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Subordination Results for Certain Subclass of Univalent Functions

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Abstract

In this paper, we consider $\frac{(1+\gamma)z^2f''(z)+zf'(z)}{(1+\gamma)zf'(z)-\gamma f(z)}$, $(0 \le \gamma < 1)$ of univalent functions $f \in \mathcal{A}$ such that

$$Re\left\{\frac{(1+\gamma)z^2f''(z)+zf'(z)}{(1+\gamma)zf'(z)-\gamma f(z)}\right\} > \alpha$$

be the class of γ-convex functions of order $\alpha(0 \le \alpha < 1)$, and analogous to the class $C_p^*(\emptyset)$, for $f \in \mathcal{A}$, we define the class $L(b,\gamma)$ of univalent functions $f \in \mathcal{A}$, then, obtaining some subordination results on functions in this class.

Keyword: univalent function, convex function, γ-convex function, differential subordination
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Introduction

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$ and let S be the subclass of $\mathcal{A}$ consisting of functions which are also univalent in U. Let $\emptyset(z)$ be an analytic function with positive real part on U that satisfies $\emptyset(0) = 1, \emptyset'(0) > 0$ and which maps the unit disk U onto a region starlike with respect to 1 and symmetric with respect to the real axis.

Ma and Minda [2] introduced and studied the class $S^*(\emptyset)$ consists of functions $f \in S$ for which

$$\frac{zf'(z)}{f(z)} \prec \emptyset(z)(z \in U).$$

Following Ma and Minda [2], Ravichandran [6] defined a more general class $S_b^*(\emptyset)$ of starlike functions of complex order consists of functions $f \in S$ for which

$$1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\prec \emptyset(z)(z\in U),$$ where $b\neq 0$ is a complex number.

The class $C_b(\emptyset)$consists of functions$f\in\mathcal{A}$ satisfying

$$1+\frac{1}{b}\frac{zf''(z)}{f'(z)}\prec\phi(z). \tag{1.2}$$

$$Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha \qquad (z\in U), \tag{1.3}$$

The function $f\in\mathcal{A}$ is said to be starlike [1] of order $\alpha(0\leq\alpha<1)$, if and this class is denoted by $S^*(\alpha)$.

Similarly the function $f\in\mathcal{A}$ is said to be convex [1] of order α $(0\leq\alpha<1)$, if

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \qquad (z\in U), \tag{1.4}$$

and this class is denoted by $C(\alpha)$.Let f and g be analytic functions in the open unit disk U. Then g is said to be subordinate to$f$, written $g\prec f$or $g(z)\prec f(z)$, if there exists a Schwarz function$w$, which is analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$, such that $g(z)=f\big(w(z)\big)$ $(z\in U)$. Furthermore, if the function $f(z)$ is univalent in U, we have the following equivalence relationship

$$g(z)\prec f(z)(z\in U)\Leftrightarrow g(0)=f(0) \text{ and } g(U)\subset f(U).$$

The general theory of differential subordinations was introduced by Miller and Mocanu [3,4].In this paper for functions $f\in\mathcal{A}$ given as in (1.1) we consider $\frac{(1+\gamma)z^2f''(z)+zf'(z)}{(1+\gamma)zf'(z)-\gamma f(z)}$, $(0\leq\gamma<1)$ such that $Re\left\{\frac{(1+\gamma)z^2f''(z)+zf'(z)}{(1+\gamma)zf'(z)-\gamma f(z)}\right\}>\alpha$ be the class of γ-convex functions of order $(0\leq\alpha<1)$.

Taking $\gamma=0$, it reduce to the class $1+\frac{zf''(z)}{f'(z)}$ with

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \qquad (0\leq\alpha<1),$$

of convex univalent functions of order α [1], and then analogous to the class $C_b(\emptyset)$, for $f\in\mathcal{A}$, we define the class $\mathcal{L}[b,\gamma](\emptyset)$of univalent functions $f\in\mathcal{A}$ as follows:

Definition (1.1) Let$b\neq 0$ be a complex number and let $\emptyset(z)$ be analytic function with positive real part on U that satisfies $\emptyset(0)=1,\emptyset'(0)>0$ and which maps the unit disk$U$ onto a region starlike with respect to 1 and symmetric with respect to the real axis.Let $\mathcal{L}[b,\gamma](\emptyset)$ be the class of functions $f\in\mathcal{A}, f'(z)\neq 0$ satisfying

$$1+\frac{1}{b}\left(\frac{(1+\gamma)z^2f''(z)+zf'(z)}{(1+\gamma)zf'(z)-\gamma f(z)}-1\right)\prec\phi(z), \quad (0\leq\gamma<1, z\in U). \tag{1.5}$$

Clearly$\mathcal{L}[b,0](\emptyset)\equiv C_b(\phi)$.

