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On a New Subclass of Univalent Function with Negative Coefficients Defined by Hadamard Product

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Abstract

In this paper, we introduce and study a new subclass of univalent functions with negative coefficient defined by Hadamard product. We obtain various important properties and characteristics properties for this class. Further we obtain partial sums for the same.

Keywords: Univalent functions, Hadamard product (or convolution), Starlike function, Convexity, Close to convexity, Partial sums.
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1. Introduction

Let € denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z : z \in \mathbb{C}, |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$.See [4], denoted by $T^*(\varphi)$and $K(\varphi), (0 \le \varphi < 1)$ the subclasses of function in €. That is starlike and convex functions of order φ respectively.

Analytically , $f \in T^*(\varphi)$ if and only if, f is of the form (1.1) and satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \varphi, \quad (0 \le \varphi < 1; z \in U).$$

Similarly, $f \in K(\varphi)$if and only if, f is of the form (1.1) and satisfies

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \varphi, (0 \le \varphi < 1; z \in U).$$

Also, let A denote the subclass of € consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \; , \quad (a_n \geq 0, n \in \mathbb{N}) \tag{1.2}$$

This class introduced and studied by Silverman [8], let $R^*(\varphi) = R \cap T^*(\varphi)$, $CV(\varphi) = R \cap K^*(\varphi)$. The Classes $R^*(\varphi)$ and $K^*(\varphi)$ possess some interesting properties and have been extensively studied by Silverman [8] and others.

The Hadamard product (or Convolution) of two power series

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \; , \; g(z) = z - \sum_{n=2}^{\infty} b_n z^n \; , \tag{1.3}$$

in A is defined (as usual) by

$$(f * g)(z) = \mathrm{f}(z) * \mathrm{g}(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n \, . \tag{1.4}$$

Motivated by Ravichandaran et al. [12] and Atshan et al. [3] and see [4],[9].

Now, we define a new subclass $A^*(g, \varphi, k, \lambda)$ of the class A.

Definition (1.1): For $0 \leq \varphi < 1, k \geq 0$ and $0 \leq \lambda \leq \frac{1}{2}$, we let $A^*(g, \varphi, k, \lambda)$ be the subclass of the class A consisting of functions of the form (1.2) and satisfying the analytic criterion

$$Re\left\{\frac{z(f*g)'(z)}{f*g(z)} + \lambda\, \frac{z^2(f*g)''(z)}{(f*g)(z)} - \varphi\right\} > k\left|\frac{z(f*g)'(z)}{f*g(z)} + \lambda\, \frac{z^2(f*g)''(z)}{(f*g)(z)} - 1\right| \tag{1.5}$$

The main object of this paper is to study some geometric properties of the class $A^*(g, \varphi, k, \lambda)$, like the coefficient bounds, extreme points, radii of starlikeness, convexity and close to convexity for the class $A^*(g, \varphi, k, \lambda)$.Further, we obtain partial sums for this class.

Atshan and Buti [3], Dziok and Srivastava [5], Goodman [6], Ruscheweyh [7], Silverman [8], Aouf and Magesh and others [2], studied the Univalent function for different classes.

2. Coefficients Inequalities

In the following theorem, we obtain necessary and sufficient condition for a function f to be in the class $A^*(g, \varphi, k, \lambda)$. We will mention some lemmas, useful for our work.

Lemma (2.1) [7]: Let $\varphi \geq 0$ and w be any complex number. Then

$Re(w) \geq \varphi$ *if and only if* $|w - (1+\varphi)| < |w + (1-\varphi)|$.

Lemma (2.2) [7]: Let $k \geq 0, 0 \leq \varphi < 1$ and $\theta \in \mathbb{R}$ .Then

$Re\,(w) > k|w - 1| + \varphi$ if and only if $Re(w(1 + ke^{i\theta}) - ke^{i\theta}) > \varphi$.

Where w be any complex number.

