Bull. Pure Appl. Math.
Vol. 1, No. 2 (2007), 113-125

DIFFERENTIAL OPERATOR OF A SUBCLASS OF *k*-UNIFORMLY *p*-VALENT STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY INTEGRAL OPERATOR

WAGGAS GALIB AATSHAN* and S. R. KULKARNI**

*Department of Mathematics,
University of Pune, Pune - 411007, India
E-Mail : waggashnd@yahoo.com

** Department of Mathematics,
Fergusson College, Pune - 411004, India
E-mail : kulkarni−ferg@yahoo.com

(Received : 06.03.07)

Abstract

In the present paper, we introduce the subclass $k - UCV_p^m(\lambda, \gamma, \beta, \delta, q)$ by using the differential operator. The aim of this paper is to study some interesting properties of this subclass of *k*-uniformly starlike and convex functions by using higher order derivatives of Taylor series expansion of some *p*-valent functions with negative coefficients defined by integral operator, we have obtained the necessary and sufficient condition for $f(z)$ to be in the class $k - UCV_p^m(\lambda, \gamma, \beta, \delta, q)$. We have also obtained the results leading to linear combination, quasi-Hadamard products, distortion theorem, integral representation and extreme points.

Keywords : Uniform p-valent starlike, Uniform p-valent convex, Differential operator, Integral operator, Coefficient bounds, Linear combination, Quasi-Hadamard product, Distortion theorem, Integral representation, Extreme points.

2000 AMS Subject Classification : Secondary 30C45.

1. Introduction

Let $S(m, p)$ denote the class of functions $f(z)$ which are analytic and p-valent in the unit disk $U = \{ z \in \mathbb{C} : |z| < 1\}$,

$$f(z) = z^p - \sum_{n=m+p}^{\infty} a_n z^n \quad (a_n \geq 0; m, p \in \mathbb{N}) \quad . \tag{1}$$

A function $f \in S(m, p)$ is said to be in $k - UST^p(\gamma)$, the class of k-uniformly p-valent starlike functions of order γ, $0 \le \gamma < p$, if f satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma \quad , \quad k \ge 0 \quad . \tag{2}$$

Replacing f in (2) by $z f'$ we obtain the condition

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right| + \gamma \quad , \quad k \ge 0 \tag{3}$$

required for the function f to be in the subclass $k - UCV^p(\gamma)$ of k-uniformly p-valent convex functions of order γ.

For $p = 1$, uniformly starlike and convex functions were introduced by Kanas, Wiśniowska [9] and Silverman [10] and then studied by various authors like Goodman [6].

Definition 1 : *Let f (z) be given by* (1), *is said to be a member of the* $k-UCV_p^m$ $(\lambda, \gamma, \beta, \delta, q)$ *if and only if it satisfies the inequality:*

$$\operatorname{Re}\left\{\frac{z(Q_\delta^\beta f(z))^{(1+q)} + \lambda z^2 (Q_\delta^\beta f(z))^{(2+q)}}{(1-\lambda)(Q_\delta^\beta f(z))^{(q)} + \lambda z (Q_\delta^\beta f(z))^{(1+q)}}\right\}$$

$$\ge k\left|\frac{z(Q_\delta^\beta f(z))^{(1+q)} + \lambda z^2 (Q_\delta^\beta f(z))^{(2+q)}}{(1-\lambda)(Q_\delta^\beta f(z))^{(q)} + \lambda z (Q_\delta^\beta f(z))^{(1+q)}} - 1\right| + \gamma \quad , \tag{4}$$

where $0 \le \gamma < p - q$, $p > q, m \in \mathbb{N}$, $q \in N_0 = \{0, 1, 2, \cdots\}$, $0 \le \lambda \le 1$, $k \ge 0$, $\beta \ge 0$, $\delta > -1$, *and* Q_δ^β *is the generalized Jung-Kim-Srivastava integral operator* [8] , *defined by*

$$Q_\delta^\beta f(z) = \frac{\Gamma(\beta+\delta+p)}{z^p \Gamma(\beta p)\Gamma(\delta+p)} \int_0^z t^{\delta-1}\left(1-\frac{t}{z}\right)^{\beta-1} f(t)dt \quad , \quad \beta \ge 0, \delta > -1$$

$$= z^p - \sum_{n=m+p}^{\infty} \frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)} a_n z^n \tag{5}$$

and for $\beta = 0$, *we have* $Q_\delta^0 f(z) = f(z)$. *For each* $f(z) \in S(m, p)$, *we have*

$$(Q_\delta^\beta f(z))^{(j)} = \frac{p!}{(p-j)!} z^{p-j} - \sum_{n=m+p}^{\infty} \frac{n!\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{(n-j)!\Gamma(\delta+p)\Gamma(\beta+\delta+n)} a_n z^{n-j} \quad , \tag{6}$$

where $p > j, m, p \in \mathbb{N}$ and $j \in N_0$.