To prove our main results the following lemmas are needed.

Lemma (1.1) [Miller and A. S. S. Mocanu[5]: Let$k,r\in\mathbb{C}$, with $k\neq 0$ and let h be convex function in the open unit disk$U$, with

$$Re(k\,h(z)+r)>0, \tag{1.6}$$

if$p$ is analytic in U with $p(0)=h(0)$ and

$$p(z)+\frac{zp'(z)}{kp(z)+r}\prec h(z), \tag{1.7}$$

then $\prec h$.

Lemma (1.2) [5]: Let h be convex function in the open unit disk$U$, $h(0)=1$. Define $F(z)$ by

$$F(z)=z\exp\left(\int_0^z\frac{h(x)-1}{x}dx\right) \tag{1.8}$$

Let $q(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be analyticin U. Then

$$1 + \frac{zq'(z)}{q(z)} \prec h(z) \tag{1.9}$$

if and only if for all $|s| \le 1, |t| \le 1$, we have

$$\frac{q(tz)}{q(sz)} \prec \frac{sF(tz)}{tF(sz)}. \tag{1.10}$$

Main Results

Theorem (2.1):A function $f \in \mathcal{L}[b,\gamma](\emptyset)$ for $z \neq 0$ if and only if

$$\left(\frac{(1+\gamma)zf'(z) - \gamma f(z)}{z}\right)^{\frac{1}{b}} = \exp\left(\int_0^z \frac{\emptyset(w(\delta)) - 1}{\delta} d\delta\right) \tag{2.1}$$

where$w(z)$ is analytic in U satisfying $w(0) = 0, |w(z)| \le 1$.

Proof: Let$f \in \mathcal{L}[b,\gamma](\emptyset)$.Then by Definition (1.1), we get

$$1 + \frac{1}{b}\left(\frac{(1+\gamma)z^2 f''(z) + zf'(z)}{(1+\gamma)zf'(z) - \gamma f(z)} - 1\right) \prec \phi(z) \tag{2.2}$$

Therefore there is a function $w(z)$ analytic in U with $w(0) = 0, |w(z)| \le 1$ such that

$$1 + \frac{1}{b}\left(\frac{(1+\gamma)z^2 f''(z) + zf'(z)}{(1+\gamma)zf'(z) - \gamma f(z)} - 1\right) = \phi(w(z)) \tag{2.3}$$

Then

$$\frac{1}{b}\left(\frac{(1+\gamma)z^2 f''(z) + zf'(z)}{(1+\gamma)zf'(z) - \gamma f(z)} - 1\right) = \phi(w(z)) - 1$$

$$\frac{1}{b}\left(\frac{(1+\gamma)zf''(z) + f'(z)}{(1+\gamma)zf'(z) - \gamma f(z)} - \frac{1}{z}\right) = \frac{\phi(w(z)) - 1}{z} \tag{2.4}$$

Integration both sides (2.4) from 0 to z gives

$$\frac{1}{b}\left[\ln\left((1+\gamma)zf'(z) - \gamma f(z)\right) - \ln z\right] = \int_0^z \frac{\emptyset(w(\delta)) - 1}{\delta} d\delta$$

$$\ln\left(\frac{(1+\gamma)zf'(z) - \gamma f(z)}{z}\right)^{\frac{1}{b}} = \int_0^z \frac{\emptyset(w(\delta)) - 1}{\delta} d\delta,$$

then,

$$\left(\frac{(1+\gamma)zf'(z) - \gamma f(z)}{z}\right)^{\frac{1}{b}} = \exp\left(\int_0^z \frac{\emptyset(w(\delta)) - 1}{\delta} d\delta\right)$$

Conversely, suppose that

$$\left(\frac{(1+\gamma)zf'(z)-\gamma f(z)}{z}\right)^{\frac{1}{b}} = \exp\left(\int_0^z \frac{\emptyset(w(\delta))-1}{\delta}\,d\delta\right) \tag{2.5}$$