Theorem (2.3) Let $f \in A$ be given by (1.2). Then $f \in A^*(g, \varphi, k, \lambda)$ if and only if

$$\sum_{n=2}^{\infty} (n(k+1)(1+(n-1)\lambda) - (k+\varphi))a_n b_n \leq (1-\varphi) \; . \tag{2.1}$$

The result is sharp for the function

$$f(z) = z + \frac{(1-\varphi)}{[n(1+k)(1+(n-1)\lambda) - (k+\varphi)]b_n} z^n, n \geq 1.$$

Proof:- Let $f \in A^*(g, \varphi, k, \lambda)$ and $|z| = 1$.Then by definition and using Lemma (2.2) it is enough to show that

$$Re\left\{\left(\frac{z(f*g)'(z)}{f*g(z)}+\lambda\ \frac{z^2(f*g)''(z)}{(f*g)(z)}\right)\left(1+ke^{i\theta}\right)-ke^{i\theta}\right\}>\varphi$$

$$=Re\left[\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)\left(1+ke^{i\theta}\right)-ke^{i\theta}(f*g)(z)}{(f*g)(z)}\right]>\varphi,$$

$$\theta\in\mathbb{R} \qquad (2.2)$$

For convenience, we let $A(z)=\ z(f*g)'(z)+\lambda z^2(f*g)''(z)\left(1+ke^{i\theta}\right)-ke^{i\theta}(f*g)(z),$ $B(z)=\ (f*g)(z)$

That is, the equation (2.2) is equivalent to $Re\left(\frac{A(z)}{B(z)}\right)\geq\varphi.$

In view of Lemma (2.1), we only need to prove that

$$|A(z)+\ (1-\varphi)B(z)|-|A(z)-(1+\varphi)B(z)|\geq 0\,.$$

Therefore $|A(z)+\ (1-\varphi)B(z)|=$

$$\left|((z-\sum_{n=2}^{\infty}a_nb_nnz^n)-\lambda(\sum_{n=2}^{\infty}a_nb_nn\,(n-1)z^n))\,\left(1+ke^{i\theta}\right)-ke^{i\theta}\left(z-\sum_{n=2}^{\infty}a_nb_nz^n\right)\right.$$
$$\left.+\,(1-\varphi)(z-\sum_{n=2}^{\infty}a_nb_nz^n)\right|$$

$$=\left|(z\left(1+ke^{i\theta}\right)-\sum_{n=2}^{\infty}a_nb_nn\,z^n)\left(1+ke^{i\theta}\right)-\lambda\sum_{n=2}^{\infty}a_nb_nn\,(n-1)z^n\left(1+ke^{i\theta}\right)-ke^{i\theta}z\right.$$
$$\left.+\,ke^{i\theta}\sum_{n=2}^{\infty}a_nb_nz^n+\ +(1-\varphi)z-(1-\varphi)\,(\sum_{n=2}^{\infty}a_nb_nz^n)\right|$$

$$\geq(2-\varphi)-\sum_{n=2}^{\infty}a_nb_n(n(1+k)(1+\,(n-1)\lambda)-\mathrm{k}+(1-\varphi)\,.$$

Also

$|A(z)-(1+\varphi)B(z)|=$

$$\left|((z-\sum_{n=2}^{\infty}a_nb_nnz^n)-\lambda(\sum_{n=2}^{\infty}a_nb_nn\,(n-1)z^n))\left(1+ke^{i\theta}\right)-ke^{i\theta}(z-\sum_{n=2}^{\infty}a_nb_nz^n)\right.$$
$$\left.-\,(1+\varphi)(z-\sum_{n=2}^{\infty}a_nb_nz^n)\right|$$

$$\leq\left(1-\,(1+\varphi)\right)-\sum_{n=2}^{\infty}a_nb_n\left(n(1+k)+n(n-1)\lambda(1+\mathrm{k})\right)-\mathrm{k}-(1+\varphi),$$

$$=(\varphi)+\textstyle\sum_{n=2}^{\infty}a_nb_n\left(n(1+k)(1+\,(n-1)\lambda)-\mathrm{k}-(1+\varphi)\right).$$

It is easy to show that

$$|A(z)+\ (1-\varphi)B(z)|-|A(z)-(1+\varphi)B(z)|$$

$$\geq(2-\varphi)-\sum_{n=2}^{\infty}a_nb_n(n(1+k)(1+\,(n-1)\lambda)-\mathrm{k}+(1-\varphi)$$
$$-\left((\varphi)+\sum_{n=2}^{\infty}a_nb_n(n(1+k)(1+\,(n-1)\lambda)-\mathrm{k}-(1+\varphi)\right).$$

$$=\ 2(1-\varphi)-2\sum_{n=2}^{\infty}a_nb_n\,(n(1+k)(1+\,(n-1)\lambda)-(\mathrm{k}+\varphi))$$

$$= (1-\varphi) - \sum_{n=2}^{\infty} a_n b_n (n(1+k)(1+(n-1)\lambda) - (\mathrm{k}+\varphi)) \geq 0,$$

by the given condition (2.1).