Now, for $\beta = 0$, we have

$$f^{(j)}(z)=\frac{p!}{(p-j)!}z^{p-j}-\sum_{n=m+p}^{\infty}\frac{n!}{(n-j)!}a_n z^{n-j} \qquad \text{, (cf. [4])} \tag{7}$$

By specializing the parameters $\lambda, k, \gamma, \beta, \delta, q,$ we obtain following different subclasses studied by various authors:

(1) If $k = 0, \beta = 0$, the family $k-UCV_p^m$ $(\lambda, \gamma, \beta, \delta, q)$ reduces to the class SC_p^m (q, λ, γ) which was studied by [7].

(2) If $k = 0, q = 0, \beta = 0$, we obtain the class $S(m, p, \lambda, \gamma)$, which is introduced by [2].

(3) If $k = 0, \lambda = 0, \beta = 0$, we get the classes $S_m(p, q, \gamma)$ and $C_m(p, q, \gamma)$, studied by [5].

(4) If $k = q = 0, \beta = 0, p = 1$, we obtain the class $p(m, \lambda, \gamma)$, studied by [1].

(5) If $\lambda = 0, q = 0, \beta = 0, p = 1$, then the class $k-UCV_1^m$ $(0, \gamma, 0, \delta, 0) \equiv k-UST(\gamma)$, studied by [4].

(6) If $\lambda = 0, q = 0, \gamma = 0, \beta = 0, p = 1$, that is $k-UST$ introduced by [9].

(7) If $\lambda = 1, q = 0, \gamma = 0, \beta = 0, p = 1$, that is $k-UCV$ introduced and studied by [9].

(8) If $\beta = 0$, then the class $k-UCV_p^m$ $(\lambda, \gamma, 0, \delta, q)$ was studied by [3].

In this paper, we first derive the coefficient inequality for the class $k-UCV_p^m$ $(\lambda, \gamma, \beta, \delta, q)$.

2. Coefficient Bounds

Theorem 1 : *The function f(z), defined by (1), be in the class* $k-UCV_p^m$ $(\lambda, \gamma, \beta, \delta, q)$ *if and only if*

$$\sum_{n=m+p}^{\infty}\frac{n!(1+\lambda(n-q-1))}{(n-q)!}\left[(n-q)(k+1)-(k+\gamma)\frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)}\right]a_n$$

$$\leq\frac{p!(p-q-\gamma)}{(p-q)!}(1+\lambda(p-q-1))\,, \tag{8}$$

where $p,m\in\mathbb{N}$ and $q\in N_0$, $n \geq m+p$, $k \geq 0$, $0 \leq \gamma < p-q$, $p > q$, $0 \leq \lambda \leq 1$, $\beta \geq 0$ *and* $\delta > -1$. *The result* (8) *is sharp for the function f(z) given by*

$$f(z) = z^p - \{[p!(m+p-q)!(p-q-\gamma)(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+m+p)z^{m+p}]$$

$$/[(m+p)!(p-q)!(1+\lambda(m+p-q-1))[(m+p-q)(k+1)-(k-\gamma)]\Gamma(\beta+\delta+p)\Gamma(\delta+m+p)]\}$$

($m,p\in\mathbb{N}$; $p>q$; $q\in N_0$; $z\in U$).

Proof : Assume $f(z)\in UCV_p^m$ $(\lambda, \gamma, \beta, \delta, q)$. Then $f(z)$ satisfies the inequality (4), which is equivalent to

$$\operatorname{Re}\left\{\frac{z(Q_\delta^\beta f(z))^{(1+q)}+\lambda z^2(Q_\delta^\beta f(z))^{(2+q)}}{(1-\lambda)(Q_\delta^\beta f(z))^{(q)}+\lambda z(Q_\delta^\beta f(z))^{(1+q)}}(1+ke^{i\theta})-ke^{i\theta}\right\}\geq\gamma \quad ,$$

($0 \le \gamma < p - q,\ p > q,\ k \ge 0;\ 0 \le \lambda \le 1;\ \beta \ge 0,\ \delta > -1$); $p \in \mathbb{N},\ q \in N_0$ and $-\pi < \theta \le \pi$.

or

$$\mathrm{Re}\{[[z(Q_\delta^\beta f(z))^{(1+q)} + \lambda z^2 (Q_\delta^\beta f(z))^{(2+q)}](1+ke^{i\theta}) - ke^{i\theta}(1-\lambda)(Q_\delta^\beta f(z))^{(q)} + \lambda z(Q_\delta^\beta f(z))^{(1+q)}]]$$

$$/[(1-\lambda)(Q_\delta^\beta f(z))^{(q)} + \lambda z(Q_\delta^\beta f(z))^{(1+q)}]\} \ge \gamma \qquad (9)$$