Now differentiating (2.5)both sides with respect to z, we obtain

$$\frac{1}{b}\left(\frac{(1+\gamma)zf'(z)-\gamma f(z)}{z}\right)^{\frac{1}{b}-1}\left[\frac{z\big((1+\gamma)zf''(z)+(1+\gamma)f'(z)-\gamma f'(z)\big)-\big((1+\gamma)zf'(z)-\gamma f(z)\big)}{z^2}\right]$$

$$= \exp\left(\int_0^z \frac{\emptyset(w(\delta))-1}{\delta}\,d\delta\right)\times\left(\left(\frac{\emptyset(w(\delta))-1}{z}\right)\right) \tag{2.6}$$

Then

$$\frac{1}{b}\left(\frac{z}{(1+\gamma)zf'(z)-\gamma f(z)}\right)\left(\frac{\big((1+\gamma)z^2f''(z)+zf'(z)-\big((1+\gamma)zf'(z)-\gamma zf(z)\big)\big)}{z^2}\right)$$
$$= \frac{\emptyset(w(z))-1}{z}. \tag{2.7}$$

Hence

$$\frac{1}{b}\left[\frac{(1+\gamma)z^2f''(z)+zf'(z)}{z\big((1+\gamma)zf'(z)-\gamma f(z)\big)}-\frac{1}{z}\right] = \frac{\emptyset(w(z))-1}{z}$$

$$1+\frac{1}{b}\left[\frac{(1+\gamma)z^2f''(z)+zf'(z)}{\big((1+\gamma)zf'(z)-\gamma f(z)\big)}-1\right] = \emptyset(w(z)),$$

then,

$$1+\frac{1}{b}\left(\frac{(1+\gamma)z^2f''(z)+zf'(z)}{(1+\gamma)zf'(z)-\gamma f(z)}-1\right) \prec \phi(z) \tag{2.8}$$

Therefore$f \in \mathcal{L}[b,\gamma](\emptyset)$.
This complete the proof of Theorem (2.1).
Taking $\gamma = 0$ in Theorem (2.1), then we have the following corollary.
Corollary (2.2): A function $f \in \mathcal{L}[b,\gamma](\emptyset)$ for $z \neq 0$ if and only if

$$(f'(z))^{\frac{1}{b}} = \exp\left(\int_0^z \frac{\emptyset(w(z))-1}{\delta}\,d\delta\right), \tag{2.9}$$

where$w(z)$ is analytic in U satisfying $w(0) = 0$ and $|w(z)| < 1$.

In the next theorem, we obtain a necessary and sufficient conditions for a function belong to the class $\mathcal{L}[b,\gamma](\emptyset)$.

Theorem (2.3): Let $\emptyset(z)$ and $F(z)$ be as in Lemma (1.2), the function $f \in \mathcal{L}[b,\gamma](\emptyset)$ if and only if for all $|\delta| \leq 1$ and $|t| \leq 1$,we have

$$\left(\frac{t[(\gamma+1)zsf'(sz)-\gamma f(sz)]}{s[(\gamma+1)ztf'(tz)-\gamma f(tz)]}\right)^{\frac{-1}{b}} \prec \frac{sF(tz)}{tF(sz)} \tag{2.9}$$

Proof: Let $f \in \mathcal{L}[b,\gamma](\emptyset)$. Define the function p by

$$p(z)=\left(\frac{z}{(1+\gamma)zf'(z)-\gamma f(z)}\right)^{-\frac{1}{b}}, \tag{2.11}$$

Differentiating (2.11)both sides with respect to z, we obtain

$$p'(z)=-\frac{1}{b}\left[\frac{(1+\gamma)zf'(z)-\gamma f(z)-z\big((1+\gamma)zf''(z)+(1+\gamma)zf'(z)-\gamma f'(z)\big)}{\big((1+\gamma)zf'(z)-\gamma f(z)\big)^2}\right]\times$$

$$\left(\frac{z}{(1+\gamma)zf'(z)-\gamma f(z)}\right)^{-\frac{1}{b}-1} \tag{2.12}$$

Dividing (2.12) by (2.11) gives

$$\frac{zp'(z)}{p(z)}=\frac{1}{b}\left[\frac{(1+\gamma)zf''(z)+zf'(z)}{(1+\gamma)zf'(z)-\gamma f(z)}-1\right] \tag{2.13}$$

$$1+\frac{zp'(z)}{p(z)}=1+\frac{1}{b}\left[\frac{(1+\gamma)zf''(z)+zf'(z)}{(1+\gamma)zf'(z)-\gamma f(z)}-1\right] \prec \emptyset(z). \tag{2.14}$$