Conversely, by Lemma (2.2), we have equation (2).

$$Re\left[\frac{[z(f*g)'(z)+\lambda z^2(f*g)''(z)](1+ke^{i\theta}) - ke^{i\theta}(f*g)(z)}{(f*g)(z)}\right] \geq \varphi \quad , \quad \theta \in \mathbb{R},$$

then,

$$Re\left[\frac{[z(f*g)'(z)+\lambda z^2(f*g)''(z)](1+ke^{i\theta}) - ke^{i\theta}(f*g)(z)}{(f*g)(z)} - \varphi\right] \geq 0$$

$$= Re\left[\frac{[z(f*g)'(z)+\lambda z^2(f*g)''(z)](1+ke^{i\theta}) - ke^{i\theta}(f*g)(z) - \varphi(f*g)(z)}{(f*g)(z)}\right] \geq 0,$$

or equivalently,

$$Re\left[\frac{(z-\sum_{n=2}^{\infty} a_n b_n n z^n)(1+ke^{i\theta}) - \lambda(\sum_{n=2}^{\infty} a_n b_n n(n-1)z^n)(1+ke^{i\theta}) - ke^{i\theta}(z-\sum_{n=2}^{\infty} a_n b_n z^n) - \varphi(z-\sum_{n=2}^{\infty} a_n b_n z^n)]}{z-\sum_{n=2}^{\infty} a_n b_n z^n}\right] \geq 0,$$

$$= Re\left[\frac{z(1-\varphi) - \sum_{n=2}^{\infty} a_n b_n z^n [n(1+ke^{i\theta})(1+(n-1)\lambda) - (ke^{i\theta}+\varphi)]}{z-\sum_{n=2}^{\infty} a_n b_n z^n}\right] \geq 0,$$

$$= Re\left[\frac{z(1-\varphi) - \sum_{n=2}^{\infty} a_n b_n z^{n-1} [n(1+ke^{i\theta})(1+(n-1)\lambda) - (ke^{i\theta}+\varphi)]}{z(1-\sum_{n=2}^{\infty} a_n b_n z^{n-1})}\right] \geq 0.$$

Since $|e^{i\theta}| = 1$, hence $Re(e^{i\theta}) \geq |e^{i\theta}| = 1$.Letting $r \to 1^-$ yields

$$Re\left[\frac{(1-\varphi) - \sum_{n=2}^{\infty} a_n b_n [n(1+k)(1+(n-1)\lambda) - (k+\varphi)]}{(1-\sum_{n=2}^{\infty} a_n b_n)}\right] \geq 0,$$

and so by the mean value Theorem, we have

$$Re\{(1-\varphi) - \sum_{n=2}^{\infty} a_n b_n (n(1+k)(1+(n-1)\lambda) - (k+\varphi))\} \geq 0,$$

so we have

$$\sum_{n=2}^{\infty} a_n b_n [n(1+k)(1+(n-1)\lambda) - (k+\varphi)] \leq (1-\varphi).$$

Finally, the result is sharp for the function

$$f(z) = z - \frac{(1-\varphi)}{[n(1+k)(1+(n-1)\lambda) - (k+\varphi)]b_n} z^n \quad .$$

The proof is complete. ■

Corollary (2.4) Let $f \in A^*(g, \varphi, k, \lambda)$.Then

$$a_n \leq \frac{(1-\varphi)}{[n(1+k)(1+(n-1)\lambda) - (k+\varphi)]b_n} \quad . \tag{2.3}$$

By taking λ=0, in Theorem (2.3) we get the following Corollary.