Let

$$x(z) = [z(Q_\delta^\beta f(z))^{(1+q)} + \lambda z^2 (Q_\delta^\beta f(z))^{(2+q)}](1+ke^{i\theta}) - ke^{i\theta}(1-\lambda)(Q_\delta^\beta f(z))^{(q)} + \lambda z(Q_\delta^\beta f(z))^{(1+q)}]$$

$$y(z) = (1-\lambda)(Q_\delta^\beta f(z))^{(q)} + \lambda z(Q_\delta^\beta f(z))^{(1+q)} \quad .$$

Then (9) is equivalent to

$$|x(z) + (1-\gamma)y(z)| \ge |x(z) - (1+\gamma)y(z)| \quad \text{for} \quad 0 \le \gamma < p - q.$$

Now

$$|x(z) + (1-\gamma)y(z)| = \Bigg\| \Bigg[\frac{p!}{(p-q-1)!} z^{p-q} - \sum_{n=m+p}^{\infty} \frac{n!}{(n-q-1)!} \frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)} a_n z^{n-q}$$

$$+ \frac{\lambda\, p!}{(p-q-2)!} z^{p-q} - \sum_{n=m+p}^{\infty} \frac{\lambda\, n!}{(n-q-2)!} \frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)} a_n z^{n-q} \Bigg](1+ke^{i\theta})$$

$$-ke^{i\theta}\Bigg[\frac{(1-\lambda)\, p!}{(p-q)!} - \sum_{n=m+p}^{\infty} \frac{(1-\lambda)\, n!}{(n-q)!} \frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)} a_n z^{n-q}$$

$$+ \frac{\lambda\, p!}{(p-q-1)!} z^{p-q} - \sum_{n=m+p}^{\infty} \frac{\lambda\, n!}{(n-q-1)!} \frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)} a_n z^{n-q} \Bigg]$$

$$+(1-\gamma)\Bigg[\frac{(1-\lambda)\, p!}{(p-q)!} z^{p-q} - \sum_{n=m+p}^{\infty} \frac{(1-\lambda)\, n!}{(n-q)!} \frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)} a_n z^{n-q}$$

$$+ \frac{\lambda\, p!}{(p-q-1)!} z^{p-q} - \sum_{n=m+p}^{\infty} \frac{\lambda\, n!}{(n-q-1)!} \frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)} a_n z^{n-q} \Bigg]\Bigg\|$$

$$\ge \frac{p!}{(p-q)!}(1+\lambda(p-q-1))(p-q+1-\gamma)|z|^{p-q} + \frac{k\, p!}{(p-q)!}[(p-q)(1+\lambda(p-q-2)-1-\lambda]|z|^{p-q}$$

$$- \sum_{n=m+p}^{\infty} \frac{n!}{(n-q)!}(1+\lambda(n-q-1))(n-q+1-\gamma) \frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)} a_n |z|^{n-q}$$

$$- \sum_{n=m+p}^{\infty} \frac{kn!}{(n-q)!}[(n-q)(1+\lambda(n-q-2))-1+\lambda] \frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)} a_n |z|^{n-q} \quad .$$

Also,

$$\left|x(z)+(1\text{-}\gamma)y(z)\right| = \left|\frac{p!}{(p-q-1)!}(1+\lambda(p-q-1))(p-q-1-\gamma)z^{p-q}\right.$$

$$+\frac{ke^{i\theta}p!}{(p-q)!}[(p-q)(1+\lambda(p-q-2))-1-\lambda]z^{p-q}-\sum_{n=m+p}^{\infty}\frac{n!}{(n-q)!}$$

$$\times(1+\lambda(n-q-1-\gamma))\frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)}a_n z^{n-q}-\sum_{n=m+p}^{\infty}\frac{ke^{i\theta}n!}{(n-q)!}$$

$$\left.\times[(n-q)(1+\lambda(n-q-2))-1+\lambda]\frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)}a_n z^{n-q}\right|$$

$$\leq\frac{p!}{(p-q)!}(1+\lambda(p-q-1))(p-q-1-\gamma)|z|^{p-q}+\frac{k\,p!}{(p-q)!}[(p-q)(1+\lambda(p-q-2)-1+\lambda]|z|^{p-q}$$

$$+\sum_{n=m+p}^{\infty}\frac{n!}{(n-q)!}(1+\lambda(n-q-1))(n-q+1-\gamma)\ \frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)}a_n|z|^{n-q}$$

$$+\sum_{n=m+p}^{\infty}\frac{kn!}{(n-q)!}[(n-q)(1+\lambda(n-q-2))-1+\lambda]\ \frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)}a_n|z|^{n-q}\ .$$

