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A NEW CLASS OF MULTIVALENT HARMONIC FUNCTIONS ASSOCIATED A LINEAR OPERATOR

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Abstract. New class of multivalent harmonic functions are introduced. Furthermore, we determine coefficient bounds, extreme points, closure theorem, convolution condition, integral operator and other property for the functions in this class.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h+\overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co- analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense - preserving in D is that $|h'(z)| > |g'(z)|$ in D, see Clunie and Sheil - Small [2].

Denote by $H(p)$ the class of functions $f = h + \overline{g}$ that are harmonic multivalent and sense - preserving in the unit disk $U = \{z : |z| < 1\}$. The class $H(p)$ was studied by Ahuja and Jahangiri [1] and the class $H(p)$ for $p = 1$ was defined and studied by Jahangiri et.al in [3].

For $f = h + \overline{g}$, we may express the analytic functions h and g as:

$$h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,\ g(z) = \sum_{n=p}^{\infty} b_n z^n,\ |b_p| < 1. \tag{1.1}$$

Let $\mathcal{A}(p)$ denote the subclass of $H(p)$ consisting of functions $f = h + \overline{g}$, where h and g are given by

$$h(z) = z^p - \sum_{n=p+1}^{\infty} |a_n| z^n,\ g(z) = -\sum_{n=p}^{\infty} |b_n| z^n,\ |b_p| < 1. \tag{1.2}$$

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Now, we define a new class $N_p(\beta,\alpha)$ of harmonic functions of the form (1.1) that satisfy the inequality

$$Re\left\{\beta+\frac{(1-\beta)z^{p-1}+\beta z(D_p(\lambda,q,\eta)f(z))''}{(D_p(\lambda,q,\eta)f(z))'}\right\}>(\beta p^2+(1-\beta))\alpha, \tag{1.3}$$

where $0\le\alpha<\frac{1}{p}, p\in\mathbb{N}=\{1,2,\cdots\}, 0\le\beta\le 1, \eta\ge 0, \lambda\ge 0, q\ge 0$ and

$$D_p(\lambda,q,\eta)f(z)=D_p(\lambda,q,\eta)h(z)+\overline{D_p(\lambda,q,\eta)g(z)}. \tag{1.4}$$

The operator $D_p(\lambda,q,\eta)$ denotes the linear operator introduced in [5]. For h and g given by (1.1) we obtain

$$D_p(\lambda,q,\eta)h(z)=z^p+\sum_{n=p+1}^{\infty}\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}a_nz^n, \tag{1.5}$$

$$D_p(\lambda,q,\eta)g(z)=\sum_{n=p}^{\infty}\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}b_nz^n, \tag{1.6}$$

where $p\in\mathbb{N}=\{1,2,\cdots\}, \lambda\ge 0, q\ge 0, \eta\ge 0$.

We further denote by $\mathcal{A}_p(\beta,\alpha)$ the subclass of $N_p(\beta,\alpha)$ that satisfies the relation

$$\mathcal{A}_p(\beta,\alpha)=\mathcal{A}(p)\cap N_p(\beta,\alpha). \tag{1.7}$$

2. Coefficient Bounds

Theorem 2.1. *Let $f=h+\overline{g}$ (h and g being given by (1.1)). If*

$$\sum_{n=p+1}^{\infty}n[(\beta p^2+(1-\beta))\alpha-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|a_n|$$

$$+\sum_{n=p}^{\infty}n[\beta p^2+(1-\beta))\alpha-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|b_n|\le p, \tag{2.1}$$

where $0\le\alpha<\frac{1}{p}, p\in\mathbb{N}=\{1,2,\cdots\}, 0\le\beta\le 1, \eta\ge 0, \lambda\ge 0, q\ge 0$, then f is harmonic p- valent sense - preserving in U and $f\in N_p(\beta,\alpha)$.