Corollary (2.5) Let $f(z) \in A$ given by (1.2).Then $f \in A^*(g, \varphi, k, \lambda)$ if and only if

$$\sum_{n=2}^{\infty} a_n b_n [n(1+k) - (k+\varphi)] \leq (1-\varphi).$$

3. Growth Theorem and Distortion Theorem

Theorem (3.1) If $f \in A^*(g, \varphi, k, \lambda)$ and $b_n \geq b_2$, then

$$r-\frac{(1-\varphi)}{(2(k+1)(1+\lambda)-(k+\varphi))b_2}r^2\leq|f(z)|\leq r+\frac{(1-\varphi)}{(2(k+1)(1+\lambda)-(k+\varphi))b_2}r^2,$$

and

$$1-\frac{2(1-\varphi)}{(2(k+1)(1+\lambda)-(k+\varphi))b_2}r\leq|f'(z)|\leq 1+\frac{2(1-\varphi)}{(2(k+1)(1+\lambda)-(k+\varphi))b_2}r.$$

$(|z|=r<1)$.

The result is sharp for

$$f(z)=z-\frac{(1-\varphi)}{(2(k+1)(1+\lambda)-(k+\varphi))b_2}\ z. \tag{3.1}$$

Proof:- Since

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n,$$

we have

$$|f(z)|\leq\left|z-\sum_{n=2}^{\infty}a_nz^n\right||z|+\sum_{n=2}^{\infty}a_n|z|^n\ \leq r+r^2\sum_{n=2}^{\infty}a_n. \tag{3.2}$$

Since for $n\geq 2,\ (2(k+1)(1+\lambda)-(k+\varphi))b_2\leq\ (n(1+k)(1+(n-1)\lambda)-(k+\varphi)]$.

Using Theorem (2.3), we have

$$(2(k+1)(1+\lambda)-(k+\varphi))b_2\sum_{n=2}^{\infty}a_n\leq\sum_{n=2}^{\infty}a_nb_n[\,(n(1+k)(1+(n-1)\lambda)-(k+\varphi)]$$
$$\leq(1-\varphi).$$

That is,

$$\sum_{n=2}^{\infty}a_n\leq\frac{1-\varphi}{(2(k+1)(1+\lambda)-(k+\varphi))b_2}.$$

Using the above equation in (3.3), we have

$$|f(z)|\leq r+\frac{(1-\varphi)}{(2(k+1)(1+\lambda)-(k+\varphi))b_2}r^2,$$

And

$$|f(z)|\geq r-\frac{(1-\varphi)}{(2(k+1)(1+\lambda)-(k+\varphi))b_2}r^2.$$

The result is sharp for

$$f(z)=z-\frac{(1-\varphi)}{(2(k+1)(1+\lambda)-(k+\varphi))b_2}\ z^2.$$

Similarly, since

$$f'(z)=1-\sum_{n=2}^{\infty}a_nnz^{n-1},$$

we have,

$$|f'(z)|=\left|1-\sum_{n=2}^{\infty}a_nnz^{n-1}\right|\leq|1|+\sum_{n=2}^{\infty}na_n|z|^{n-1}\ \leq 1+2r\sum_{n=2}^{\infty}a_n. \tag{3.3}$$

Since for $n\geq 2,\ (2(k+1)(1+\lambda)-(k+\varphi))b_2\leq\ (n(1+k)(1+(n-1)\lambda)-(k+\varphi)]b_n$,

Using Theorem (2.3), we have

$$(2(k+1)(1+\lambda)-(k+\varphi))b_2\sum_{n=2}^{\infty}a_n \le \sum_{n=2}^{\infty}a_n b_n[(n(1+k)(1+(n-1)\lambda)-(k+\varphi)]$$
$$\le (1-\varphi).$$

That is

$$\sum_{n=2}^{\infty}a_n \le \frac{1-\varphi}{(2(k+1)(1+\lambda)-(k+\varphi))b_2}.$$

Using the above equation in (3.3), we have

$$|f'(z)| \le 1+\frac{2(1-\varphi)}{(2(k+1)(1+\lambda)-(k+\varphi))b_2}r,$$

and

$$|f'(z)| \ge 1-\frac{2(1-\varphi)}{(2(k+1)(1+\lambda)-(k+\varphi))b_2}r.$$

The result is sharp for

$$f(z)=1-\frac{2(1-\varphi)}{(2(k+1)(1+\lambda)-(k+\varphi))b_2}\,z.$$

The proof is complete. ■

4. Extreme Point

Theorem (4.1) Let $f_1(\mathrm{z})=\mathrm{z}$,

$$f_n(\mathrm{z})=\mathrm{z}-\frac{(1-\varphi)}{[n(1+k)(1+(n-1)\lambda)-(k+\varphi)]b_n}z^n. \quad (\mathrm{n}\ge 2) \tag{4.1}$$

where $(\mathrm{n}\in\mathbb{N}, 0\le\varphi<1, k\ge 0)$.