Therefore,

$$\left|x(z)+(1\text{-}\gamma)y(z)\right|-\left|x(z)-(1+\gamma)y(z)\right|\ \geq\frac{2p!}{(p-q)!}(1+\lambda(p-q-1))(p-q-\gamma)$$

$$-\sum_{n=m+p}^{\infty}\frac{2n!}{(n-q)!}[(1+\lambda(n-q-1))(n-q-\gamma)+k((n-q)(1+\lambda(n-q-2))-1+\lambda)]$$

$$\times\frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)}a_n\geq 0\quad .$$

Hence,

$$\sum_{n=m+p}^{\infty}\frac{n!(1+\lambda(n-q-1))}{(n-q)!}[(n-q)(k+1)-(k-\gamma)]\ \frac{\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{\Gamma(\delta+p)\Gamma(\beta+\delta+n)}a_n$$

$$\leq\frac{p!(p-q-\gamma)}{(p-q)!}(1+\lambda(p-q-1))$$

Conversely, by considering (6) we must show that

$$\operatorname{Re}\{[[z(Q_\delta^\beta f(z))^{(1+q)}+\lambda z^2(Q_\delta^\beta f(z))^{(2+q)}](1+ke^{i\theta})-(ke^{i\theta}+\gamma)(1-\lambda)(Q_\delta^\beta f(z))^{(q)}+\lambda z(Q_\delta^\beta f(z))^{(1+q)}]]$$

$$/[(1-\lambda)(Q_\delta^\beta f(z))^{(q)}+\lambda z(Q_\delta^\beta f(z))^{(1+q)}]\}\geq 0\quad . \tag{10}$$

Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$, $\mathrm{Re}(-e^{i\theta}) \ge \left|e^{i\theta}\right| = -1$ and letting $r \to 1^{-}$, we conclude to (10) by using (6) in the left hand of (10).

Corollary 1 : *Let f(z) be in* $k-\ UCV_p^m\ (\lambda, \gamma, \beta, \delta, q)$ *, then*

$$a_n \le \frac{p!(n-q)!(1+\lambda(p-q-1)(p-q-\gamma)\Gamma(\delta+p)\Gamma(\beta+\delta+n)}{n!(p-q)!(1+\lambda(n-q-1)[(n-q)(k+1)-(k-\gamma)]\Gamma(\beta+\delta+p)\Gamma(\delta+n)} ,$$

where $m,p \in \mathbb{N}$, $q, \in N_0$ $0 \le \gamma < p-q,\ p > q, k \ge 0, n \ge m+p, 0 \le \lambda \le 1, \beta \ge 0$ *and* $\delta > -1$.

This corollary is believed to be new.

3. Linear Combination

In the following theorem we prove linear combination for the class $k-\ UCV_p^m\ (\lambda, \gamma, \beta, \delta, q)$,.

Theorem 2 : *Let* $f_i(z)(i \in I = \{1,2,\ldots,t\})$ *defined by*

$$f_i(z) = z^p - \sum_{n=m+p}^{\infty} a_{n,i} z^n \qquad (a_{n,i} \ge 0 ,, p, m \in \mathbb{N}) \tag{11}$$

be in the class $k-\ UCV_p^m\ (\lambda, \gamma, \beta, \delta, q)$ *. Then the function g(z) defined by*

$$g(z) = \sum_{i=1}^{t} X_i f_i(z)$$

(*where* $X_i \ge 0$ *and* $\sum_{i=1}^{t} X_i = 1$*) is also in the class* $k-\ UCV_p^m\ (\lambda, \gamma, \beta, \delta, q)$.

Proof : By (11) we have

$$g(z) = \sum_{i=1}^{t} X_i f_i(z) = z^p - \sum_{n=m+p}^{\infty}\left[\sum_{i=1}^{t} X_i a_{n,i}\right] z^n = z^p - \sum_{n=m+p}^{\infty} d_n z^n ,$$

where $d_n = \sum_{i=1}^{t} X_i a_{n,i}$. By using Theorem 1, we have

$$\sum_{n=m+p}^{\infty} \frac{n!(1+\lambda(n-q-1)[(n-q)(k+1)-(k-\gamma)]\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{(n-q)!\Gamma(\delta+p)\Gamma(\beta+\delta+p)} d_n$$

$$= \sum_{i=1}^{t} X_i \left\{ \sum_{n=m+p}^{\infty} \frac{n!(1+\lambda(n-q-1)[(n-q)(k+1)-(k-\gamma)]\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{(n-q)!\Gamma(\delta+p)\Gamma(\beta+\delta+p)} a_{n,i} \right\}$$

$$= \sum_{i=1}^{t} X_i \left[\frac{p!(1+\lambda(p-q-1))(p-q-\gamma)}{(p-q)!} \right] \le \frac{p!(1+\lambda(p-q-1))(p-q-\gamma)}{(p-q)!} \quad .$$

Thus, this proves that $g(z)$ is in the class $k-UCV_p^m$ (λ ($\lambda, \gamma, \beta, \delta, q$).