Proof. Let

$$w(z)=\left\{\beta+\frac{(1-\beta)z^{p-1}+\beta z(D_p(\lambda,q,\eta)f(z))''}{(D_p(\lambda,q,\eta)f(z))'}\right\}.$$

Using the fact that $Re\{w(z)\}>(\beta p^2+(1-\beta))\alpha$ if and only if

$$|w(z)-(1+(\beta p^2+(1-\beta))\alpha)|\le|w(z)+(1-(\beta p^2+(1-\beta))\alpha)|, \tag{2.2}$$

it is suffices to show the inequality (2.2).

Substituting for w and making use of (1.4) to (1.6), and resorting to simple calculation, we find that

$$|w(z)-(1+(\beta p^2+(1-\beta))\alpha)|\le[(\beta p^2+(1-\beta))-p-(\beta p^2+(1-\beta))\alpha p]$$

$$+\sum_{n=p+1}^{\infty} n[1+(\beta p^2+(1-\beta))\alpha-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|a_n||z^{n-p}|$$

$$+\sum_{n=p+1}^{\infty} n[1+(\beta p^2+(1-\beta))\alpha-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|b_n||z^{n-p}| \tag{2.3}$$

and

$$|w(z)+(1-(\beta p^2+(1-\beta))\alpha)| \geq [(\beta p^2+(1-\beta))+p-(\beta p^2+(1-\beta))\alpha p]$$

$$-\sum_{n=p+1}^{\infty} n[(\beta p^2+(1-\beta))\alpha-1-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|a_n||z^{n-p}|$$

$$-\sum_{n=p+1}^{\infty} n[(\beta p^2+(1-\beta))\alpha-1-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|b_n||z^{n-p}| \tag{2.4}$$

Evidently (2.3) and (2.4) in conjunction with (2.1) yields

$$|w(z)-(1+(\beta p^2+(1-\beta))\alpha)|-|w(z)+(1-(\beta p^2+(1-\beta))\alpha)| \leq 0.$$

The harmonic functions

$$f(z)=z^p+\sum_{n=p+1}^{\infty}\frac{x_n}{n[(\beta p^2+(1-\beta))\alpha-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}}z^n$$

$$+\sum_{n=p}^{\infty}\frac{\overline{y}_n}{n[(\beta p^2+(1-\beta))\alpha-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}}(\overline{z})^n \tag{2.5}$$

where

$$\left(\sum_{n=p+1}^{\infty}|x_n|+\sum_{n=p}^{\infty}|\overline{y}_n|=p\right),$$

show that the coefficients bounds given by (2.1) is sharp.

The functions of the form (2.5) are in $N_p(\beta,\alpha)$ because in view of (2.5), we infer that

$$\sum_{n=p+1}^{\infty} n[(\beta p^2+(1-\beta))\alpha-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|a_n|$$

$$+\sum_{n=p}^{\infty} n[(\beta p^2+(1-\beta))\alpha-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|b_n|$$

$$\sum_{n=p+1}^{\infty}|x_n|+\sum_{n=p}^{\infty}|\overline{y}_n|=p.$$

The restriction placed in Theorem 2.1 on the moduli of coefficients of $f=h+\overline{g}$ implies that for arbitrary rotation of the coefficients of f, the resulting functions would still be harmonic multivalent and $f\in N_p(\beta,\alpha)$. □

The following theorem shows that the condition (2.1) is also necessary for function f to belong to $\mathcal{A}_p(\beta,\alpha)$.