By Definition (1.1) and applying Lemma (1.2), we get that

$$1+\frac{zp'(z)}{p(z)}=1+\frac{1}{b}\left[\frac{(1+\gamma)zf''(z)+zf'(z)}{(1+\gamma)zf'(z)-\gamma f(z)}-1\right] \prec \emptyset(z) \tag{2.15}$$

if and only if for all $|s|\le 1$ and $|t|\le 1$, we have

$$\frac{p(tz)}{p(sz)}=\left(\frac{t[(\gamma+1)zsf'(sz)-\gamma f(sz)]}{s[(\gamma+1)ztf'(tz)-\gamma f(tz)]}\right)^{\frac{-1}{b}} \prec \frac{sF(tz)}{tF(sz)} \tag{2.16}$$

and this complete the proof of the theorem.

If we take $\gamma=0$ in Theorem (2.3). Then, we have the following corollary.

Corollary (2.4) : Let $\emptyset(z)$ and F(z) be as in Lemma (1.2). Then the function $f \in \mathcal{L}[b,\gamma](\emptyset)$ if and only if for all $|s|\le 1$ and $|t|\le 1$, we have

$$\left(\frac{f'(sz)}{f'(tz)}\right)^{\frac{-1}{b}} \prec \frac{sF(tz)}{tF(sz)} \tag{2.17}$$

Theorem (2.5): Let h be a convex in the open unit disk U, with $Re(h(z))>0$, and p is analytic in U with $p(0)=h(0)$, if $f\in\mathcal{A}$ satisfies the subordination

$$\frac{(\gamma+1)z^2f'''(z)+(\gamma+2)zf''(z)}{(\gamma+1)zf''(z)+f'(z)}+1 \prec h(z) \tag{2.18}$$

Then

$$\frac{(\gamma+1)z^2f''(z)+zf'(z)}{(\gamma+1)zf'(z)-\gamma f(z)} \prec h(z) \tag{2.19}$$

Proof: Define the function p by

$$p(z)=\frac{(\gamma+1)z^2f''(z)+zf'(z)}{(\gamma+1)zf'(z)-\gamma f(z)} \tag{2.20}$$

Differentiating (2.1) with respect to z we obtain

$$\frac{zp'(z)}{p(z)}=\frac{(\gamma+1)z^2f'''(z)+(\gamma+2)zf''(z)}{(\gamma+1)zf'(z)+f'(z)}+1-\frac{(\gamma+1)z^2f''(z)+zf'(z)}{(\gamma+1)zf'(z)-\gamma f(z)} \tag{2.21}$$

so

$$p(z)+\frac{zp'(z)}{p(z)}=\frac{(\gamma+1)z^2f'''(z)+(\gamma+2)zf''(z)}{(\gamma+1)zf''(z)+f'(z)}+1 \tag{2.22}$$

Therefore by Lemma (1.1) and if, we setting $k=1, r=0$, we get $p(z)\prec h(z)$ and by using (2.20), we obtain the result.

If we take $\gamma=0$ in the Theorem (2.5), then we have the following corollary:

Corollary (2.6): Let h be a convex in open unit disk U, with $Re\big(h(z)\big)>0$, if p is analytic in U with $p(0)=h(0)$, if $f\in\mathcal{A}$ satisfies the subordination

$$\frac{z^2f'''(z)+2zf''(z)}{zf''(z)+f'(z)}+1\prec h(z).$$

Then

$$\frac{zf''(z)}{f'(z)}+1\prec h(z).$$

Theorem 2 as We give two examples for Theorem (2.5) as follows:

Example(2.7): Let $h(z)=\frac{1+z}{1-z}$, $f\in\mathcal{A}$ satisfies the subordination

$$\frac{(\gamma+1)z^2f'''(z)+(\gamma+2)zf''(z)}{(\gamma+1)zf''(z)+f'(z)}+1\prec\frac{1+z}{1-z}.$$

Then

$$\frac{(\gamma+1)z^2f''(z)+zf'(z)}{(\gamma+1)zf'(z)-\gamma f(z)}\prec\frac{1+z}{1-z}.$$

Example(2.8): Let $h(z)=e^z$, $f\in\mathcal{A}$ satisfies the subordination

$$\frac{z^2 f'''(z) + 2zf''(z)}{zf''(z) + f'(z)} + 1 \prec e^z.$$

Then

$$\frac{z^2 f''(z) + zf'(z)}{zf'(z)} \prec e^z.$$

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