4. Quasi-Hadamard Product

Let $f_i(z)(i \in I)$ defined by $f_i(z) = z^p - \sum_{n=m+p}^{\infty} a_{n,i} z^n \, (i \in I, m, p \in \mathbb{N})$, be in the class $k-UCV_p^m$ ($\lambda, \gamma, \beta, \delta, q$), then the Quasi-Hadamard product of the functions $(f_1, f_2, \ldots, f_t)$ denoted by $(f_1 * f_2 * \ldots * f_t)(z)$, defined by

$$(f_1 * f_2 * \ldots * f_t)(z) = z^p - \sum_{n=m+p}^{\infty} (a_{n,1}, a_{n,2}, \ldots a_{n,t}) z^n \quad .$$

Theorem 3 : *Let* $f_i(z) \in k-UCV_p^m(\lambda, \gamma_i, \beta, \delta, q)(i \in I)$. *Then*

$$(f_1 * f_2 * \ldots * f_t)(z) \in k-UCV_p^m(\lambda, \alpha, \beta, \delta, q) \quad ,$$

where

$$0 < \alpha \le p\text{-}q\text{-}\frac{m+k(n\text{-}q\text{-}1)}{H(m+p,t)} \tag{12}$$

and

$$H(m+p,t) = \left[\frac{(p-q)!(m+p)!(1+\lambda(m+p-q-1))\Gamma(\beta+\delta+p)\Gamma(\delta+m+p)}{p!(m+p-q)!(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+m+p)} \right]^{t-1}$$

$$\times \prod_{i=1}^{t} \left[\frac{(m+p-q)(k+1)-(k+\gamma_i)}{p-q-\gamma_i} \right] - 1 \quad ,$$

($p > q$ *;* $m,p \in \mathbb{N}$, $q \in N_0$, $i \in I; 0 \le \gamma_i < p-q$). *The result is sharp.*

Proof : This proof follows by induction on t.

We note that the assertion of the theorem holds true when t = 1 (where $\alpha = \gamma_1$). For t = 2, the coefficient inequality (8) gives us

$$\sum_{n=m+p}^{\infty} \frac{(p-q)!n!(1+\lambda(n-q-1)[(n-q)(k+1)-(k-\lambda_i)]\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{p!(n-q)!(p-q-\gamma_i)(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+n)} a_{n,i} \le 1 \tag{13}$$

(i = 1, 2; $m,p \in \mathbb{N}$), which (13) is equivalent to

$$\sum_{n=m+p}^{\infty}\frac{(p-q)!n!(1+\lambda(n-q-1)\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{p!(n-q)!(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+n)}\sqrt{\prod_{i=1}^{2}\frac{(n-q)(k+1)-(k+\gamma_i)}{(p-q-\gamma_i)}a_{n,i}}\le 1 \quad .$$

We have only to find the largest α such that

$$\sum_{n=m+p}^{\infty}\frac{(p-q)!n!(1+\lambda(n-q-1)[(n-q)(k+1)-(k+\alpha)]\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{p!(n-q)!(p-q-\alpha)(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+n)}a_{n,1}a_{n,2}\le 1$$

($n \ge m+p$; $m,p \in \mathbb{N}$) such that

$$\frac{(n-q)(1+k)-(k-\alpha)}{p-q-\alpha}\sqrt{a_{n,1}a_{n,2}}\le\sqrt{\prod_{i=1}^{2}\frac{(n-q)(k+1)-(k+\gamma_i)}{(p-q-\gamma_i)}} \quad .$$

Consequently, we have to find α such that

$$\frac{(n-q)(1+k)-(k-\alpha)}{(p-q-\alpha)}\le\frac{(p-q)!n!(1+\lambda(n-q-1))\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{p!(n-q)!(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+n)}\times\prod_{i=1}^{2}\left[\frac{(n-q)(k+1)-(k+\gamma_i)}{p-q-\gamma_i}\right] \quad .$$

which is equivalent to

$$0\le\alpha\le\alpha(n)=p-q-\frac{m+k(n-q-1)}{\frac{(p-q)!n!(1+\lambda(n-q-1))\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{p!(n-q)!(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+n)}\prod_{i=1}^{2}\left[\frac{(n-q)(1+k)-(k+\gamma_i)}{p-q-\gamma_i}\right]-1} \tag{14}$$

($n \ge m+p$; $m,p \in \mathbb{N}$) .