Then $f \in A^*(\mathrm{g}, \varphi, \mathrm{k}, \lambda)$, if and only if it can be expressed in the form

$$f(z)=\sum_{n=1}^{\infty}\mu_n f_n(\mathrm{z}), \tag{4.2}$$

where $[\mu_n \ge 0, \sum_{n=1}^{\infty}\mu_n=1$ or $1=\mu_1+\sum_{n=2}^{\infty}\mu_n]$.

<u>Proof:-</u> Let $f(z)$ can be expressed as in (4.2). Then

$$f(z)=\sum_{n=1}^{\infty}\mu_n f_n(\mathrm{z})$$
$$=\mu_1 z_1+\sum_{n=2}^{\infty}\mu_n f_n(z)$$
$$=\mu_1 z+\sum_{n=2}^{\infty}\mu_n\left(\mathrm{z}-\frac{(1-\varphi)}{(n(1+k)(1+(n-1)\lambda)-(k+\varphi))b_n}z^n\right).$$
$$=\mu_1 z+\sum_{n=2}^{\infty}\mu_n z-\sum_{n=2}^{\infty}\mu_n\left(\frac{(1-\varphi)}{(n(1+k)(1+(n-1)\lambda)-(k+\varphi))b_n}z^n\right)$$
$$=z\left(\mu_1+\sum_{n=2}^{\infty}\mu_n\right)-\sum_{n=2}^{\infty}\frac{(1-\varphi)}{(n(1+k)(1+(n-1)\lambda)-(k+\varphi))b_n}\mu_n z^n$$
$$=z-\sum_{n=2}^{\infty}\frac{(1-\varphi)}{(n(1+k)(1+(n-1)\lambda)-(k+\varphi))b_n}\mu_n z^n,$$

$$= z - \sum_{n=2}^{\infty} a_n z^n,$$

where

$$a_n = \frac{(1-\varphi)}{(n(1+k)(1+(n-1)\lambda)-(k+\varphi))b_n}\mu_n ,$$

therefore $f \in A^*(g, \varphi, k, \lambda)$, since

$$\frac{\sum_{n=2}^{\infty} a_n b_n\,(n(1+k)(1+(n-1)\lambda)-(k+\varphi)}{(1-\varphi)} < 1.$$

Hence,

$$\sum_{n=2}^{\infty} \mu_n \frac{(1-\varphi)}{[n(1+k)(1+(n-1)\lambda)-(k+\varphi)]b_n} \times \frac{[n(1+k)(1+(n-1)\lambda)-(k+\varphi)]b_n}{(1-\varphi)} z^n .$$

$$= \sum_{n=2}^{\infty} \mu_n = 1-\mu_1 < 1.$$

So by Theorem (2.3), $f \in A^*(g, \varphi, k, \lambda)$.

Conversely, we suppose $f \in A^*(g, \varphi, k, \lambda)$.Then by (2.1), we may set

$$a_n \leq \frac{(1-\varphi)}{(n(1+k)(1+(n-1)\lambda)-(k+\varphi))b_n}, \qquad (n \geq 2)$$

we set,

$$\mu_n = \frac{(n(1+k)(1+(n-1)\lambda)-(k+\varphi))b_n}{(1-\varphi)} a_n , \qquad (n \geq 2)$$

and

$$\mu_1 = 1 - \sum_{n=1}^{\infty} \mu_n ,$$

then

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n,$$

$$= z - \sum_{n=1}^{\infty} \frac{(1-\varphi)}{(n(1+k)(1+(n-1)\lambda)-(k+\varphi))b_n}\mu_n z^n. \tag{4.3}$$

Therefore,

$$z^n = \frac{(n(1+k)(1+(n-1)\lambda)-(k+\varphi))b_n}{(1-\varphi)}(z - f_n(z)).$$