Theorem 2.2. *Let $f = h + \overline{g}$ with h and g are given by (1.2). Then $f \in \mathcal{A}_p(\beta,\alpha)$ if and only if*

$$\sum_{n=p+1}^{\infty} n[(\beta p^2 + (1-\beta))\alpha - n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|a_n|$$
$$+\sum_{n=p}^{\infty} n[(\beta p^2 + (1-\beta))\alpha - n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|b_n| \le p \tag{2.6}$$

where $0 \le \alpha < \frac{1}{p}, p \in \mathbb{N} = \{1,2,\cdots\}, 0 \le \beta \le 1, \eta \ge 0, \lambda \ge 0, q \ge 0.$

Proof. By noting that $\mathcal{A}_p(\beta,\alpha) \subset N_p(\beta,\alpha)$, the sufficiency part of Theorem 2.2 follows at once from Theorem 2.1. To prove the necessary part, let us assume that $f \in \mathcal{A}_p(\beta,\alpha)$. Using (1.3), we get

$$Re\left\{\beta + \frac{(1-\beta)z^{p-1} + \beta z(D_p(\lambda,q,\eta)h(z))'' + \overline{\beta z(D_p(\lambda,q,\eta)g(z))''}}{(D_p(\lambda,q,\eta)h(z))' + \overline{(D_p(\lambda,q,\eta)g(z))'}}\right\}$$

$$= Re\left\{\frac{(\beta p^2+(1-\beta)) - \sum\limits_{n=p+1}^{\infty} n^2\beta\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|a_n|z^{n-p} - \sum\limits_{n=p}^{\infty} n^2\beta\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|b_n|(\overline{z})^{n-p}}{p - \sum\limits_{n=p+1}^{\infty} n\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|a_n|z^{n-p} - \sum\limits_{n=p}^{\infty} n\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|b_n|(\overline{z})^{n-p}}\right\}$$

$> (\beta p^2 + (1-\beta))\alpha.$

If we choose z to be real and let $z \to 1^-$, we obtain the condition (2.6) which completes the proof of Theorem 2.2. □

3. Extreme Points

Next, we determine the extreme points of the closed convex hull of $\mathcal{A}_p(\beta,\alpha)$, denoted by clco $\mathcal{A}_p(\beta,\alpha)$.

Theorem 3.1. *$f \in clco\ \mathcal{A}_p(\beta,\alpha)$ if and only if*

$$f(z) = \sum_{n=p}^{\infty}(\mu_n h_n + \delta_n g_n), \tag{3.1}$$

where $z \in U, h_p(z) = z^p,$

$$h_n(z) = z^p - \frac{p}{n[(\beta p^2+(1-\beta))\alpha - n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}}z^n, \tag{3.2}$$

$(n = p+1, p+2, \cdots)$

$$g_n(z) = z^p - \frac{p}{n[(\beta p^2+(1-\beta))\alpha - n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}}(\overline{z})^n, \tag{3.3}$$

$(n = p, p+1, \cdots)$ *and*

$$\sum_{n=p}^{\infty}(\mu_n + \delta_n) = 1, \ (\mu_n \geq 0, \delta_n \geq 0).$$

In particular, the extreme points of $\mathcal{A}_p(\beta, \alpha)$ *are* $\{h_n\}$ *and* $\{g_n\}$.