So, for $n \ge m+p$ we have $\alpha'(n)\ge 0(\alpha'(n)$ denotes the derivative of $\alpha(n)$), we can put $n=m+p$ in (14), we get

$$0\le\alpha\le\alpha(m+p)\le p-q-\frac{m+k(m+p-q-1)}{\frac{(p-q)!(m+p)!(1+\lambda(m+p-q-1))\Gamma(\beta+\delta+p)\Gamma(\delta+m+p)}{p!(m+p-q)!(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+m+p)}\prod_{i=1}^{2}\left[\frac{(m+p-q)(1+k)-(k+\gamma_i)}{p-q-\gamma_i}\right]-1} \; .$$

Therefore, the result is true for $t = 2$.

Now, assume the result is true for fixed natural number t. We must show that the result is true for t + 1 natural number, that is

$$(f_1 * f_2 * \ldots * f_t)(z)\in k-UCV_p^m(\lambda,\mathcal{E},\beta,\delta,q) \quad ,$$

where $\mathcal{E}$ satisfies the condition

$$0<\mathcal{E}\le p-q-\frac{m+k(m+p\text{-}q\text{-}1)}{D} \quad , \tag{15}$$

where

$$D = \frac{(m+p)!(p-q)!(1+\lambda(m+p-q-1))\Gamma(\beta+\delta+p)\Gamma(\delta+m+p)}{p!(m+p-q)!(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+m+p)}$$

$$\times \frac{[(m+p-q)(1+k)-(k+\alpha)][(m+p-q)(1+k)-(k+\gamma_{t+1})]}{(p-q-\alpha)(p-q-\gamma_{t+1})} - 1$$

and α given by (12) then from (15), we have

$$0 < \mathcal{E} \le p - q - \frac{m+k(m+p\text{-}q\text{-}1)}{H(m+p,t+1)} \quad , \quad (p > q; k \ge 0; m, p, t \in \mathbb{N};\ q \in N_0)$$

Therefore, the result is true for $t + 1$, then the result is true for any positive integer.

Now, from (15), we can take the Hadamard product of the functions $f_i(z)(i \in I)$, that is

$$(f_1 * f_2 * \ldots * f_t)(z) = z^p - H_{m+p} z^{m+p} \quad ,$$

where

$$H_{m+p} = \prod_{i=1}^{t} \{[p!(m+p-q)!(p-q-\gamma_i)(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+m+p)$$

$$/[(m+p)!(p-q)!(1+\lambda(p+m-q-1))\ [(m+p-q)(1+k)-(k+\gamma_i)]\Gamma(\beta+\delta+p)\Gamma(\delta+m+p)]\},$$

$(0 \le \gamma_i < p-q\ ;\ p > q\ ; t, m, p \in \mathbb{N}$, q, $\in N_0\ i \in I; 0 \le \lambda \le 1, \beta \ge 0$ and $\delta > -1$).

Then, we have

$$\sum_{n=m+p}^{\infty} [[n!(p-q)!(1+\lambda(n-q-1))[(n-q)(1+k)-(k+\alpha)]\Gamma(\beta+\delta+p)\Gamma(\delta+n)]$$

$$/[p!n-q)!(p-q-\alpha)(1+\lambda(p-q-1))\ \Gamma(\delta+p)\Gamma(\beta+\delta+n)]]H_n,$$

$$[[(m+p)!(p-q)!(1+\lambda(m+p-q-1))[(m+p-q)(1+k)-(k+\alpha)]\Gamma(\beta+\delta+p)\Gamma(\delta+m+p)]$$

$$/[p!(m+p-q)!(p-q-\alpha)(1+\lambda(p-q-1))\ \Gamma(\delta+p)\Gamma(\beta+\delta+m+p)]]H_{m+p} = 1,$$

Thus, the result is sharp for the functions $f_i(z)(i \in I)$, given by

$$f_i(z) = [p!(m+p-q)!(1+\lambda(p-q-1))(p-q-\gamma)\Gamma(\delta+p)\Gamma(\beta+\delta+m+p)z^{m+p}]$$

$$/[(m+p)!(p-q)!(1+\lambda(p-q-1))\ [(m+p-q)(1+k)-(k-\gamma)]\Gamma(\beta+\delta+p)\Gamma(\delta+m+p)]$$

Remark : For $k = 0$ and $\beta = 0$, we get a special case which studied by Irmak et al. [7].

5. Distortion Theorem

Theorem 4: *Assume that* $f \in k-\ UCV_p^m(\lambda,\gamma,\beta,\delta,q)$*, then*

$$\left|f^{(j)}(z)\right| \le [\frac{p!}{(p-j)} + \{p!(m+p-q)!(p-q-\gamma)(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+m+p)|z|^m]$$

$$/[(m+p-j)!(p-q)!(1+\lambda(m+p-q-1))\ [(m+p-q)(1+k)-(k-\gamma)]\Gamma(\beta+\delta+p)\Gamma(\delta+m+p)]\}]|z|^{p-j}$$

$$\left|f^{(j)}(z)\right| \ge [\frac{p!}{(p-j)!} - \{p!(m+p-q)!(p-q-\gamma)(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+m+p)|z|^m]$$

$$/[(m+p-j)!(p-q)!(1+\lambda(m+p-q-1))\ [(m+p-q)(1+k)-(k-\gamma)]\Gamma(\beta+\delta+p)\Gamma(\delta+m+p)]\}]|z|^{p-j}$$