Putting in (4.3), we get

$$f(z) = z - \sum_{n=2}^{\infty} \frac{(1-\varphi)}{(n(1+k)(1+(n-1)\lambda)-(k+\varphi))b_n}\mu_n \times \frac{(n(1+k)(1+(n-1)\lambda)-(k+\varphi))b_n}{(1-\varphi)} (z - f_n(z))$$

$$= z - \sum_{n=2}^{\infty} \mu_n z + \sum_{n=2}^{\infty} \mu_n f_n(z) = z\left(1 - \sum_{n=2}^{\infty} \mu_n\right) + \sum_{n=2}^{\infty} \mu_n f_n(z) = z\mu_1 + \sum_{n=2}^{\infty} \mu_n f_n(z)$$

$$= \sum_{n=1}^{\infty} \mu_n f_n(z).$$

The proof is complete. ■

5. Radii of Univalent Starlikeness, Convexity and Close to Convexity

Theorem (5.1) If $f \in A^*(g, \varphi, k, \lambda)$ then f is starlike of order $\delta\ (0 \le \delta < 1)$ in the disc $|z| < r_1(\varphi, \lambda, \delta, k)$, where

$$r_1(\varphi,\lambda,\delta,k) = inf_n \left\{ \frac{(1-\delta)[n(1+(n-1)\lambda)(k+1)-(k+\varphi)]b_n}{(n-\delta)(1-\varphi)} \right\}^{\frac{1}{n-1}} ,n \ge 2. \quad (5.1)$$

Proof:- It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1-\delta, \qquad (0 \le \delta < 1)$$

for $|z| < r_1(\varphi,\lambda,\delta,k)$.

We have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{Z(1-\sum_{n=2}^{\infty} a_n n z^{n-1})}{z - \sum_{n=2}^{\infty} a_n z^{n-1}} - 1\right| = \left|\frac{Z(1-\sum_{n=2}^{\infty} a_n n z^{n-1})}{Z(1-\sum_{n=2}^{\infty} a_n z^{n-1})} - 1\right|$$

$$= \left|\frac{(1-\sum_{n=2}^{\infty} a_n n z^{n-1}) - (1-\sum_{n=2}^{\infty} a_n z^{n-1})}{(1-\sum_{n=2}^{\infty} a_n z^{n-1})}\right|$$

$$= \left|\frac{(-\sum_{n=2}^{\infty} a_n (n-1) z^{n-1})}{(1-\sum_{n=2}^{\infty} a_n z^{n-1})}\right| \le \frac{(\sum_{n=2}^{\infty} a_n (n-1)|z|^{n-1})}{(1-\sum_{n=2}^{\infty} a_n |z|^{n-1})} \le 1-\delta.$$

If

$$\frac{\sum_{n=2}^{\infty}(n-\delta)\, a_n |z|^{n-1}}{1-\delta} \le 1. \quad (5.2)$$

Hence, by Theorem (2.3),(5.2) will be true if

$$\frac{(n-\delta)}{1-\delta}|z|^{n-1} \le \frac{(n(1+k)(1+(n-1)\lambda)-(k+\varphi))b_n}{(1-\varphi)},$$

and hence,

$$|z| \le \left\{ \frac{(1-\delta)[n(1+(n-1)\lambda)(k+1)-(k+\varphi)]b_n}{(n-\delta)(1-\varphi)} \right\}^{\frac{1}{n-1}}.$$

Setting $|z| = r_1(\varphi,\lambda,\delta,k)$, we get the desired result. The proof is complete. ∎

Theorem (5.2) If $f \in A^*(g, \varphi, k, \lambda)$, then f is convex of order $\delta\ (0 \le \delta < 1)$ in the disc $|z| < r_2(\varphi,\lambda,\delta,k)$, where

$$r_2(\varphi,\lambda,\delta,k) = inf_n \left\{ \frac{(1-\delta)[n(1+(n-1)\lambda)(k+1)-(k+\varphi)]b_n}{n(n-\delta)(1-\varphi)} \right\}^{\frac{1}{n-1}} ,n \ge 2. \quad (5.3)$$

Proof:- It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \le 1-\delta, \qquad (0 \le \delta < 1)$$

For $|z| < r_2(\varphi,\lambda,\delta,k)$.