Proof. Suppose f is of the form (3.1). Using (3.2) and (3.3) , we get

$$\begin{aligned} f(z) &= \sum_{n=p}^{\infty}(\mu_n h_n + \delta_n g_n) \\ &= \sum_{n=p}^{\infty}(\mu_n + \delta_n)z^n - \sum_{n=p+1}^{\infty} \frac{p}{n[(\beta p^2 + (1-\beta))\alpha - n\beta]\left[1 + \frac{(n-p)\lambda}{p+q}\right]^{\eta}}\mu_n z^n \\ &\quad - \sum_{n=p}^{\infty} \frac{p}{n[(\beta p^2 + (1-\beta))\alpha - n\beta]\left[1 + \frac{(n-p)\lambda}{p+q}\right]^{\eta}}\delta_n (\overline{z})^n \\ &= z^p - \sum_{n=p+1}^{\infty} \frac{p}{n[(\beta p^2 + (1-\beta))\alpha - n\beta]\left[1 + \frac{(n-p)\lambda}{p+q}\right]^{\eta}}\mu_n z^n \\ &\quad - \sum_{n=p}^{\infty} \frac{p}{n[(\beta p^2 + (1-\beta))\alpha - n\beta]\left[1 + \frac{(n-p)\lambda}{p+q}\right]^{\eta}}\delta_n (\overline{z})^n. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=p+1}^{\infty} n[(\beta p^2 + (1-\beta))\alpha - n\beta]\left[1 + \frac{(n-p)\lambda}{p+q}\right]^{\eta} \times \\ &\frac{p}{n[(\beta p^2 + (1-\beta))\alpha - n\beta]\left[1 + \frac{(n-p)\lambda}{p+q}\right]^{\eta}}\mu_n \\ &+ \sum_{n=p}^{\infty} n[(\beta p^2 + (1-\beta))\alpha - n\beta]\left[1 + \frac{(n-p)\lambda}{p+q}\right]^{\eta} \times \\ &\frac{p}{n[(\beta p^2 + (1-\beta))\alpha - n\beta]\left[1 + \frac{(n-p)\lambda}{p+q}\right]^{\eta}}\delta_n \\ = \ & p\left(\sum_{n=p}^{\infty}(\mu_n + \delta_n) - \mu_p\right) = p(1 - \mu_p) \leq p \end{aligned}$$

which implies that $f \in clco\ \mathcal{A}_p(\beta, \alpha)$.

Conversely, assume that $f \in \mathcal{A}_p(\beta, \alpha)$. Putting

$$\begin{aligned} \mu_n &= \left(\frac{n}{p}\right)[(\beta p^2 + (1-\beta))\alpha - n\beta]\left[1 + \frac{(n-p)\lambda}{p+q}\right]^{\eta}|a_n|, (n = p+1, p+2, \cdots), \\ \delta_n &= \left(\frac{n}{p}\right)[(\beta p^2 + (1-\beta))\alpha - n\beta]\left[1 + \frac{(n-p)\lambda}{p+q}\right]^{\eta}|b_n|, (n = p, p+1, \cdots), \end{aligned}$$

we get

$$f(z)=\sum_{n=p}^{\infty}(\mu_n h_n+\delta_n g_n)$$

and this completes the proof of Theorem 3.1. □

4. Closure Theorem

Theorem 4.1. *The class $\mathcal{A}_p(\beta,\alpha)$ is a convex set.*

Proof. Let the function $f_{n,j}$ $(j=1,2)$ defined by

$$f_{n,j}(z)=z^p-\sum_{n=p+1}^{\infty}|a_{n,j}|z^n-\overline{\sum_{n=p}^{\infty}|b_{n,j}|z^n}$$

be in the class $\mathcal{A}_p(\beta,\alpha)$.

It is sufficient to prove that the function

$$H(z)=\gamma f_{n,1}(z)+(1-\gamma)f_{n,2}(z),\ (0\leq\gamma<1),$$

is also in the class $\mathcal{A}_p(\beta,\alpha)$. Since for $0\leq\gamma<1$,

$$\begin{aligned}H(z) &= z^p-\sum_{n=p+1}^{\infty}(\gamma|a_{n,1}|+(1-\gamma)|a_{n,2}|)z^n\\ &\quad -\sum_{n=p}^{\infty}(\gamma|b_{n,1}|+(1-\gamma)|b_{n,2}|)(\overline{z})^n\end{aligned}$$