$m,p \in \mathbb{N},\ p > \max\{q, j\},\ q \in N_0, j \in N_0, z \in U$.

$$f(z) = z^p - \{p!(m+p-q)!(p-q-\gamma)(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+m+p)z^{m+p}]$$

$$/[(m+p)!(p-q)!(1+\lambda(m+p-q-1))\ [(m+p-q)(1+k)-(k-\gamma)]\Gamma(\beta+\delta+p)\Gamma(\delta+m+p)]] .$$

The result is sharp for the function f(z) given by

Proof : $f \in k-\ UCV_p^m(\lambda,\gamma,\beta,\delta,q)$, then by Theorem 1 we can write

$$\sum_{n=m+p}^{\infty} n!a_n \le [(m+p-q)!\,p!(p-q-\gamma)(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+m+p)]$$

$$/[(p-q)!(1+\lambda(m+p-q-1))[(m+p-q)(1+k)-(k-\gamma)]\Gamma(\beta+\delta+p)\Gamma(\delta+m+p)] .$$

for $|z| < 1$, we have

$$\left|f^{(i)}(z)\right| \le \left[\frac{p!}{(p-j)!} + \sum_{n=m+p}^{\infty}\frac{n!}{(n-j)!}a_n|z|^m\right]|z|^{p-j}$$

$$\le [\frac{p!}{(p-j)!} + \{[(m+p-j)!\,p!(p-q-\gamma)(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+m+p)|z|^m]$$

$$/[(m+p-j)!(p-q)!(1+\lambda(m+p-q-1))\ [(m+p-q)(1+k)-(k-\gamma)]\Gamma(\beta+\delta+p)\Gamma(\delta+m+p)]\}]|z|^{p-j}$$

In a similar way we can get the left hand side.

6. Integral Representation

Theorem 5 : *Let* $f \in k-\ UCV_p^m(\lambda,\gamma,\beta,\delta,q)$*, then*

$$(Q_\delta^\beta f(z))^{(q)} = \exp\left(\int_0^z \frac{k-\psi(t)\gamma}{t(k-\psi(t))}dt\right), \quad |\psi(z)| < 1, z \in U \quad .$$

Also we have

$$(Q_\delta^\beta f(z))^{(q)} = z\exp\left(\int_X \log(k-xz)^{-(1+\gamma)}\, d\mu(x)\right),$$

where μ(x) is the probability measure on X = {*x* : |*x*| = 1}.

Proof : The case $k = 0$ is obvious. Let $k \neq 0$, for $f \in k-\ UCV_p^m(0,\gamma,\beta,\delta,q)$ and $\omega = \frac{z Q_\delta^\beta f(z))^{(1+q)}}{z Q_\delta^\beta f(z))^{(q)}}$ we have $\operatorname{Re}\omega > k|\omega-1|+\gamma$. Therefore, $\left|\frac{\omega-1}{\omega-\gamma}\right| < \frac{1}{k}$ or equivalently $\frac{\omega-1}{\omega-\gamma} = \frac{\psi(z)}{k}$ where $|\psi(z)|<1, z\in U$. This yields

$$\frac{(Q_\delta^\beta f(z))^{(1+q)}}{(Q_\delta^\beta f(z))^{(q)}} = \frac{k-\psi(z)\gamma}{z(k-\psi(z))}.$$

So we obtain the first representation. For second, let us set $X = \{x : |x| = 1\}$. Then we have $\frac{\omega-1}{\omega-\gamma} = \frac{1}{k}xz$,, $x \in X$ and we obtain

$$\frac{(Q_\delta^\beta f(z))^{(1+q)}}{(Q_\delta^\beta f(z))^{(q)}} = \frac{k-\psi\gamma xz}{z(k-xz)} \to \log\frac{(Q_\delta^\beta f(z))^{(q)}}{z} = -(1+\gamma)\log(k-xz).$$

If $\mu(x)$ is the probability measure on X, then

$$(Q_\delta^\beta f(z))^{(q)} = z\exp\left(\int_X \log(k-xz)^{-(1+\gamma)}\, d\mu(x)\right),$$

which is the required result.

7. Extreme Points

In the following theorem, we obtain extreme points for $k-\ UCV_p^m(\lambda,\gamma,\beta,\delta,q)$.

