) = \frac{1}{z} - \frac{3(k(1-2\lambda) - 2\lambda) + (2+\gamma)}{(k+\gamma)\Gamma_1(\alpha_1)b_1} z^1.$$

Proof: Let

$$f(z) = \frac{1}{z} - \sum_{n=1}^{\infty} a_n z^n.$$

In the class $A^*(g,\gamma,k,\lambda)$. Then

$$F(z) = c\int_0^1 u^c f(uz)du$$

$$= c\int_0^1 \left(\frac{u^{c-1}}{z} - \sum_{n=1}^{\infty} a_n u^{n+c} z^n\right) du \ = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n z^n. \tag{28}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c(n(n-1)(1+k)\big(1+\lambda(n-2)\big) + (k+\Omega))\Gamma_n(\alpha_1)b_n a_n}{(c+n+1)(3(k(1-2\lambda)-2\lambda)+(2+\Omega))} \le 1\,. \tag{29}$$

Since $f \in A^*(g,\gamma,k,\lambda)$, We have

$$\sum_{n=1}^{\infty} \frac{\Big(n(n-1)(1+k)\big(1+\lambda(n-2)\big) + (k+\gamma)\Big)\Gamma_n(\alpha_1)a_n b_n}{3(k(1-2\lambda)-2\lambda)+(2+\gamma)} \le 1\,.$$

Note that (27) it satisfied if

$$\frac{c(n(n-1)(1+k)\big(1+\lambda(n-2)\big) + (k+\Omega))\Gamma_n(\alpha_1)b_n a_n}{(c+n+1)(3(k(1-2\lambda)-2\lambda)+(2+\Omega))} \le$$

$$\frac{\Big(n(n-1)(1+k)\big(1+\lambda(n-2)\big) + (k+\gamma)\Big)\Gamma_n(\alpha_1)a_n b_n}{3(k(1-2\lambda)-2\lambda)+(2+\gamma)}$$

Rewriting the inequality, we have

$$(c+n+1)(3(k(1-2\lambda)-2\lambda)+(2+\Omega))\Big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\Big)$$

$$\le \big(3(k(1-2\lambda)-2\lambda)+(2+\gamma)\big)c\Big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\Omega)\Big).$$

Solving for Ω, we have

$$\Omega \le \frac{cw(k+n(n-1)(1+k)\big(1+\lambda(n-2)\big) - y(c+n+1)(2+3(k(1-2\lambda)-2\lambda)}{y(c+n+1)-cw} \tag{30}$$

$$= F(n)\,,$$

Where

$$w = 3(k(1-2\lambda)-2\lambda)+(2+\gamma)\ \ , y = n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(2+\gamma).$$

A simple computation will show that $F(n)$ is increasing $F(n) \ge F(1)$.

Using this, the results follows

<u>Theorem 9:</u> Let the function f given by (1) be in the class $A^*(g,\gamma,k,\lambda)$.

Then the function ***F*** defined by (28) is convex in the disk $|z| < R_1$, where

$$R_1 = inf_n \left\{\frac{(c+n+1)\Big(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\Big)\Gamma_n(\alpha_1)b_n}{cn(n+1)(3(k(1-2\lambda)-2\lambda)+(2+\gamma))}\right\}^{\frac{1}{n+1}}. \tag{31}$$

<u>Proof:</u> We show that

$$\left|\frac{zF''(z)}{F'(z)} + 2\right| \le 1 \qquad in \quad |z| < R_1\,, \tag{32}$$

R_1 is given by (31) . In view of (28), we have

$$\left|\frac{zF''(z)+2F'(z)}{F'(z)}\right| = \left|\frac{-\sum_{n=1}^{\infty}\frac{cn^2}{c+n+1}a_n z^{n+1}}{1+\sum_{n+1}^{\infty}\frac{cn}{c+n+1}a_n z^{n+1}}\right|$$

$$\le \frac{\sum_{n=1}^{\infty}\frac{cn^2}{c+n+1}a_n|z|^{n+1}}{1-\sum_{n+1}^{\infty}\frac{cn}{c+n+1}a_n|z|^{n+1}} \le 1\,.$$

Hence

$$\sum_{n=1}^{\infty}\frac{cn^2}{c+n+1}a_n|z|^{n+1}\le 1 .$$

This is enough to consider

$$|z|^{n+1}\le\frac{(c+n+1)\left(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\right)\Gamma_n(\alpha_1)b_n}{cn(n+1)\big(3(k(1-2\lambda)-2\lambda)+(2+\gamma)\big)}.$$

Therefore,

$$|z|\le\left\{\frac{(c+n+1)\left(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+(k+\gamma)\right)\Gamma_n(\alpha_1)b_n}{cn(n+1)\big(3(k(1-2\lambda)-2\lambda)+(2+\gamma)\big)}\right\}^{\frac{1}{n+1}},$$

For $n\ge 1, n\in N$. The result follows by setting $|z|=R_1$.

Theorem 10: Let $f\in A^*(g,\gamma,k,\lambda)$. Then the integral operator

$$F(z)=c\int_0^1 u^c f(uz)du ,\qquad (0<u\le 1 ,\qquad 0<c<\infty),$$

is in the class $A^*\left(g,\frac{1+\gamma c}{2+c},k,\lambda\right)$.

The result is sharp for

$$f_n(z)=\frac{1}{z}-\frac{3(k(1-2\lambda)-2\lambda)+(2+\frac{1+\gamma c}{2+\gamma})}{\left(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+\left(k+\frac{1+\gamma c}{2+\gamma}\right)\right)\Gamma_n(\alpha_1)b_n}z^n.$$

Proof: By definition of F, we obtain

$$F(z)=c\int_0^1 u^c f(uz)du =\frac{1}{z}-\sum_{n+1}^{\infty}\frac{cn}{c+n+1}a_n z^n .$$

By Theorem 1, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{\left(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+\left(k+\frac{1+\gamma c}{2+c}\right)\right)\Gamma_n(\alpha_1)b_n}{(c+n+1)(3(k(1-2\lambda)-2\lambda)+(2+\frac{1+\gamma c}{2+c}))}a_n\le 1, \tag{33}$$

Since, If $f\in A^*(g,\gamma,k,\lambda)$, then (33) satisfied if

$$\frac{c}{(c+n+1)(3(k(1-2\lambda)-2\lambda)+(2+\frac{1+\gamma c}{2+\gamma}))}\le\frac{1}{3(k(1-2\lambda)-2\lambda)+(2+\gamma)}$$

or equivalent, when

$$\emptyset(n,c,k,\lambda,\gamma)=\frac{c(3(k(1-2\lambda)-2\lambda)+(2+\gamma))}{(c+n+1)(3(k(1-2\lambda)-2\lambda)+(2+\frac{1+\gamma c}{2+c}))}\le 1 .$$

Since $\emptyset(n,c,k,\lambda,\gamma)$ is a decreasing function of $n(n\ge 1)$, then the proof is completed . The result is sharp for the function

$$f_n(z)=\frac{1}{z}-\frac{3(k(1-2\lambda)-2\lambda)+(2+\frac{1+\gamma c}{2+c})}{\left(n(n-1)(1+k)\big(1+\lambda(n-2)\big)+\left(k+\frac{1+\gamma c}{2+c}\right)\right)b_n}z^n.$$

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