with the aid of Theorem 2.2, we have

$$\begin{aligned}&\sum_{n=p+1}^{\infty}n[(\beta p^2+(1-\beta))\alpha-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}(\gamma|a_{n,1}|+(1-\gamma)|a_{n,2}|)\\ &+\sum_{n=p}^{\infty}n[(\beta p^2+(1-\beta))\alpha-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}(\gamma|b_{n,1}|+(1-\gamma)|b_{n,2}|)\\ =\ &\gamma\left[\sum_{n=p+1}^{\infty}n[(\beta p^2+(1-\beta))\alpha-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|a_{n,1}|\right.\\ &\left.+\sum_{n=p}^{\infty}n[(\beta p^2+(1-\beta))\alpha-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|b_{n,1}|\right]\\ &+(1-\gamma)\left[\sum_{n=p+1}^{\infty}n[(\beta p^2+(1-\beta))\alpha-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|a_{n,2}|\right.\\ &\left.+\sum_{n=p}^{\infty}n[(\beta p^2+(1-\beta))\alpha-n\beta]\left[1+\frac{(n-p)\lambda}{p+q}\right]^{\eta}|b_{n,2}|\right]\\ \leq\ &\gamma p+(1-\gamma)p=p.\end{aligned}$$

Hence, $H(z) \in \mathcal{A}_p(\beta, \alpha)$. This completes the proof of Theorem 4.1. □

5. Convolution Condition

For harmonic functions

$$f(z) = z^p - \sum_{n=p+1}^{\infty} |a_n| z^n - \sum_{n=p}^{\infty} |b_n| (\overline{z})^n \tag{5.1}$$

and

$$F(z) = z^p - \sum_{n=p+1}^{\infty} |A_n| z^n - \sum_{n=p}^{\infty} |B_n| (\overline{z})^n \tag{5.2}$$

we define the convolution of f and F as

$$(f * F)(z) = z^p - \sum_{n=p+1}^{\infty} |a_n A_n| z^n - \sum_{n=p}^{\infty} |b_n B_n| (\overline{z})^n. \tag{5.3}$$

In the following theorem we examine the convolution property of the class $\mathcal{A}_p(\beta, \alpha)$.

Theorem 5.1. *If f and F are in $\mathcal{A}_p(\beta, \alpha)$, then $(f * F) \in \mathcal{A}_p(\beta, \alpha)$.*

Proof. Let f and F of the forms (5.1) and (5.2) belongs to $\mathcal{A}_p(\beta, \alpha)$. Then the convolution of f and F is given by (5.3). Note that $|A_n| \leq 1$ and $|B_n| \leq 1$, since $F \in \mathcal{A}_p(\beta, \alpha)$.

Then we can write

$$\begin{aligned}
& \sum_{n=p+1}^{\infty} n[(\beta p^2 + (1-\beta))\alpha - n\beta] \left[1 + \frac{(n-p)\lambda}{p+q}\right]^{\eta} |a_n||A_n| \\
& + \sum_{n=p}^{\infty} n[(\beta p^2 + (1-\beta))\alpha - n\beta] \left[1 + \frac{(n-p)\lambda}{p+q}\right]^{\eta} |b_n||B_n| \\
\leq \; & \sum_{n=p+1}^{\infty} n[(\beta p^2 + (1-\beta))\alpha - n\beta] \left[1 + \frac{(n-p)\lambda}{p+q}\right]^{\eta} |a_n| \\
& + \sum_{n=p}^{\infty} n[(\beta p^2 + (1-\beta))\alpha - n\beta] \left[1 + \frac{(n-p)\lambda}{p+q}\right]^{\eta} |b_n|.
\end{aligned}$$

The right hand side of the above inequality is bounded by p because $f \in \mathcal{A}_p(\beta, \alpha)$. Therefore $(f * F) \in \mathcal{A}_p(\beta, \alpha)$. □

6. Integral Operator

Definition 6.1. The Jung-Kim-Srivastava integral operator [4] is defined by

$$\mathcal{J}^{\sigma} K(z) = \frac{(p+1)^{\sigma}}{z\Gamma(\sigma)} \int_0^z \left(\log \frac{z}{t}\right)^{\sigma} K(t)dt, \ \sigma > 0. \tag{6.1}$$

If

$$K(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n,$$

then

$$\mathcal{J}^\sigma K(z) = z^p - \sum_{n=p+1}^{\infty} \left(\frac{p+1}{n+1}\right)^\sigma a_n z^n, \tag{6.2}$$

also $\mathcal{J}^\sigma$ is a linear operator.