We have

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{z(-\sum_{n=2}^{\infty} a_n n(n-1) z^{n-2})}{1-\sum_{n=2}^{\infty} a_n n z^{n-1}}\right| = \left|\frac{-\sum_{n=2}^{\infty} a_n n(n-1) z^{n-1}}{1-\sum_{n=2}^{\infty} a_n n z^{n-1}}\right|$$

$$\le \frac{\sum_{n=2}^{\infty} a_n n(n-1)|z|^{n-1}}{1-\sum_{n=2}^{\infty} a_n n |z|^{n-1}} \le 1-\delta.$$

If

$$\frac{\sum_{n=2}^{\infty} n(n-\delta)}{(1-\delta)} a_n |z|^{n-1} \le 1. \quad (5.4)$$

Hence, by Theorem (2.3),(5.4) will be true if

$$\frac{n(n-\delta)}{1-\delta}|z|^{n-1} \le \frac{[n(1+(n-1)\lambda)(k+1)-(k+\varphi)]b_n}{(1-\varphi)},$$

and hence,

$$|z| \le \left\{\frac{(1-\delta)[n(1+(n-1)\lambda)(k+1)-(k+\varphi)]b_n}{n(n-\delta)(1-\varphi)}\right\}^{\frac{1}{n-1}}.$$

Setting $|z| = r_2(\varphi,\lambda,\delta,k)$, we get the desired result. The proof is complete. ■

Theorem(5.3) Let a function $f \in A^*(g, \varphi, k, \lambda)$. Then f is close to convex of order δ $(0 \le \delta < 1)$ in the disc $|z| < r_3(\varphi,\lambda,\delta,k)$, where

$$r_3(\varphi,\lambda,\delta,k) = \inf_n \left\{\frac{(1-\delta)[n(1+(n-1)\lambda)(k+1)-(k+\varphi)]b_n}{n(1-\varphi)}\right\}^{\frac{1}{n-1}} \quad ,n \ge 2. \tag{5.5}$$

Proof:- It is sufficient to show that

$|f'(z)-1| \le 1-\delta, \quad (0 \le \delta < 1)$

for $|z| \le r_3(\varphi,\lambda,\delta,k)$.

We have,

$$|f'(z)-1| = \left|1-\sum_{n=2}^{\infty} a_n n z^{n-1} - 1\right| = \left|-\sum_{n=2}^{\infty} a_n n z^{n-1}\right| \le \sum_{n=2}^{\infty} a_n n |z|^{n-1} \le 1-\delta,$$

If

$$\frac{\sum_{n=2}^{\infty} n}{(1-\delta)} a_n |z|^{n-1} \le 1. \tag{5.6}$$

Hence, by Theorem (2.3), (5.6) will be true if

$$\frac{n}{1-\delta}|z|^{n-1} \le \frac{[n(1+(n-1)\lambda)(k+1)-(k+\varphi)]b_n}{(1-\varphi)},$$

and hence,

$$|z| \le \left\{\frac{(1-\delta)[n(1+(n-1)\lambda)(k+1)-(k+\varphi)]b_n}{n(1-\varphi)}\right\}^{\frac{1}{n-1}}.$$

Setting $|z| = r_3(\varphi,\lambda,\delta,k)$, we get the desired result. The proof is complete. ■

6. The Closure Theorem

Theorem(6.1) Let the function $f_j(z) \in A^*(g, \varphi, k, \lambda)$ for every $j = 1,2,\cdots,l$. Then the function $h(z)$ defined by $h(z) = \sum_{j=1}^{\infty} c_j f_j(z)$ and $\sum_{j=1}^{\infty} c_j = 1$, $c_j \ge 0$, in the class $A^*(g, \varphi, k, \lambda)$.

Proof:- By definition of $h(z)$, we have

$$h(z) = \left[\sum_{j=1}^{\infty} c_j\right] z - \sum_{n=2}^{\infty}\left[\sum_{j=1}^{\infty} c_j\, a_{n,j} b_{n,j}\right] z^n. \tag{6.1}$$

Further, since $f_j(z)$ are in the class $A^*(g, \varphi, k, \lambda)$ for every $j = 1,2,\cdots,l$.