Theorem 6 : *Let* $f_1(z) = z^p$ *and*

$$f_n(z) = z^p - \frac{p!(n-q)!(p-q-\gamma)(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+n)}{n!(p-q)!(1+\lambda(n-q-1))[(n-q)(k+1)-(k-\gamma)\Gamma(\beta+\delta+p)\Gamma(\delta+n)} z^n$$

where ($n \ge m+p$; $m,p \in \mathbb{N}$; $q \in N_0$; $0 \le \gamma < p-q$; $k \ge 0$; $0 \le \lambda \le 1$, $\beta \ge 0$; $\delta \ge -1$ *and* $z \in$ *U). Then f (z) is in the class* $k-\ UCV_p^m(\lambda,\gamma,\beta,\delta,q)$ *, if and only if it can be expressed in the form*

$$f(z) = \sigma_1 f_1(z) + \sum_{n=m+p}^{\infty}\sigma_n f_n(z)$$

where $(\sigma_1 + \sum_{n=m+p}^{\infty}\sigma_n = 1, (\sigma_1 \ge 0, p, m \in \mathbb{N}))$.

Proof : Let $f(z)=\sigma_1 f_1(z)+\sum_{n=m+p}^{\infty}\sigma_n f_n(z)$, where ($\sigma_1 \geq 0, \sigma_n \geq 0$ and $\sigma_1+\sum_{n=m+p}^{\infty}\sigma_n=1$).

Therefore,

$$f(z)=\sigma_1 z^p+\sum_{n=m+p}^{\infty}\Big[\sigma_n z^p -$$

$$\frac{p!(n-q)!(p-q-\gamma)(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+n)\sigma_n}{n!(p-q)!(1+\lambda(n-q-1))[(n-q)(k+1)-(k+\gamma)\Gamma(\beta+\delta+p)\Gamma(\delta+n)}z^n\Big]=z^p-\sum_{n=m+p}^{\infty}y_n z^n ,$$

where

$$y_n=\frac{p!(n-q)!(p-q-\gamma)(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+n)\sigma_n}{n!(p-q)!(1+\lambda(n-q-1))[(n-q)(k+1)-(k+\gamma)\Gamma(\beta+\delta+p)\Gamma(\delta+n)},$$

Since

$$\sum_{n=m+p}^{\infty}\frac{n!(p-q)!(1+\lambda(n-q-1))[(n-q)(k+1)-(k-\gamma)]\Gamma(\beta+\delta+p)\Gamma(\delta+n)\sigma_n}{p!(n-q)!(p-q-\gamma)(1+\lambda(p-q-1))\Gamma(\delta+p)\Gamma(\beta+\delta+n)}y_n ,$$

$$=\sum_{n=m+p}^{\infty}\sigma_n=1-\sigma_1\leq 1. \quad \text{(by Theorem 1)}$$

therefore, $f\in k-UCV_p^m(\lambda,\gamma,\beta,\delta,q)$,

Conversely, suppose that $f(z)$ of the form (1) belongs to $k-UCV_p^m(\lambda,\gamma,\beta,\delta,q)$, then by (8), we have

$$a_n\leq\frac{p!(n-q)!(1+\lambda(p-q-1))(p-q-\gamma)\Gamma(\delta+p)\Gamma(\beta+\delta+n)}{n!(p-q)!(1+\lambda(n-q-1))[(n-q)(k+1)-(k+\gamma)]\Gamma(\beta+\delta+p)\Gamma(\delta+n)},$$

($n\geq m+p$; $m, p\in\mathbb{N}$, $q\in N_0$, $p>q$).

Setting

$$\sigma_n=\frac{n!(p-q)!(1+\lambda(n-q-1))[(n-q)(k+1)-(k+\gamma)]\Gamma(\beta+\delta+p)\Gamma(\delta+n)}{p!(n-q)!(1+\lambda(p-q-1))(p-q-\gamma)\Gamma(\delta+p)\Gamma(\beta+\delta+n)}a_n$$

and $\sigma_1=1-\sum_{n=m+p}^{\infty}\sigma_n$. Then

Then $f(z)=z^p-\sum_{n=m+p}^{\infty}a_n z^n=z^p$

$$-\sum_{n=m+p}^{\infty}\frac{p!(n-q)!(1+\lambda(p-q-1))(p-q-\gamma)\Gamma(\delta+p)\Gamma(\beta+\delta+n)\sigma_n}{n!(p-q)!(1+\lambda(n-q-1))[(n-q)(k+1)-(k+\gamma)]\Gamma(\beta+\delta+p)\Gamma(\delta+n)}z^n$$

$$= z^p - \sum_{n=m+p}^{\infty} \sigma_n (z^p - f_n(z))$$

$$= (1 - \sum_{n=m+p}^{\infty} \sigma_n) z^p + \sum_{n=m+p}^{\infty} \sigma_n f_n(z)$$

$$= \sigma_n f_1(z) + \sum_{n=m+p}^{\infty} \sigma_n f_n(z) \ .$$

This completes the proof of the theorem.

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