Remark 6.1. If $f(z) = h(z) + \overline{g(z)}$, where

$$h(z) = z^p - \sum_{n=p+1}^{\infty} |a_n| z^n, \ g(z) = -\sum_{n=p}^{\infty} |b_n| z^n, \ |b_p| < 1,$$

then

$$\mathcal{J}^\sigma f(z) = \mathcal{J}^\sigma h(z) + \overline{\mathcal{J}^\sigma g(z)}. \tag{6.3}$$

Theorem 6.1. *If $f \in \mathcal{A}_p(\beta, \alpha)$, then $\mathcal{J}^\sigma f$ is also in $\mathcal{A}_p(\beta, \alpha)$.*

Proof. By (6.2) and (6.3), we obtain

$$\begin{aligned}\mathcal{J}^\sigma f(z) &= \mathcal{J}^\sigma \left(z^p - \sum_{n=p+1}^{\infty} |a_n| z^n - \sum_{n=p}^{\infty} |b_n| (\overline{z})^n\right) \\ &= z^p - \sum_{n=p+1}^{\infty} \left(\frac{p+1}{n+1}\right)^\sigma |a_n| z^n - \sum_{n=p}^{\infty} \left(\frac{p+1}{n+1}\right)^\sigma |b_n| (\overline{z})^n,\end{aligned}$$

since $f \in \mathcal{A}_p(\beta, \alpha)$, then by Theorem 2.2, we have

$$\begin{aligned}&\sum_{n=p+1}^{\infty} n[(\beta p^2 + (1-\beta))\alpha - n\beta] \left[1 + \frac{(n-p)\lambda}{p+q}\right]^\eta |a_n| \\ &+ \sum_{n=p}^{\infty} n[(\beta p^2 + (1-\beta))\alpha - n\beta] \left[1 + \frac{(n-p)\lambda}{p+q}\right]^\eta |b_n| \le p,\end{aligned} \tag{6.4}$$

we must show

$$\begin{aligned}&\sum_{n=p+1}^{\infty} n[(\beta p^2 + (1-\beta))\alpha - n\beta] \left[1 + \frac{(n-p)\lambda}{p+q}\right]^\eta \left(\frac{p+1}{n+1}\right)^\sigma |a_n| \\ &+ \sum_{n=p}^{\infty} n[(\beta p^2 + (1-\beta))\alpha - n\beta] \left[1 + \frac{(n-p)\lambda}{p+q}\right]^\eta \left(\frac{p+1}{n+1}\right)^\sigma |b_n| \le p.\end{aligned} \tag{6.5}$$

But in view of (6.4) the inequality in (6.5) holds true if

$$\left(\frac{p+1}{n+1}\right)^\sigma \le 1,$$

since $\sigma > 0$ and $p \le n$, therefore (6.5) holds true and this gives the result. $\square$

References

[1] O. P. Ahuja and J. M. Jahangiri, Multivalent harmonic starlike functions, Ann. Univ. Marie-Curie Sklodowska Sect. A, 55(1) (2001), 1-13.

[2] J. Clunie and T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A. I. Math., 9 (1984), 3-25.

[3] J. M. Jahangiri, G. Murugusundaramoorthy, K. Vijaya, Salagean-type harmonic univalent functions, South. J. Pure Appl.Math., 2 (2002), 77-82.

[4] I. B. Jung, Y. C. Kim and H. M. Srivastava, The Hardy space of analytic functions associated with one-parameter families of integral operators, J. Math. Appl. 176 (1993), 138-197.

[5] H. Mahzoon and S. Latha, Neighborhoods of multivalent functions. Int. Journal of Math. Analysis, 3(30) (2009), 1501-1507.

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