Hence, we can see that

$$\sum_{n=1}^{\infty}[(n(1+k)(1+(n-1)\lambda)-(k+\varphi)]\,[\textstyle\sum_{j=1}^{\infty} c_j a_{n,j} b_{n,j}],$$

$$= \sum_{j=1}^{\infty} c_j \left[\sum_{n=1}^{\infty}[(n(1+k)(1+(n-1)\lambda)-(k+\varphi)]a_{n,j}\, b_{n,j}\right].$$

$\le (1-\varphi)\sum_{j=1}^{\infty} c_j = (1-\varphi)$, since $\sum_{j=1}^{\infty} c_j = 1$.

This proves that $h(z) \in A^*(g, \varphi, k, \lambda)$. The proof is complete. ■

7. Partial Sums

Let $f \in A$ be a function of the form (1.2). Motivated by Silverman [10] and Silvia [11]

See also [2], [1], we define the partial sums f_m defined by

$$f_m(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (m \in \mathbb{N}). \tag{7.1}$$

Theorem (7.1) Let $f \in A^*(g, \varphi, k, \lambda)$ be given by (1.2) and define a partial sums $f_1(z)$ and $f_m(z)$ by

$$f_1(z) = z \quad \text{and} \quad f_m(z) = z - \sum_{n=2}^{\infty} a_n z^n \ , \ (m \in \mathbb{N}\backslash\{1\}). \tag{7.2}$$

Suppose also that

$$\sum_{n=2}^{\infty} d_n a_n \leq 1 ,$$

Where

$$d_n \geq \begin{cases} 1 & for\ n=2,3,\ldots\ldots m. \\ \frac{(n(1+k)(1+(n-1)\lambda)-(k+\varphi))b_n}{(1-\varphi)} & for\ n=m+1,m+2,m+3,.. \end{cases} , \tag{7.3}$$

then $f \in A^*(g, \varphi, k, \lambda)$.Furthermore,

$$Re\left(\frac{f(z)}{f_m(z)}\right) > 1 - \frac{1}{d_{m+1}} \ , \tag{7.4}$$

and

$$Re\left(\frac{f_m(z)}{f(z)}\right) > \frac{d_{m+1}}{1 + d_{m+1}}. \tag{7.5}$$

Proof:- For the coefficients d_n given by (7.3), it has not difficult to verify that

$$d_{n+1} > d_n > 1 \quad . \tag{7.6}$$

Therefor we have

$$\sum_{n=2}^{\infty} a_n + d_{m+1} \sum_{n=m+1}^{\infty} a_n \leq \sum_{n=2}^{\infty} d_n a_n \leq 1 . \tag{7.7}$$

By using the hypothesis (7.3), by setting $g_1(z) = d_{m+1}\left(\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right)\right)$,

$$= 1 + \frac{d_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}} \ ,$$

then it suffices show that $Re(g_1(z)) \geq 0 (z \in U)$,

and applying(7.7), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq 1 \quad . \ (z \in U)$$

We applying (7.7), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_{m+1} \sum_{n=m+1}^{\infty} a_n}{2 - 2\sum_{n=2}^{\infty} a_n - d_{m+1} \sum_{n=m+1}^{\infty} a_n} \leq 1 \quad . \ (z \in U)$$

Which readily yields the assertion (7.4) of Theorem (7.1),in order to see that

$$f(z) = z - \frac{z^{m+1}}{d_{m+1}}, \tag{7.8}$$

gives sharp result, we observe that for $z = re^{i\pi/m}$ that

$$\frac{f(z)}{f_m(z)} = 1 - \frac{r^{m+1}}{d_{m+1}} \Rightarrow 1 - \frac{1}{d_{m+1}} \qquad as\ r \to 1^-.$$

Similarly, if we take

$$g_2(z) = (1 + d_{m+1}) \left(\frac{f_m(z)}{f(z)} - \frac{d_{m+1}}{1 + d_{m+1}} \right),$$

and making use of (7.7), we deduce that

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + d_{m+1}) \sum_{n=m+1}^{\infty} a_n}{2 - 2\sum_{n=2}^{\infty} a_n - (1 - d_{m+1}) \sum_{n=m+1}^{\infty} a_n} \leq 1.$$

Which leads us immediately to the assertion (7.5) of Theorem (7.1),the bound in (7.5) is sharp for each $m \in N$ with the extremal function f(z) given by (7.8). The proof is complete. ■

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